DERIVED CATEGORIES OF QUARTIC DOUBLE FIVEFOLDS

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Abstract. We construct singular quartic double fivefolds whose Kuznetsov component admits a crepant categorical resolution of singularities by a twisted Calabi–Yau threefold. We also construct rational specializations of these fivefolds where such a resolution exists without a twist. This confirms an instance of a higher-dimensional version of Kuznetsov's rationality conjecture, and of a noncommutative version of Reid's fantasy on the connectedness of the moduli of Calabi–Yau threefolds.

1. Introduction

We work over an algebraically closed field k of characteristic 0. Let X be a smooth prime Fano variety of index r, i.e. $\mathrm{Pic}(X) \cong \mathbf{Z}$ is generated by an ample line bundle $\mathcal{O}_X(1)$ such that $\omega_X \cong \mathcal{O}_X(-r)$. Then the bounded derived category of coherent sheaves on X admits a semiorthogonal decomposition

$$\mathrm{D^b}(X) = \langle \mathcal{K}u(X), \mathcal{O}_X, \mathcal{O}_X(1), \dots, \mathcal{O}_X(r-1) \rangle \tag{1.1}$$

where $\mathcal{K}u(X) \subset \mathrm{D^b}(X)$ is the *Kuznetsov component*, defined explicitly as the full subcategory

$$\mathcal{K}u(X) = \{ F \in \mathrm{D^b}(X) \mid \mathrm{Ext}^\bullet(\mathcal{O}_X(i), F) = 0 \text{ for } 0 \leq i \leq r-1 \}.$$

The category $\mathcal{K}u(X)$ should be thought of as the "interesting part" of $\mathrm{D^b}(X)$ obtained as the orthogonal to some tautological pieces coming from the polarization of X.

Remark 1.1. In some cases, $\mathcal{K}u(X)$ can be refined by taking the orthogonal to some additional tautological objects on X (see [Kuz14b] for examples), but in this paper we will only be concerned with $\mathcal{K}u(X)$ as defined above.

Kuznetsov components have recently been very influential in algebraic geometry, due in part to their close connections to birational geometry. The most famous example is when $X \subset \mathbf{P}^5$ is a cubic fourfold, in which case Kuznetsov [Kuz10] conjectured that X is rational if and only if $\mathcal{K}u(X)$ is equivalent to the derived category of a K3 surface. This has been verified for all known rational cubic fourfolds, and there is now a precise Hodge-theoretic characterization of when $\mathcal{K}u(X)$ is equivalent to the derived category of a K3 surface [AT14, BLM+21], but in general the conjecture remains tantalizingly open.

The heuristics behind Kuznetsov's rationality conjecture suggest more generally that if $\mathcal{K}u(X)$ is a Calabi–Yau category of dimension $\dim(X) - 2$ and X is rational, then $\mathcal{K}u(X)$ is equivalent to the derived category of a smooth projective Calabi–Yau variety (see [Kuz16]). Most work on this problem has been confined to the case $\dim(X) = 4$. The purpose of this paper is to investigate an interesting 5-dimensional example.

Namely, we take X to be a quartic double fivefold, i.e. a double cover $X \to \mathbf{P}^5$ branched along a quartic hypersurface, whose Kuznetsov component is defined by the semiorthogonal decomposition

$$\mathrm{D^b}(X) = \langle \mathcal{K}u(X), \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2), \mathcal{O}_X(3) \rangle . \tag{1.2}$$

By [Kuz19, Corollary 4.6] the category $\mathcal{K}u(X)$ is Calabi–Yau of dimension 3. Therefore, Kuznetsov's philosophy suggests the following.

Conjecture 1.2. *If X is a smooth quartic double fivefold which is rational, then there exists an equivalence $\mathcal{K}u(X) \simeq \mathrm{D^b}(W)$ for a smooth projective Calabi–Yau threefold W.*

This conjecture appears to be quite difficult. In fact, it follows from a Hochschild homology computation that $\mathcal{K}u(X)$ is not equivalent to the derived category of any smooth projective variety (see [Per21, Lemma 6.9]), so the conjecture is equivalent to the irrationality of every smooth quartic double fivefold. This remains open despite many recent breakthroughs on the rationality problem. In fact, at the moment no quartic double fivefold is known to be irrational, and with the current techniques the best one could hope to prove is irrationality in the very general case.

In this paper, we instead investigate the situation for certain singular quartic double fivefolds X, which are more tractable. In this situation, the subcategory $\mathcal{K}u(X) \subset \mathrm{D^b}(X)$ may be defined by the same semiorthogonal decomposition (1.2) as in the smooth case. Our first main result is as follows.

Theorem 1.3. *Let $X \to \mathbf{P}^5$ be a double cover branched along a quartic hypersurface $Y \subset \mathbf{P}^5$ which is singular along a line $L \subset \mathbf{P}^5$.*

(i) *For general such Y, there is a crepant categorical resolution of singularities of $\mathcal{K}u(X)$ by $\mathrm{D^b}(W^+, \mathcal{A}^+)$, where W^+ is a 3-dimensional Calabi–Yau algebraic space which is not projective, and $\mathcal{A}^+$ is an Azumaya algebra on W^+.*

(ii) *If in the situation of (i) the Brauer class of $\mathcal{A}^+$ is trivial, then X is rational.*

Remark 1.4. The notion of a crepant categorical resolution[1] is due to Kuznetsov [Kuz08b] and abstracts the properties of the derived category of a crepant resolution of singularities of a variety with rational singularities.

The proof of Theorem 1.3 is based on a study of the derived category of a natural resolution of singularities $\widetilde{X} \to X$. Roughly, we define a Kuznetsov component $\widetilde{\mathcal{K}u}(X) \subset \mathrm{D^b}(\widetilde{X})$ which is a crepant categorical resolution of $\mathcal{K}u(X)$, and use the quadric bundle structure on $\widetilde{X}$ induced by linear projection from L to identify $\widetilde{\mathcal{K}u}(X)$ with the derived category of an associated pair $(W^+, \mathcal{A}^+)$.

The result is motivated by a version of Conjecture 1.2 allowing singularities, which for rational X asks for a crepant resolution of $\mathcal{K}u(X)$ by a Calabi–Yau threefold, instead of a derived equivalence with one (which is impossible when X is singular). Theorem 1.3 verifies a case of this statement, or alternatively a case of Kuznetsov's original rationality heuristics on the resolution $\widetilde{X}$, with two caveats. First, we do not know an example where the Brauer class of $\mathcal{A}^+$ vanishes, so that X is rational; in fact, we expect that this class either always or never vanishes (Remark 3.7). Second, even if the Brauer class of $\mathcal{A}^+$ vanishes, our construction

[1] To be precise, we use the term "crepant categorical resolution" for what is called a "weakly crepant categorical resolution" in [Kuz08b].

produces a *non-projective* Calabi–Yau threefold W^+, in contrast to what is expected from Conjecture 1.2.

To produce a sharper example, we specialize further to a situation where the quadric bundle $\widetilde{X}$ mentioned above admits a section. Our second main result is as follows.

Theorem 1.5. *Let $X \to \mathbf{P}^5$ be a double cover branched along a quartic hypersurface $Y \subset \mathbf{P}^5$. Assume that Y is singular along a line $L \subset \mathbf{P}^5$ and that there exists a 3-plane $P \subset \mathbf{P}^5$ complementary to L which is tangent to Y along a smooth quadric surface.*

(i) *For general such Y, there is a crepant categorical resolution of singularities of $\mathcal{K}u(X)$ by $\mathrm{D^b}(W^{\#})$, where $W^{\#}$ is a projective Calabi–Yau threefold.*

(ii) *In the situation of (i), X is rational.*

Theorems 1.3 and 1.5 give a procedure for connecting the CY3 category $\mathcal{K}u(X)$ of a smooth X first to a twisted geometric Calabi–Yau threefold $(W^+, \mathcal{A}^+)$, and then to a geometric Calabi–Yau threefold $W^{\#}$, where each step proceeds by degenerating the CY3 category and then taking a crepant resolution; we expect this to be useful for studying the category $\mathcal{K}u(X)$ and its moduli spaces of objects by deformation from the geometric case. There is a classical geometric version of this procedure, the simplest example being a conifold transition, where a Calabi–Yau threefold is degenerated and then crepantly resolved to obtain another; such constructions have been widely studied in support of "Reid's fantasy" [Rei87] that all Calabi–Yau threefolds can be connected in this way. Theorem 1.3 can similarly be regarded as evidence for the noncommutative version of Reid's fantasy raised in [KP23].

Let us also note that Theorem 1.5 provides an analog for quartic double fivefolds of [Kuz10, Theorem 5.2], which gives a crepant resolution of the Kuznetsov component of a nodal (necessarily rational) cubic fourfold by the derived category of a K3 surface.

Remark 1.6 (Higher dimensions)**.** Quartic double fivefolds are the first in a series of higher-dimensional examples with similar properties. Namely, if $X \to \mathbf{P}^{4m+1}$ is a double cover branched along a quartic hypersurface, then by [Kuz19, Corollary 4.6] the category $\mathcal{K}u(X)$ is Calabi–Yau of dimension $2m+1$. We expect our arguments to also be useful for proving analogs of Theorems 1.3 and 1.5 when $m > 1$.

Outline. Some basic facts about the geometry of quartic double fivefolds X arising from quartics singular along a line are worked out in §2, after which Theorems 1.3 and 1.5 are proven in §3 and §4, respectively.

Conventions. All functors are derived. In particular, for a morphism $f\colon X \to Y$ we write f^* and f_* for the derived pullback and pushforward, and for $E, F \in \mathrm{D^b}(X)$ we write $E \otimes F$ for their derived tensor product.

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2. Quartic double fivefolds singular along a line

In this section we analyze the geometry of a quartic double fivefold singular along a line. Throughout, V is a 6-dimensional vector space and $\mathbf{P}V \cong \mathbf{P}^5$ is the associated projective space of lines. Let $U \subset V$ be a 2-dimensional subspace corresponding to a projective line $L := \mathbf{P}U$. Set $\overline{V} := V/U$, choose a splitting $V \cong U \oplus \overline{V}$, let $P \subset \mathbf{P}V$ be the corresponding projective 3-space complementary to L, and fix a smooth quadric surface $S \subset \mathbf{P}\overline{V}$.

2.1. **Linear projection from the line.** Projection away from L induces a rational map $\mathbf{P}V \dashrightarrow \mathbf{P}\overline{V}$, which is resolved on the blow up $b\colon \widetilde{\mathbf{P}}V \to \mathbf{P}V$ along L. Write $E \hookrightarrow \widetilde{\mathbf{P}}V$ for the exceptional divisor. This data fits into a diagram

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d, "b_E"] & \widetilde{\mathbf{P}}V \arrow[r, "a"'] \arrow[d, "b"] & \mathbf{P}\overline{V} \\ L \arrow[r, hook] & \mathbf{P}V. \end{tikzcd}$$

Let H be the hyperplane class on $\mathbf{P}V$ and h the hyperplane class on $\mathbf{P}\overline{V}$. Some useful standard facts about the situation are as follows:

Lemma 2.1. *Let $\mathcal{F} := (a_*\mathcal{O}_{\widetilde{\mathbf{P}}V}(H))^\vee$. Then*

(i) $\mathcal{F} \cong \mathcal{O}_{\mathbf{P}\overline{V}} \otimes U \oplus \mathcal{O}_{\mathbf{P}\overline{V}}(-h)$;
(ii) $a\colon \widetilde{\mathbf{P}}V \to \mathbf{P}\overline{V}$ *is isomorphic to the projective bundle* $\mathbf{P}\mathcal{F} \to \mathbf{P}\overline{V}$;
(iii) $h = H - E$ *in* $\mathrm{Pic}(\widetilde{\mathbf{P}}V)$; *and*
(iv) *the canonical divisor of* $\widetilde{\mathbf{P}}V$ *is* $K_{\widetilde{\mathbf{P}}V} = -6H + 3E = -3H - 3h$.

2.2. **Quartic fourfolds.** Within the complete linear system $|4H|$ of quartic fourfolds in $\mathbf{P}V$, consider the linear systems

$$\mathfrak{a} := \{\, Y \in |4H| : Y \text{ is singular along } L \,\} \text{ and}$$
$$\mathfrak{b} := \{\, Y \in \mathfrak{a} : Y \cap P = 2S \,\},$$

of codimensions 21 and 56, respectively, consisting of quartics that are singular along the line L, and those which are furthermore tangent to the 3-space P along the quadric S. Let $Y \hookrightarrow \mathbf{P}V$ be a member of $\mathfrak{a}$, and let $\widetilde{Y} \hookrightarrow \widetilde{\mathbf{P}}V$ be the strict transform of Y along the blow up b. Since L has multiplicity at least 2 in Y, $\widetilde{Y}$ is a member of the complete linear system $|4H - 2E| = |2H + 2h|$ on $\widetilde{\mathbf{P}}V$. Moreover, the projection $\widetilde{Y} \to \mathbf{P}\overline{V}$ exhibits $\widetilde{Y}$ as a conic bundle in $\mathbf{P}\mathcal{F} \to \mathbf{P}\overline{V}$ corresponding to a section $\theta\colon \mathcal{O}_{\mathbf{P}\overline{V}} \to \mathrm{Sym}^2(\mathcal{F}^\vee) \otimes \mathcal{O}_{\mathbf{P}\overline{V}}(2h)$. Every such conic bundle arises in this way, and the ones corresponding to $Y \in \mathfrak{b}$ are those whose $\mathcal{O}_{\mathbf{P}\overline{V}}(4h)$ component of θ is an equation for $2S \subset P \cong \mathbf{P}\overline{V}$.

The following describes the singularities of $\widetilde{Y}$ when it arises from a general member of $\mathfrak{a}$ or $\mathfrak{b}$. In the latter case, the surface $S \hookrightarrow Y$ may be viewed as a subscheme of $\widetilde{Y}$ since S is disjoint from L. Recall that an isolated singularity is called an *ordinary double point* or a *node* if it is a hypersurface singularity and the exceptional divisor E of the blow up therein is a smooth quadric with normal bundle $\mathcal{O}_E(E) \cong \mathcal{O}_E(-1)$.

Lemma 2.2. *The strict transform $\widetilde{Y}$ along the blow up $b\colon \widetilde{\mathbf{P}}V \to \mathbf{P}V$*

(i) *is smooth for general $Y \in \mathfrak{a}$, and*
(ii) *has only 18 nodes along $S \hookrightarrow \widetilde{Y}$ for general $Y \in \mathfrak{b}$.*

Proof. For (i), if Y is a general member of $\mathfrak{a}$, then $\widetilde{Y}$ is a general member of $|2H + 2h|$; since $H + h$ is ample, Bertini's theorem implies $\widetilde{Y}$ is smooth. For (ii), choose linear forms y_1 and y_2 on $\mathbf{P}V$ which together cut out the 3-plane P. Then a quartic Y in the linear system $\mathfrak{b}$ is defined by an equation of the form

$$Y = \mathrm{V}(\beta_{11}y_1^2 + \beta_{12}y_1y_2 + \beta_{22}y_2^2 + \alpha_1 y_1 + \alpha_2 y_2 + q^2)$$

where $\beta_{11}, \beta_{12}, \beta_{22} \in \mathrm{H}^0(P, \mathcal{O}_P(2))$, $\alpha_1, \alpha_2 \in \mathrm{H}^0(P, \mathcal{O}_P(3))$, and q is an equation of S in P; here, functions on P are viewed as functions on $\mathbf{P}V$ via the splitting $V \cong U \oplus \overline{V}$. The base points of $\mathfrak{b}$ are contained in the line L and the quadric surface S, and since y_1 and y_2 span the space of linear functions on L, the base points of the strict transform of $\mathfrak{b}$ on $\widetilde{\mathbf{P}}V$ are contained in S. Therefore the singularities of $\widetilde{Y}$ for general $Y \in \mathfrak{b}$ are points along S at which the above equation vanishes to order at least 2, and these are the points where both cubics α_1 and α_2 vanish. It remains to observe that when these two cubics intersect the quadric q in the projective 3-space P at 18 reduced points, the corresponding singularities on $\widetilde{Y}$ are nodes: indeed, this means that the linear terms of y_1, y_2, α_1, α_2, and q in formal local coordinates are linearly independent, and so the tangent cone therein is a full rank quadric. $\square$

2.3. **Double quartic fivefolds.** Let $f\colon X \to \mathbf{P}V$ and $\tilde{f}\colon \widetilde{X} \to \widetilde{\mathbf{P}}V$ be the double covers branched along Y and $\widetilde{Y}$. Lemma 2.2 shows that $\widetilde{X}$ is smooth for general $Y \in \mathfrak{a}$, and has only 18 nodes along the preimage of S for general $Y \in \mathfrak{b}$. In both cases, the canonical morphism $b_X\colon \widetilde{X} \to X$ resolves the singularities along L. Writing $Z := b_X^{-1}(L)$ for the exceptional divisor and $\pi := a \circ \tilde{f}\colon \widetilde{X} \to \mathbf{P}\overline{V}$, there is a commutative diagram

$$\begin{tikzcd} Z \arrow[r, hook, "i"'] \arrow[d, "b_Z"] & \widetilde{X} \arrow[r, "\pi"'] \arrow[d, "b_X"] & \mathbf{P}\overline{V} \\ L \arrow[r, hook] & X. & \end{tikzcd}$$

Note that, for general Y in either $\mathfrak{a}$ or $\mathfrak{b}$, the singularities of X are rational: those along L are because $Z \to L$ is a quadric threefold fibration, as explained in Lemma 3.2; and the remaining singularities are simply nodes.

To simplify notation, write H and h for the pullback under $\widetilde{X} \to \widetilde{\mathbf{P}}V$ of the corresponding divisor classes. The essential point is that $\widetilde{X}$ is a quadric surface bundle over $\widetilde{\mathbf{P}}V$:

Lemma 2.3. *Let $\mathcal{E} := (\pi_*\mathcal{O}_{\widetilde{X}}(H))^\vee$. Then*

(i) *$\mathcal{E} \cong \mathcal{F} \oplus \mathcal{O}_{\mathbf{P}\overline{V}}(h)$;*
(ii) *$\widetilde{X}$ embeds into $\mathbf{P}\mathcal{E}$ as a hypersurface of class $2H + 2h$; and*

(iii) *the canonical class of* $\widetilde{X}$ *is* $K_{\widetilde{X}} = -2H - 2h$.

Proof. For (i), compute using the projection formula and Lemma 2.1(iii):

$$\pi_*\mathcal{O}_{\widetilde{X}}(H) = a_*\widetilde{f}_*\widetilde{f}^*\mathcal{O}_{\widetilde{\mathbf{P}}V}(H) = a_*\big(\mathcal{O}_{\widetilde{\mathbf{P}}V}(H) \oplus \mathcal{O}_{\widetilde{\mathbf{P}}V}(-H+E)\big) = \mathcal{F}^\vee \oplus \mathcal{O}_{\mathbf{P}\overline{V}}(-h).$$

For (ii), note that $\widetilde{X} \to \widetilde{\mathbf{P}}V$ is a double cover of $\mathbf{P}^2$ branched along a conic Zariski-locally over $\mathbf{P}\overline{V}$, so the canonical map $\widetilde{X} \to \mathbf{P}\mathcal{E}$ is an embedding as a quadric surface. Since the relative hyperplane class of $\mathbf{P}\mathcal{E} \to \mathbf{P}\overline{V}$ is the pullback of H, and since $\mathrm{Pic}(\mathbf{P}\mathcal{E}) \cong \mathbf{Z}H \oplus \mathbf{Z}h$, the class of $\widetilde{X}$ in $\mathbf{P}\mathcal{E}$ is $2H + nh$ for some integer n. To determine n, consider the sequence

$$0 \to \mathcal{O}_{\mathbf{P}\mathcal{E}}(-nh) \to \mathcal{O}_{\mathbf{P}\mathcal{E}}(2H) \to \mathcal{O}_{\widetilde{X}}(2H) \to 0.$$

Pushing forward to $\mathbf{P}\overline{V}$ and computing as in (i) gives

$$0 \to \mathcal{O}_{\mathbf{P}\overline{V}}(-nh) \to \mathrm{Sym}^2(\mathcal{E}^\vee) \to \mathrm{Sym}^2(\mathcal{F}^\vee) \oplus \mathcal{F}^\vee(-h) \to 0.$$

Since $\mathrm{Sym}^2(\mathcal{E}^\vee) \cong \mathrm{Sym}^2(\mathcal{F}^\vee) \oplus \mathcal{F}^\vee(-h) \oplus \mathcal{O}_{\mathbf{P}\overline{V}}(-2h)$ and, as $\mathcal{F}$ is a sum of line bundles, the sequence necessarily splits and thus $n = 2$.

For (iii), use Lemma 2.1(iv) together with the description of $\widetilde{X}$ as a double cover of $\widetilde{\mathbf{P}}V$ branched along a divisor of type $2H + 2h$:

$$K_{\widetilde{X}} = \widetilde{f}^*K_{\widetilde{\mathbf{P}}V} + \frac{1}{2}(2H+2h) = (-3H-3h) + (H+h) = -2H - 2h. \qquad \square$$

Let $D \hookrightarrow \mathbf{P}\overline{V}$ be the discriminant locus of the quadric surface bundle $\pi\colon \widetilde{X} \to \mathbf{P}\overline{V}$: as usual, this is the subscheme over which fibers are singular quadrics, or equivalently, the locus over which the associated bilinear form has corank at least 1. When π is generically smooth, D is a surface and, by [ABB14, Proposition 1.2.5], its singular locus consists of the subscheme D_0 over which the bilinear form furthermore has corank at least 2 together with the image of the singular locus of $\widetilde{X}$. In the situation at hand, D is as follows:

Lemma 2.4. *For general quartics* Y *in either* $\mathfrak{a}$ *or* $\mathfrak{b}$*, the discriminant locus* D *of* π *is a surface of degree* 8 *with singular locus consisting of*

(i) *only* 72 *nodes along* D_0 *for general* $Y \in \mathfrak{a}$*; and*
(ii) *only* 18 *additional nodes corresponding to those of* $\widetilde{X}$ *for general* $Y \in \mathfrak{b}$.

Proof. The fibers of $\pi\colon \widetilde{X} \to \mathbf{P}\overline{V}$ are double planes branched over the conic fibers of $\widetilde{Y} \to \mathbf{P}\overline{V}$, so D is also the discriminant locus of the latter conic bundle. Since $\widetilde{Y}$ is defined by a section of $\mathrm{Sym}^2(\mathcal{F}^\vee) \otimes \mathcal{O}_{\mathbf{P}\overline{V}}(2h)$, writing $c_i := c_i(\mathcal{F}^\vee \otimes \mathcal{O}_{\mathbf{P}\overline{V}}(h))$, [HT84, Theorem 10] or [Ful98, Example 14.4.11] apply to give identities

$$[D] = 2c_1 \quad \text{and} \quad [D_0] = 4\det\begin{pmatrix} c_2 & c_3 \\ c_0 & c_1 \end{pmatrix}$$

in the Chow ring of $\mathbf{P}\overline{V}$ whenever D and D_0 are of expected dimensions 2 and 0, respectively. These have degrees 8 and 72, respectively, using

$$c(\mathcal{F}^\vee \otimes \mathcal{O}_{\mathbf{P}\overline{V}}(h)) = (1+h)^2(1+2h) = 1 + 4h + 5h^2 + 2h^3.$$

For general Y in either $\mathfrak{a}$ or $\mathfrak{b}$, $\widetilde{Y}$ is generically smooth and so D is a surface of degree 8. Since the vector bundle $\mathrm{Sym}^2(\mathcal{F}^\vee) \otimes \mathcal{O}_{\mathbf{P}\overline{V}}(2h)$ defining $\widetilde{Y}$ is globally generated, and since the vector space underlying $\mathfrak{a}$ is canonically identified with its space of sections, a Bertini-type argument as in [Bar80, Lemma 4] shows that the singular locus of D consists of nodes supported on the 0-dimensional locus D_0, giving (i).

Similarly, since $\mathrm{Sym}^2(\mathcal{F}^\vee) \otimes \mathcal{O}_{\mathbf{P}\overline{V}}(2h)$ is generated away from $S \hookrightarrow \mathbf{P}\overline{V}$ by its sections corresponding to members of $\mathfrak{b}$, the Bertini argument also shows that, for general $Y \in \mathfrak{b}$, D has only nodes along D_0 away from S. Thus to prove (ii), it remains to show that D_0 is disjoint from S and that D has only nodes along S for general $Y \in \mathfrak{b}$. For this, and for later use, note that for any $Y \in \mathfrak{b}$, a symmetric bilinear form defining the quadric surface bundle $\pi\colon \widetilde{X} \to \mathbf{P}\overline{V}$ may be written with the notation of Lemma 2.2 as

$$A := \frac{1}{2}\begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & 2\beta_{11} & \beta_{12} & \alpha_1 \\ 0 & \beta_{12} & 2\beta_{22} & \alpha_2 \\ 0 & \alpha_1 & \alpha_2 & 2q^2 \end{pmatrix} : \mathcal{E} \to \mathcal{E}^\vee \otimes \mathcal{O}_{\mathbf{P}\overline{V}}(2h) \tag{2.1}$$

where the matrix is with respect to the decompositions

$$\mathcal{E} \cong \mathcal{O}_{\mathbf{P}\overline{V}}(h) \oplus \mathcal{O}_{\mathbf{P}\overline{V}} \oplus \mathcal{O}_{\mathbf{P}\overline{V}} \oplus \mathcal{O}_{\mathbf{P}\overline{V}}(-h)$$
$$\mathcal{E}^\vee \otimes \mathcal{O}_{\mathbf{P}\overline{V}}(2h) \cong \mathcal{O}_{\mathbf{P}\overline{V}}(h) \oplus \mathcal{O}_{\mathbf{P}\overline{V}}(2h) \oplus \mathcal{O}_{\mathbf{P}\overline{V}}(2h) \oplus \mathcal{O}_{\mathbf{P}\overline{V}}(3h).$$

Therefore an equation for D in $\mathbf{P}\overline{V}$ is given by

$$-4\det(A) = \alpha_2^2\beta_{11} - \alpha_1\alpha_2\beta_{12} + \alpha_1^2\beta_{22} - q^2(4\beta_{11}\beta_{22} - \beta_{12}^2).$$

The singularities of D away from D_0 lie in the image of the singular locus of $\widetilde{X}$, which by Lemma 2.2, is the subscheme $\mathrm{V}(\alpha_1, \alpha_2, q)$. This is disjoint from D_0 since none of β_{11}, β_{12}, β_{22}, nor $4\beta_{11}\beta_{22} - \beta_{12}^2$ vanish there for general $Y \in \mathfrak{b}$. This moreover implies that the tangent cone to D at points therein is a quadric defined by products of linear terms of α_1, α_2, and q, and so, arguing as in Lemma 2.2, D has only nodes precisely when the hypersurfaces defined by α_1, α_2, and q intersect transversally in $\mathbf{P}\overline{V}$, yielding (ii). $\square$

Since $\mathbf{P}\overline{V}$ is, of course, smooth, a node $x \in \widetilde{X}$ must be contained in the nonsmooth locus of the map π. So $\pi(x) \in D$ and it is a singular point of D. Since the proof of Lemma 2.4(ii) shows that, for general $Y \in \mathfrak{b}$, the singularities of D corresponding to those of $\widetilde{X}$ lie away from the corank 2 locus D_0, this implies:

Corollary 2.5. *For general $Y \in \mathfrak{b}$, each node of $\widetilde{X}$ is the cone point of a corank 1 fiber of π.*

2.4. **Sections of the singular quadric surface bundles.** Consider the singular quadric surface bundles $\pi\colon \widetilde{X} \to \mathbf{P}\overline{V}$ arising from a general member Y of the linear system $\mathfrak{b}$. Since the 3-plane P intersects the branch locus Y doubly along the quadric surface S, its preimage along the double cover $f\colon X \to \mathbf{P}V$ is reducible, and so is a union $f^{-1}(P) = P^+ \cup P^-$ of two components, each isomorphic to P. As P is disjoint from the line L, the $P^\pm$ may be identified as subschemes of $\widetilde{X}$, providing two sections

$$\sigma^\pm\colon \mathbf{P}\overline{V} \xrightarrow{\sim} P^\pm \hookrightarrow \widetilde{X}$$

to π; in particular, this implies that $\widetilde{X}$ is rational. Moreover, upon examining the bilinear form (2.1), $\sigma^\pm$ are seen to correspond to the line subbundles

$$\mathcal{N}^\pm := \mathrm{image}\big((\pm q, 0, 0, 1)^t\colon \mathcal{O}_{\mathbf{P}\overline{V}}(-h) \hookrightarrow \mathcal{E}\big).$$

The basic fact about the geometry of these sections is:

Lemma 2.6. *Each section $\sigma^{\pm}$ passes through every node of $\widetilde{X}$, and are otherwise contained in the smooth locus of π.*

Proof. Lemma 2.2(ii) implies that the nodes of $\widetilde{X}$ lie over $Y \cap P$, so $\sigma^{\pm}$ must pass through all of them by construction. That $\sigma^{\pm}$ are otherwise contained in the smooth locus of π follows from noting that $\mathcal{N}^{\pm} \hookrightarrow \mathcal{E}$ does not intersect the kernel of the bilinear form (2.1) at points where at least one of α_1, α_2, or q is nonvanishing. □

3. General situation: twisted geometric component

This section is concerned with the double quartic fivefolds that arise from a general quartic fourfold singular along the line L. So fix a general member Y in the linear system $\mathfrak{a}$, and continue with the notation in §2.

3.1. **Crepant resolution of $\mathcal{K}u(X)$.** First we construct a crepant resolution of the Kuznetsov component of X using the geometric resolution of singularities $b_X \colon \widetilde{X} \to X$. Recall that we define the Kuznetsov component of a smooth prime Fano variety by the semiorthogonal decomposition (1.1). This semiorthogonal decomposition still exists in many situations when the Fano variety is singular, and in particular in our setting:

Lemma 3.1. *There is a semiorthogonal decomposition*

$$\mathrm{D^b}(X) = \langle \mathcal{K}u(X), \mathcal{O}_X, \mathcal{O}_X(H), \mathcal{O}_X(2H), \mathcal{O}_X(3H) \rangle,$$

where $\mathcal{K}u(X) \subset \mathrm{D^b}(X)$ is the full subcategory defined by

$$\mathcal{K}u(X) = \{\, F \in \mathrm{D^b}(X) \mid \mathrm{Hom}^\bullet(\mathcal{O}_X(iH), F) = 0 \text{ for } 0 \le i \le 3 \,\}.$$

Proof. This is a special case of [KP17, Lemma 5.1], but for convenience we recall the proof in our situation. Since X is a double cover of $\mathbf{P}V$ branched along a quartic,

$$\mathrm{H}^\bullet(X, \mathcal{O}_X(mH)) = \mathrm{H}^\bullet(\mathbf{P}V, \mathcal{O}_{\mathbf{P}V}(mH) \oplus \mathcal{O}_{\mathbf{P}V}((m-2)H))$$

for all integers m. Taking $m = 0$ shows that $\mathcal{O}_X$ is an exceptional object; hence the same applies to all of the line bundles $\mathcal{O}_X(nH)$. Similarly, it is easy to see that the displayed cohomology group vanishes for $-3 \le m \le -1$, so the objects $\mathcal{O}_X, \mathcal{O}_X(H), \mathcal{O}_X(2H), \mathcal{O}_X(3H)$ are semiorthogonal. Since $\mathcal{K}u(X)$ is by definition the right orthogonal to the admissible subcategory generated by these objects, the result follows. □

To construct a resolution of $\mathcal{K}u(X)$, we will need a suitable semiorthogonal decomposition of the exceptional divisor $Z \subset \widetilde{X}$:

Lemma 3.2. *There is a semiorthogonal decomposition*

$$\mathrm{D^b}(Z) = \langle b_Z^*\mathrm{D^b}(L) \otimes \mathcal{O}_Z(2Z), b_Z^*\mathrm{D^b}(L) \otimes \mathcal{O}_Z(Z), \mathcal{D} \rangle$$

where $\mathcal{D} = \langle b_Z^*\mathrm{D^b}(L), \mathrm{D^b}(L, \mathcal{B}_0') \rangle$ for $\mathcal{B}_0'$ the even parts of a sheaf of Clifford algebras on L.

Proof. The point is that $b_Z \colon Z \to L$ itself is a quadric threefold fibration: This map factors through $E \cong \mathbf{P}\overline{V} \times L \to L$ and $Z \to E$ is a double cover branched along $\widetilde{Y} \cap E$, which is of class $2H + 2h$. Thus $Z \to L$ is fiberwise a double covering of $\mathbf{P}^3$ branched along a quadric surface, and so, as in Lemma 2.3, is a quadric threefold fibration. The claimed semiorthogonal decomposition now follows by applying Serre duality to the quadric bundle decomposition [Kuz08a, Theorem 4.2] and observing that $Z = H - h$ by Lemma 2.1(iii). □

Now we can give the promised resolution of $\widetilde{\mathcal{K}u}(X)$.

Lemma 3.3. *There is a semiorthogonal decomposition*

$$\begin{aligned}\mathrm{D^b}(\widetilde{X}) = \langle i_* b_Z^* \mathrm{D^b}(L) \otimes \mathcal{O}_Z(2Z), i_* b_Z^* \mathrm{D^b}(L) \otimes \mathcal{O}_Z(Z),\\ \widetilde{\mathcal{K}u}(X), \mathcal{O}_{\widetilde{X}}, \mathcal{O}_{\widetilde{X}}(H), \mathcal{O}_{\widetilde{X}}(2H), \mathcal{O}_{\widetilde{X}}(3H)\rangle,\end{aligned} \tag{3.1}$$

where $\widetilde{\mathcal{K}u}(X)$ is a crepant categorical resolution of singularities of $\mathcal{K}u(X)$. More precisely, writing $\mathcal{K}u(X)^{\mathrm{perf}}$ for the subcategory of $\mathcal{K}u(X)$ consisting of perfect complexes, pullback and pushforward along $b_X \colon \widetilde{X} \to X$ restrict to functors

$$b_X^* \colon \mathcal{K}u(X)^{\mathrm{perf}} \to \widetilde{\mathcal{K}u}(X) \qquad \textit{and} \qquad b_{X*} \colon \widetilde{\mathcal{K}u}(X) \to \mathcal{K}u(X)$$

which are mutually left and right adjoint.

Proof. Apply [Kuz08b, Theorem 1] to the resolution of singularities $b_X \colon \widetilde{X} \to X$ and the decomposition of $\mathrm{D^b}(Z)$ from Lemma 3.2. It is easy to see that the assumptions of this theorem are satisfied, so that it gives a semiorthogonal decomposition

$$\mathrm{D^b}(\widetilde{X}) = \langle i_* b_Z^* \mathrm{D^b}(L) \otimes \mathcal{O}_Z(2Z), i_* b_Z^* \mathrm{D^b}(L) \otimes \mathcal{O}_Z(Z), \widetilde{\mathcal{D}}\rangle$$

where $\widetilde{\mathcal{D}}$ is a crepant categorical resolution of singularities of $\mathrm{D^b}(X)$. In particular, b_X^* fully faithfully embeds the category of perfect complexes on X into $\widetilde{\mathcal{D}}$, so $\widetilde{\mathcal{D}}$ contains the objects $b_X^* \mathcal{O}_X(mH) = \mathcal{O}_{\widetilde{X}}(mH)$ for $m = 0, 1, 2, 3$, and they remain a semiorthogonal exceptional collection. Therefore, we obtain a semiorthogonal decomposition

$$\widetilde{\mathcal{D}} = \langle \widetilde{\mathcal{K}u}(X), \mathcal{O}_{\widetilde{X}}, \mathcal{O}_{\widetilde{X}}(H), \mathcal{O}_{\widetilde{X}}(2H), \mathcal{O}_{\widetilde{X}}(3H)\rangle,$$

where $\widetilde{\mathcal{K}u}(X)$ is the right orthogonal to the subcategory of $\widetilde{\mathcal{D}}$ generated by the displayed line bundles. Putting these decompositions together now gives the statement. $\square$

3.2. **Clifford algebra description of $\widetilde{\mathcal{K}u}(X)$.** Since $\pi \colon \widetilde{X} \to \mathbf{P}\overline{V}$ is a quadric surface fibration by Lemma 2.3, [Kuz08a, Theorem 4.2] gives a semiorthogonal decomposition

$$\mathrm{D^b}(\widetilde{X}) = \langle \mathrm{D^b}(\mathbf{P}\overline{V}, \mathcal{B}_0), \pi^* \mathrm{D^b}(\mathbf{P}\overline{V}) \otimes \mathcal{O}_{\widetilde{X}}, \pi^* \mathrm{D^b}(\mathbf{P}\overline{V}) \otimes \mathcal{O}_{\widetilde{X}}(H)\rangle, \tag{3.2}$$

where $\mathcal{B}_0$ is the sheaf on $\mathbf{P}\overline{V}$ of even Clifford algebras associated with $\pi \colon \widetilde{X} \to \mathbf{P}\overline{V}$. This subsection aims to prove:

Proposition 3.4. *There is an equivalence of categories $\widetilde{\mathcal{K}u}(X) \simeq \mathrm{D^b}(\mathbf{P}\overline{V}, \mathcal{B}_0)$.*

We shall compare the semiorthogonal decomposition from Lemma 3.3 with that of (3.2) via a sequence of mutations. Recall that given a semiorthogonal decomposition

$$\mathcal{T} = \langle \mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n\rangle$$

of a triangulated category with admissible components, there are functors $\mathbf{L}_{\mathcal{A}_i}, \mathbf{R}_{\mathcal{A}_j} \colon \mathcal{T} \to \mathcal{T}$, for $1 \leq i \leq n-1$ and $2 \leq j \leq n$, which give semiorthogonal decompositions of $\mathcal{T}$ of the form

$$\langle \mathcal{A}_1, \dots, \mathcal{A}_{i-1}, \mathbf{L}_{\mathcal{A}_i}(\mathcal{A}_{i+1}), \mathcal{A}_i, \dots, \mathcal{A}_n\rangle \quad \text{and} \quad \langle \mathcal{A}_1, \dots, \mathcal{A}_j, \mathbf{R}_{\mathcal{A}_j}(\mathcal{A}_{j-1}), \mathcal{A}_{j+1}, \dots, \mathcal{A}_n\rangle,$$

called the *left mutation through* $\mathcal{A}_i$ and *right mutation through* $\mathcal{A}_j$, respectively. The following summarizes some basic properties of these functors that we will freely use below; see [Bon89, BK89, Kuz10] for details.

Lemma 3.5. *Let $\mathcal{T} = \langle \mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \rangle$ be a semiorthogonal decomposition with admissible components.*

(i) *If $\mathcal{A}_k$ and $\mathcal{A}_{k+1}$ are completely orthogonal, meaning* $\mathrm{Hom}^\bullet(E, F) = 0$ *for $E \in \mathcal{A}_k$ and $F \in \mathcal{A}_{k+1}$, then*
$$\mathbf{L}_{\mathcal{A}_k}(\mathcal{A}_{k+1}) = \mathcal{A}_{k+1} \quad \textit{and} \quad \mathbf{R}_{\mathcal{A}_{k+1}}(\mathcal{A}_k) = \mathcal{A}_k.$$

(ii) *If $\mathcal{A}_k$ is generated by an exceptional object E, then the associated mutation functors $\mathbf{L}_E$ and $\mathbf{R}_E$ are given by*
$$\mathbf{L}_E(F) = \mathrm{Cone}(\mathrm{Hom}^\bullet(E, F) \otimes E \to F) \quad \textit{and} \quad \mathbf{R}_E(F) = \mathrm{Cone}(F \to \mathrm{Hom}^\bullet(F, E)^\vee \otimes E)[-1].$$

(iii) *If $\mathcal{T} = \mathrm{D^b}(Y)$ for a smooth projective variety Y, then*
$$\mathbf{L}_{\langle \mathcal{A}_1, \dots, \mathcal{A}_{n-1} \rangle}(\mathcal{A}_n) = \mathcal{A}_n \otimes \omega_Y \quad \textit{and} \quad \mathbf{R}_{\langle \mathcal{A}_2, \dots, \mathcal{A}_n \rangle}(\mathcal{A}_1) = \mathcal{A}_1 \otimes \omega_Y^{-1}.$$

We now proceed with the proof of Proposition 3.4. Using the standard Beilinson decomposition for the derived category of projective space, the categories to the right of $\mathrm{D^b}(\mathbf{P}\overline{V}, \mathcal{B}_0)$ in (3.2) are generated by the exceptional collection
$$\langle \mathcal{O}_{\widetilde{X}}(-h), \mathcal{O}_{\widetilde{X}}, \mathcal{O}_{\widetilde{X}}(h), \mathcal{O}_{\widetilde{X}}(2h), \mathcal{O}_{\widetilde{X}}(H), \mathcal{O}_{\widetilde{X}}(H+h), \mathcal{O}_{\widetilde{X}}(H+2h), \mathcal{O}_{\widetilde{X}}(H+3h) \rangle. \qquad ⓪$$
Similarly, after mutating $\widetilde{\mathcal{K}u}(X)$ to the far left of the decomposition from Lemma 3.3, the categories to its right are generated by the exceptional collection
$$\langle i_*\mathcal{O}_Z(2Z-2H), i_*\mathcal{O}_Z(2Z-H), i_*\mathcal{O}_Z(Z), i_*\mathcal{O}_Z(Z+H), \mathcal{O}_{\widetilde{X}}, \mathcal{O}_{\widetilde{X}}(H), \mathcal{O}_{\widetilde{X}}(2H), \mathcal{O}_{\widetilde{X}}(3H) \rangle. \qquad ⑧$$
It suffices to find a sequence of mutations which takes the exceptional collection ⓪ to ⑧. We explain the steps below; see also Figure 1 for a summary.

Step 1. Mutate the subcategory $\langle \mathcal{O}_{\widetilde{X}}(H+2h), \mathcal{O}_{\widetilde{X}}(H+3h) \rangle$ on the right end of ⓪ to the far left. Left mutating $\mathrm{D}(\mathbf{P}\overline{V}, \mathcal{B}_0)$ through the resulting category $\langle \mathcal{O}_{\widetilde{X}}(-H), \mathcal{O}_{\widetilde{X}}(-H+h) \rangle$ results in the decomposition with exceptional objects
$$\langle \mathcal{O}_{\widetilde{X}}(-H), \mathcal{O}_{\widetilde{X}}(-H+h), \mathcal{O}_{\widetilde{X}}(-h), \mathcal{O}_{\widetilde{X}}, \mathcal{O}_{\widetilde{X}}(h), \mathcal{O}_{\widetilde{X}}(2h), \mathcal{O}_{\widetilde{X}}(H), \mathcal{O}_{\widetilde{X}}(H+h) \rangle. \qquad ①$$

Step 2. The pairs $\langle \mathcal{O}_{\widetilde{X}}(-H+h), \mathcal{O}_{\widetilde{X}}(-h) \rangle$ and $\langle \mathcal{O}_{\widetilde{X}}(2h), \mathcal{O}_{\widetilde{X}}(H) \rangle$ are completely orthogonal. Indeed, by Lemma 2.3 we have
$$\pi_*\mathcal{O}_{\widetilde{X}}(H-2h) \cong \mathcal{E}^\vee(-2h) \cong \mathcal{O}_{\mathbf{P}\overline{V}}(-h) \oplus \mathcal{O}_{\mathbf{P}\overline{V}}(-2h)^{\oplus 2} \oplus \mathcal{O}_{\mathbf{P}\overline{V}}(-3h),$$
and hence $\mathrm{H}^\bullet(\mathcal{O}_{\widetilde{X}}(H-2h)) = 0$. Simultaneously transposing these pairs yield the collection
$$\langle \mathcal{O}_{\widetilde{X}}(-H), \mathcal{O}_{\widetilde{X}}(-h), \mathcal{O}_{\widetilde{X}}(-H+h), \mathcal{O}_{\widetilde{X}}, \mathcal{O}_{\widetilde{X}}(h), \mathcal{O}_{\widetilde{X}}(H), \mathcal{O}_{\widetilde{X}}(2h), \mathcal{O}_{\widetilde{X}}(H+h) \rangle. \qquad ②$$

Step 3. Simultaneously perform right mutations in the pairs
$$\langle \mathcal{O}_{\widetilde{X}}(-H), \mathcal{O}_{\widetilde{X}}(-h) \rangle \quad \text{and} \quad \langle \mathcal{O}_{\widetilde{X}}(-H+h), \mathcal{O}_{\widetilde{X}} \rangle,$$
and left mutations in the pairs
$$\langle \mathcal{O}_{\widetilde{X}}(h), \mathcal{O}_{\widetilde{X}}(H) \rangle \quad \text{and} \quad \langle \mathcal{O}_{\widetilde{X}}(2h), \mathcal{O}_{\widetilde{X}}(H+h) \rangle.$$
For each pair, the space of morphisms from the left object to the right is $\mathrm{H}^\bullet(\mathcal{O}_{\widetilde{X}}(H-h)) = k[0]$, which can be computed as in the previous step. The nonzero section corresponds to the

$$
\begin{array}{lllllllll}
\textcircled{0} & (0,-1) & (0,0) & (0,1) & (0,2) & (1,0) & (1,1) & \multicolumn{2}{l}{\underset{\xrightarrow{\qquad\qquad\qquad}}{(1,2) \qquad (1,3)}} \\
\textcircled{1} & (-1,0) & (-1,1) \longleftrightarrow & (0,-1) & (0,0) & (0,1) & (0,2) \longleftrightarrow & (1,0) & (1,1) \\
\textcircled{2} & (-1,0) \longrightarrow & (0,-1) & (-1,1) \longrightarrow & (0,0) & (0,1) \longleftarrow & (1,0) & (0,2) \longleftarrow & (1,1) \\
\textcircled{3} & \underset{\xleftarrow{\qquad\quad}}{(0,-1)} & i_*(0,-1) & (0,0) & i_*(0,0) & i_*(1,0) & (0,1) & i_*(1,1) & (0,2) \\
\textcircled{4} & i_*(0,-1) & (0,0) \longrightarrow & i_*(0,0) & i_*(1,0) & (0,1) & i_*(1,1) \leftarrow & (0,2) & (2,1) \\
\textcircled{5} & i_*(0,-1) & i_*(0,0) & (-1,1) \longleftrightarrow & i_*(1,0) & (0,1) & (1,1) & i_*(1,1) \leftrightarrow & (2,1) \\
\textcircled{6} & i_*(0,-1) & i_*(0,0) & i_*(1,0) & (-1,1) & (0,1) & (1,1) & (2,1) & \underset{\xrightarrow{\qquad\quad}}{i_*(1,1)} \\
\textcircled{7} & i_*(-1,-1) & i_*(0,-1) & i_*(0,0) & i_*(1,0) & (-1,1) & (0,1) & (1,1) & (2,1) \\
\textcircled{8} & i_*(0,-2) & i_*(1,-2) & i_*(1,-1) & i_*(2,-1) & (0,0) & (1,0) & (2,0) & (3,0)
\end{array}
$$

FIGURE 1. This diagram summarizes the sequence of mutations we perform to transform the exceptional collection coming from (3.2) to the exceptional collection coming from the decomposition of Lemma 3.3. The symbol (a, b) represents the object $\mathcal{O}_{\widetilde{X}}(aH + bh)$, and $i_*(a, b)$ represents the object $i_*\mathcal{O}_Z(aH + bh)$. An arrow indicates that the next row is obtained by mutating the object at the tail of the arrow through the object at the head of the arrow. Arrows that point off the sides indicate the application of Serre duality to flip underlined objects to the other side.

equation of $Z = H - h$. Thus each mutation gives the structure sheaf of Z, possibly with a twist; for example,

$$\mathbf{L}_{\mathcal{O}_{\widetilde{X}}(h)}(\mathcal{O}_{\widetilde{X}}(H)) := \mathrm{Cone}(\mathrm{Hom}^\bullet(\mathcal{O}_{\widetilde{X}}(h), \mathcal{O}_{\widetilde{X}}(H)) \otimes \mathcal{O}_{\widetilde{X}}(h) \to \mathcal{O}_{\widetilde{X}}(H)) \cong i_*\mathcal{O}_Z(H).$$

Similarly computing for the others finally yields the collection

$$\langle \mathcal{O}_{\widetilde{X}}(-h), i_*\mathcal{O}_Z(-h), \mathcal{O}_{\widetilde{X}}, i_*\mathcal{O}_Z, i_*\mathcal{O}_Z(H), \mathcal{O}_{\widetilde{X}}(h), i_*\mathcal{O}_Z(H+h), \mathcal{O}_{\widetilde{X}}(2h)\rangle. \qquad \textcircled{3}$$

Step 4. Right mutate $\mathrm{D}(\mathbf{P}\overline{V}, \mathcal{B}_0)$ through $\mathcal{O}_{\widetilde{X}}(-h)$, and then mutate $\mathcal{O}_{\widetilde{X}}(-h)$ to the far right side, resulting in

$$\langle i_*\mathcal{O}_Z(-h), \mathcal{O}_{\widetilde{X}}, i_*\mathcal{O}_Z, i_*\mathcal{O}_Z(H), \mathcal{O}_{\widetilde{X}}(h), i_*\mathcal{O}_Z(H+h), \mathcal{O}_{\widetilde{X}}(2h), \mathcal{O}_{\widetilde{X}}(2H+h)\rangle. \qquad \textcircled{4}$$

Step 5. Simultaneously right mutate $\mathcal{O}_{\widetilde{X}}$ through $i_*\mathcal{O}_Z$, and left mutate $\mathcal{O}_{\widetilde{X}}(2h)$ through $i_*\mathcal{O}_Z(H+h)$. For the right mutation, we have

$$\mathrm{Hom}^\bullet(\mathcal{O}_{\widetilde{X}}, i_*\mathcal{O}_Z) = \mathrm{H}^\bullet(\mathcal{O}_Z) = \mathrm{H}^\bullet(\mathcal{O}_E \oplus \mathcal{O}_E(-H-h)) = k[0]$$

since $Z \to E \cong \mathbf{P}\overline{V} \times L$ is a double cover branched along a divisor of class $2H + 2h$, as discussed in Lemma 3.2. Thus the only morphism is the canonical quotient map, so

$$\mathbf{R}_{i_*\mathcal{O}_Z}(\mathcal{O}_{\widetilde{X}}) = \mathcal{O}_{\widetilde{X}}(-Z) = \mathcal{O}_{\widetilde{X}}(-H+h),$$

using Lemma 2.1(iii). Likewise, for the left mutation, by Grothendieck duality for $i\colon Z \to \widetilde{X}$,

$$\begin{aligned}\mathrm{Hom}^\bullet(i_*\mathcal{O}_Z(H+h), \mathcal{O}_{\widetilde{X}}(2h)) &= \mathrm{Hom}^\bullet(\mathcal{O}_Z(H+h), \mathcal{O}_{\widetilde{X}}(Z+2h)[-1]) \\ &= \mathrm{H}^\bullet(\mathcal{O}_Z[-1]) \\ &= k[-1].\end{aligned}$$

This map corresponds to the exact sequence

$$0 \to \mathcal{O}_{\widetilde{X}}(2h) \to \mathcal{O}_{\widetilde{X}}(H+h) \to i_*\mathcal{O}_Z(H+h) \to 0,$$

and therefore

$$\mathbf{L}_{i_*\mathcal{O}_Z(H+h)}(\mathcal{O}_{\widetilde{X}}(2h)) = \mathcal{O}_{\widetilde{X}}(H+h).$$

In total, the exceptional collection has now become

$$\langle i_*\mathcal{O}_Z(-h), i_*\mathcal{O}_Z, \mathcal{O}_{\widetilde{X}}(-H+h), i_*\mathcal{O}_Z(H), \mathcal{O}_{\widetilde{X}}(h), \mathcal{O}_{\widetilde{X}}(H+h), i_*\mathcal{O}_Z(H+h), \mathcal{O}_{\widetilde{X}}(2H+h)\rangle. \quad (5)$$

Step 6. A computation as in the previous step shows that the pairs $\langle \mathcal{O}_{\widetilde{X}}(-H+h), i_*\mathcal{O}_Z(H)\rangle$ and $\langle i_*\mathcal{O}_Z(H+h), \mathcal{O}_{\widetilde{X}}(2H+h)\rangle$ are completely orthogonal. Transposing the objects in each pair results in the collection

$$\langle i_*\mathcal{O}_Z(-h), i_*\mathcal{O}_Z, i_*\mathcal{O}_Z(H), \mathcal{O}_{\widetilde{X}}(-H+h), \mathcal{O}_{\widetilde{X}}(h), \mathcal{O}_{\widetilde{X}}(H+h), \mathcal{O}_{\widetilde{X}}(2H+h), i_*\mathcal{O}_Z(H+h)\rangle. \quad (6)$$

Step 7. Mutate $i_*\mathcal{O}_Z(H+h)$ to the far left, and left mutate $\mathrm{D^b}(\mathbf{P}\overline{V}, \mathcal{B}_0)$ through the resulting object $i_*\mathcal{O}_Z(-H-h)$. This yields the collection

$$\langle i_*\mathcal{O}_Z(-H-h), i_*\mathcal{O}_Z(-h), i_*\mathcal{O}_Z, i_*\mathcal{O}_Z(H), \mathcal{O}_{\widetilde{X}}(-H+h), \mathcal{O}_{\widetilde{X}}(h), \mathcal{O}_{\widetilde{X}}(H+h), \mathcal{O}_{\widetilde{X}}(2H+h)\rangle. \quad (7)$$

Step 8. Finally, twist the collection (7) by $\mathcal{O}_{\widetilde{X}}(H-h)$. The resulting collection is exactly that appearing in (8) above. This completes the proof of Proposition 3.4. □

3.3. **A twisted Calabi–Yau threefold.** Combining Lemma 3.3 and Proposition 3.4 shows that $\mathcal{K}u(X)$ admits a crepant resolution by the category $\mathrm{D^b}(\mathbf{P}\overline{V}, \mathcal{B}_0)$. Thus, to prove part (i) of Theorem 1.3, it suffices to identify $\mathrm{D^b}(\mathbf{P}\overline{V}, \mathcal{B}_0)$ with the twisted derived category of a non-projective Calabi–Yau threefold. Below, we show how this as well as the rationality criterion in part (ii) of Theorem 1.3 follow from the results of [Kuz14a], modulo the non-projectivity of the threefold which we show in §3.4.

It will be convenient to work more generally and to consider any quadric surface bundle $Q \to \mathbf{P}\overline{V}$ of class $2H + 2h$ in $\mathbf{P}\mathcal{E}$ with smooth total space and discriminant locus D an octic surface with only 72 nodes as singularities, as in Lemma 2.4(i); call any such quadric surface bundle *good*. Writing $\mathcal{B}_0$ for its sheaf of even Clifford algebras on $\mathbf{P}\overline{V}$, we have:

Lemma 3.6. *If $Q \to \mathbf{P}\overline{V}$ is good, then there is an equivalence of categories*

$$\mathrm{D^b}(\mathbf{P}\overline{V}, \mathcal{B}_0) \simeq \mathrm{D^b}(W^+, \mathcal{A}^+)$$

where W^+ is a 3-dimensional Calabi–Yau algebraic space with an Azumaya algebra $\mathcal{A}^+$. Moreover, the Brauer class of $\mathcal{A}^+$ is trivial if and only if $Q \to \mathbf{P}\overline{V}$ has a rational section.

Proof. Let $\mu\colon M \to \mathbf{P}\overline{V}$ be the relative Fano scheme of lines of $Q \to \mathbf{P}\overline{V}$. The fiber of μ over a point of $\mathbf{P}\overline{V} \setminus D$ is $\mathbf{P}^1 \sqcup \mathbf{P}^1$, corresponding to the two rulings on the smooth quadric surface fiber of $Q \to \mathbf{P}\overline{V}$, whereas over a point of D, the fiber is a $\mathbf{P}^1$ with multiplicity 2. Thus the Stein factorization of μ gives a double covering $\tau\colon W \to \mathbf{P}\overline{V}$ branched along D. When $Q \to \mathbf{P}\overline{V}$ is good, then W has only 72 nodes as singularities, exactly over those of D.

Now [Kuz14a, Section 4] gives a diagram

$$\begin{tikzcd} M \arrow[r, dashed, leftrightarrow, "\text{flip}"'] \arrow[d] & M^+ \arrow[d] \\ W & W^+ \arrow[l] \end{tikzcd}$$

where M^+ is an algebraic space obtained as a flip of M, and $W^+ \to W$ is a small resolution of singularities. Since D is an octic, W and W^+ have trivial canonical bundles. By [Kuz14a, Propositions 4.4 and 5.5], the morphism $M^+ \to W^+$ is a $\mathbf{P}^1$-bundle with Brauer class represented by an explicit Azumaya algebra $\mathcal{A}^+$ on W^+. The calculations on [Kuz14a, p.670] then gives the equivalence of categories

$$\mathrm{D}^{\mathrm{b}}(\mathbf{P}\overline{V}, \mathcal{B}_0) \simeq \mathrm{D}^{\mathrm{b}}(W^+, \mathcal{A}^+).$$

The Brauer class of $\mathcal{A}^+$ vanishes if and only if the $\mathbf{P}^1$-bundle $M^+ \to W^+$ admits a rational section. Upon undoing the birational modifications, this is equivalent to $M \to W$ admitting a rational section. Since a rational section of $M \to W$ gives one line in each of the two rulings on the generic fiber of $Q \to \mathbf{P}\overline{V}$, taking their intersection gives a rational section of the quadric surface bundle $Q \to \mathbf{P}\overline{V}$. Conversely, a rational section of $Q \to \mathbf{P}\overline{V}$ distinguishes the unique pair of lines passing through this point, and thus gives a rational section of $M \to W$. ☐

Remark 3.7. We do not know if any good quadric bundle $Q \to \mathbf{P}\overline{V}$ admits a rational section; in particular, we do not know if any of the bundles $\widetilde{X} \to \mathbf{P}\overline{V}$ arising from quartic double fivefolds with $Y \in \mathfrak{a}$ are rational. However, it seems reasonable to expect the following dichotomy: either all or none of these quadric bundles admit a rational section. Indeed, the exponential sequence shows that the Brauer group of a strict (meaning that $h^i(\mathcal{O}) = 0$ for $0 < i < n$) complex Calabi–Yau n-fold is nothing but the torsion in degree 3 cohomology when $n \geq 3$, and hence a topological invariant. Thus, if it were possible to construct the pair $(W^+, \mathcal{A}^+)$ in families, then it would follow that the Brauer class of $\mathcal{A}^+$ either vanishes everywhere or nowhere, and by Lemma 3.6 the vanishing is equivalent to the existence of rational section of $Q \to \mathbf{P}\overline{V}$. This argument is not complete, because a priori it is not clear that the choice of the small resolution of nodal singularities in the construction of $(W^+, \mathcal{A}^+)$ can be made simultaneously in a family.

3.4. **Non-projectivity.** To complete the proof of Theorem 1.3, we just need to show that the Calabi–Yau algebraic space W^+ appearing in §3.3 is not projective. We show that it has no ample divisors via the following non-projectivity criterion:

Lemma 3.8. *Let T be a projective threefold with only nodal singularities. If $\delta := b_4(T) - b_2(T)$ vanishes, then no small resolution of T can be projective.*

Proof. Let $T^+ \to T$ be any small resolution of T. Cohomology of the pair T^+ with the union of its exceptional curves $C_1, \ldots, C_n$ gives a sequence

$$0 \to \mathrm{H}^2(T) \to \mathrm{H}^2(T^+) \to \bigoplus\nolimits_{i=1}^n \mathrm{H}^2(C_i) \to \mathrm{H}^3(T) \to \mathrm{H}^3(T^+) \to 0$$

and an isomorphism $\mathrm{H}^4(T) \cong \mathrm{H}^4(T^+)$. Thus $\delta = 0$ if and only if any class in $\mathrm{H}^2(T^+)$ restricts trivially to $\mathrm{H}^2(C_i)$ for all $i = 1, \ldots, n$. In particular, each of the curves C_i are numerically trivial in T^+, and so there are no ample divisors. See [Cle83, §1], [Wer87, pp.5–6 and Chapter III], and [Add09, p.43] for more. $\square$

The integer δ is called the *defect* of the nodal threefold. In the case T is a double solid, Clemens provides a formula in [Cle83, (3.17)] for δ in terms of the number of independent conditions on certain polynomials imposed by the position of the nodes; see also [Cyn01] for a generalization and an algebraic proof.

Lemma 3.9. *If T is a double cover of $\mathbf{P}^3$ branched along a nodal surface B of degree $2d$, then, writing $\mathcal{I}$ for the ideal sheaf of the nodes of B viewed as a subscheme of $\mathbf{P}^3$,*

$$\delta = \dim \mathrm{H}^1(\mathbf{P}^3, \mathcal{I} \otimes \mathcal{O}_{\mathbf{P}^3}(3d-4)).$$

In the situation at hand, the nodes of the discriminant surface parameterize points where a bilinear form has corank at least 2, so its ideal sheaf in $\mathbf{P}\overline{V}$ is locally generated by the minors of a symmetric matrix. Such an ideal can be resolved as follows:

Lemma 3.10. *Let $\varphi \colon \mathcal{V}^\vee \to \mathcal{V} \otimes \mathcal{L}$ be a symmetric morphism between locally free modules of rank r on a locally Noetherian scheme, where $\mathcal{L}$ is a line bundle. Then there is a complex*

$$0 \to \mathcal{L}^{\vee,\otimes 2} \otimes \wedge^2 \mathcal{V}^\vee \to \mathcal{L}^\vee \otimes (\mathcal{V}^\vee \otimes \mathcal{V})_0 \to \mathrm{Sym}^2(\mathcal{V}) \to \mathcal{I} \otimes \mathcal{L}^{\otimes r-1} \otimes \det(\mathcal{V})^{\otimes 2} \to 0$$

where $(\mathcal{V}^\vee \otimes \mathcal{V})_0 := \ker(\mathrm{ev} \colon \mathcal{V}^\vee \otimes \mathcal{V} \to \mathcal{O}_S)$ and $\mathcal{I}$ is the sheaf of ideals locally generated by the size $r-1$ minors of φ. If $\mathcal{I}$ furthermore has its maximal depth 3, then the complex is exact.

Proof. Up to twisting by a power of $\mathcal{L}$, the first three terms essentially comprise of a generalized Eagon–Northcott complex associated with $\varphi \colon \mathcal{V}^\vee \to \mathcal{V} \otimes \mathcal{L}$, see [Laz04, ($\mathrm{EN}_2$) on p.323]. The point here is to identify the cokernel on the right, and this is done locally by [Józ78, Theorem 3.1] and [GT77]. Therefore it remains to globalize Józefiak's description of the rightmost differential: View the map $\wedge^{r-1}\varphi$ as a bilinear form

$$\wedge^{r-1}\mathcal{V}^\vee \otimes \wedge^{r-1}\mathcal{V}^\vee \to \mathcal{L}^{\otimes r-1}.$$

Locally, this is given by a matrix consisting of size $r-1$ minors of φ, so this map is symmetric, and has image the twisted ideal sheaf $\mathcal{I} \otimes \mathcal{L}^{\otimes r-1}$. Upon identifying $\wedge^{r-1}\mathcal{V}^\vee$ with $\mathcal{V} \otimes \det(\mathcal{V}^\vee)$ via the isomorphism induced by wedge products, this gives a surjective map

$$\mathrm{Sym}^2(\mathcal{V}) \otimes \det(\mathcal{V}^\vee)^{\otimes 2} \to \mathcal{I} \otimes \mathcal{L}^{\otimes r-1}.$$

Twisting by $\det(\mathcal{V})^{\otimes 2}$ gives the desired map. $\square$

Proposition 3.11. *No small resolution of the double cover $\tau \colon W \to \mathbf{P}\overline{V}$ is projective.*

Proof. Let $Q \to \mathbf{P}\overline{V}$ be a good quadric bundle of class $2H + 2h$ in $\mathbf{P}\mathcal{E}$ with discriminant D. The ideal sheaf $\mathcal{I}$ of the nodes of D, viewed as a subscheme of $\mathbf{P}\overline{V}$, is locally generated by the size 3 minors of the symmetric morphism $\mathcal{E} \to \mathcal{E}^\vee \otimes \mathcal{O}_{\mathbf{P}\overline{V}}(2h)$ associated with the bilinear form defining Q. Since $\mathcal{I}$ defines a set of reduced points in $\mathbf{P}\overline{V}$, it has depth 3, so Lemma 3.10 provides a resolution of $\mathcal{I} \otimes \mathcal{O}_{\mathbf{P}\overline{V}}(8h)$ of the form

$$\begin{aligned} 0 \to \mathcal{O}(-3)^2 \oplus \mathcal{O}(-2)^2 \oplus \mathcal{O}(-1)^2 \to \mathcal{O}(-2) \oplus \mathcal{O}(-1)^4 \oplus \mathcal{O}^5 \oplus \mathcal{O}(1)^4 \oplus \mathcal{O}(2) \\ \to \mathcal{O} \oplus \mathcal{O}(1)^2 \oplus \mathcal{O}(2)^4 \oplus \mathcal{O}(3)^2 \oplus \mathcal{O}(4) \to \mathcal{I} \otimes \mathcal{O}(8) \to 0, \end{aligned}$$

where we abbreviate $\mathcal{O}_{\mathbf{P}\overline{V}}(kh)^{\oplus r}$ to $\mathcal{O}(k)^r$ for $k, r \in \mathbf{Z}$. Since none of the terms in the resolution have any higher cohomology, $\mathrm{H}^1(\mathbf{P}\overline{V}, \mathcal{I} \otimes \mathcal{O}_{\mathbf{P}\overline{V}}(8h)) = 0$, so Lemma 3.9 shows that the defect of W vanishes. Lemma 3.8 then applies to give the result. $\square$

4. Special rational examples

Specialize further and consider quartic double fivefolds arising from quartics that are singular along the line L, and which are tangent to the complementary 3-plane P along the smooth quadric surface S. Such quartic double fivefolds are rational since, as observed in §2.4, P gives rise to a section of the associated quadric surface bundle. In this section, we construct a further crepant resolution of the Kuznetsov component of such fivefolds, and show that, this time, it is equivalent to a geometric Calabi–Yau 3-category. Note that Lemma 3.1 holds for a double cover $X \to \mathbf{P}^5$ branched along any quartic hypersurface, regardless of the singularities, so $\mathcal{K}u(X)$ is indeed still defined in this more singular setting.

Throughout, fix a general member Y of the linear system $\mathfrak{b}$, and continue with the notation of §2.

4.1. **Projection from a section.** As in §2.4, the quadric surface bundle $\pi\colon \widetilde{X} \to \mathbf{P}\overline{V}$ admits two distinguished sections. Fix one of them, call it $\sigma\colon \mathbf{P}\overline{V} \to \widetilde{X}$, and let $\mathcal{N} \subset \mathcal{E}$ be the corresponding line subbundle. Relative linear projection centered along $\mathbf{P}\mathcal{N}$ defines rational maps $\widetilde{X} \dashrightarrow \mathbf{P}\bar{\mathcal{E}}$ and $\mathbf{P}\mathcal{E} \dashrightarrow \mathbf{P}\bar{\mathcal{E}}$, the former birational, which are resolved on the blow ups $\widehat{X}$ and $\widehat{\mathbf{P}}\mathcal{E}$ along $\mathbf{P}\mathcal{N}$. These maps fit into a commutative diagram

$$\begin{tikzcd}
Z' \arrow[r, "i'"'] \arrow[d] & \widehat{X} \arrow[r, hook] \arrow[d, "\hat{b}_{\widetilde{X}}"] & \widehat{\mathbf{P}}\mathcal{E} \arrow[r, "\hat{a}"'] \arrow[d, "\hat{b}"] & \mathbf{P}\bar{\mathcal{E}} \arrow[d, "\bar{\pi}"] \\
\mathbf{P}\mathcal{N} \arrow[r, hook] & \widetilde{X} \arrow[r, hook] & \mathbf{P}\mathcal{E} \arrow[r] & \mathbf{P}\overline{V}
\end{tikzcd}$$

where $i'\colon Z' \hookrightarrow \widehat{X}$ is the exceptional divisor of $\hat{b}_{\widetilde{X}}$. Let $\hat{a}_{\widehat{X}}\colon \widehat{X} \to \mathbf{P}\bar{\mathcal{E}}$ denote the birational map given by the restriction of $\hat{a}$ to $\widehat{X}$.

To describe the basic geometry of the situation, abuse notation and write H and h for the hyperplane classes from $\mathbf{P}V$ and $\mathbf{P}\overline{V}$, respectively, pulled back to $\widehat{\mathbf{P}}\mathcal{E}$. Let ξ be the relative hyperplane class of $\bar{\pi}\colon \mathbf{P}\bar{\mathcal{E}} \to \mathbf{P}\overline{V}$, and write $\hat{\pi} := \bar{\pi} \circ \hat{a}\colon \widehat{X} \to \mathbf{P}\overline{V}$.

Lemma 4.1. *Let $\mathcal{G} := (\hat{a}_*\mathcal{O}_{\widehat{\mathbf{P}}\mathcal{E}}(H))^\vee$. Then*

- (i) *$\mathcal{G}$ fits into a short exact sequence $0 \to \bar{\pi}^*\mathcal{N} \to \mathcal{G} \to \mathcal{O}_{\mathbf{P}\bar{\mathcal{E}}}(-\xi) \to 0$;*
- (ii) *$\widehat{\mathbf{P}}\mathcal{E} \to \mathbf{P}\bar{\mathcal{E}}$ is isomorphic to the projective bundle $\mathbf{P}\mathcal{G} \to \mathbf{P}\bar{\mathcal{E}}$;*
- (iii) *the class of $\widehat{X}$ in $\mathbf{P}\mathcal{G}$ is $H + \xi + 2h$; and*
- (iv) *$\xi = H - Z'$ and $K_{\widehat{X}} = -H - \xi - 2h$ in $\mathrm{Pic}(\widehat{X})$.*

Proof. Items (i) and (ii) are relative versions of the corresponding parts from Lemma 2.1, and are standard. Let E' be the exceptional divisor of the blow up $\mathbf{P}\mathcal{G} \to \mathbf{P}\mathcal{E}$. Then $\xi = H - E'$ as divisor classes on $\mathbf{P}\mathcal{G}$. Since the 3-plane $\mathbf{P}\mathcal{N}$ lying at the center of the blow up generically has multiplicity 1 in $\widetilde{X}$, the class of $\widehat{X}$ in $\mathbf{P}\mathcal{G}$ is $2H + 2h - E' = H + \xi + 2h$ by Lemma 2.3(ii), yielding (iii). From this, (iv) is a straightforward computation. $\square$

The projection formula along $\hat{a}$ gives a canonical isomorphism

$$\mathrm{H}^0(\mathbf{P}\mathcal{G}, \mathcal{O}_{\mathbf{P}\mathcal{G}}(H + \xi + 2h)) \cong \mathrm{H}^0(\mathbf{P}\bar{\mathcal{E}}, \mathcal{G}^\vee(\xi + 2h)).$$

Thus an equation of $\widehat{X}$ in $\mathbf{P}\mathcal{G}$ induces a canonical section $w\colon \mathcal{O}_{\mathbf{P}\bar{\mathcal{E}}} \to \mathcal{G}^\vee(\xi+2h)$ which vanishes on the subscheme $W^\#$ over which $\hat{a}_{\widehat{X}}\colon \widehat{X} \to \mathbf{P}\bar{\mathcal{E}}$ is not an isomorphism. The scheme $W^\#$ in fact admits a useful modular interpretation:

Lemma 4.2. *The scheme $W^\#$ canonically embeds into the relative Fano scheme of lines of $\pi\colon \widetilde{X} \to \mathbf{P}\overline{V}$ as the subscheme of those lines incident with the section σ.*

Proof. The modular description of projective bundles identifies $\mathbf{P}\bar{\mathcal{E}}$ as the subscheme in the relative Fano scheme of lines of $\mathbf{P}\mathcal{E} \to \mathbf{P}\overline{V}$ parameterizing lines incident with the section σ, and such that $\hat{a}\colon \mathbf{P}\mathcal{G} \to \mathbf{P}\bar{\mathcal{E}}$ is the universal family. Since $W^\#$ may be characterized as the subscheme of $\mathbf{P}\bar{\mathcal{E}}$ over which the entire fiber of $\hat{a}$ is contained in $\widehat{X}$, the result follows. $\square$

This description allows us to determine the dimension of $W^\#$, and thereby identify the birational morphism $\hat{a}_{\widehat{X}}\colon \widehat{X} \to \mathbf{P}\bar{\mathcal{E}}$ with what it naturally should be:

Lemma 4.3. *The morphism $\hat{a}_{\widehat{X}}\colon \widehat{X} \to \mathbf{P}\bar{\mathcal{E}}$ is isomorphic to the blow up along $W^\#$.*

Proof. Observe that $W^\#$ is of its expected dimension 3: Lemma 4.2 together with Lemma 2.6 implies that $W^\# \to \mathbf{P}\overline{V}$ is finite of degree 2 away from the singularities of the discriminant surface D, and otherwise has 1-dimensional fibers. The ideal sheaf $\mathcal{I}$ of $W^\#$ in $\mathbf{P}\bar{\mathcal{E}}$ therefore admits a Koszul resolution

$$0 \to \mathcal{O}_{\mathbf{P}\bar{\mathcal{E}}} \xrightarrow{w} \mathcal{G}^\vee(\xi + 2h) \to \mathcal{I} \otimes \det(\mathcal{G}^\vee(\xi + 2h)) \to 0.$$

The blow up of $\mathbf{P}\overline{V}$ along $W^\#$ is canonically isomorphic to the Proj of the Rees algebra associated with the twisted ideal sheaf on the right. This sequence then embeds the blow up into $\mathbf{P}\mathcal{G}$ as the relative hyperplane corresponding to the section w. But this is precisely $\widetilde{X}$ by the construction of the section w, yielding the result. $\square$

Remark 4.4. The construction of $W^\# \to \mathbf{P}\overline{V}$ from the quadric surface bundle $\pi\colon \widetilde{X} \to \mathbf{P}\overline{V}$ and the section σ might be viewed as a singular variant of *hyperbolic reduction*: Typically, this is a construction that takes a flat quadric bundle equipped with a smooth section, and produces a quadric bundle of dimension two less whose homological properties are closely related to those of the original quadric bundle. See, for example, [ABB14, §1.4], [KS18, §2.3], [Kuz22], and [Xie23, §4].

4.2. **Conifold transition.** Let $\tau\colon W \to \mathbf{P}\overline{V}$ be, as in §3.3, the double cover branched along the discriminant surface D. When Y is a general member of $\mathfrak{b}$, W has 90 nodes by Lemma 2.4(ii): 72 corresponding to the corank 2 fibers of π, and an additional 18 corresponding to those of $\widetilde{X}$. Construct a small resolution of W in two steps. Begin with the small resolution $W^+ \to W$ of the 72 nodes over D_0 where the M^+ in Lemma 3.6 is obtained by flipping the planes in M parameterizing lines in the planes of $\pi^{-1}(D_0) \hookrightarrow \widetilde{X}$ *not* incident with the section σ. Combined with Lemma 2.6, this ensures that $W^\#$, embedded in M via Lemma 4.2, is disjoint from indeterminancy locus of the birational map $M \dashrightarrow M^+$, providing an embedding $W^\# \hookrightarrow M^+$. Next, composing with $M^+ \to W^+$ provides a morphism $W^\# \to W^+$. This resolves the remaining nodes:

Lemma 4.5. *The morphism $W^{+\!\!+} \to W^+$ is a small resolution of singularities.*

Proof. Lemma 4.2 implies that $W^{+\!\!+} \to W^+$ is an isomorphism away from the nodes of D. Over a node $t \in D_0$, the arguments of [Kuz14a, §4] identify the map $M^+ \to W^+$ over t with the projection of the Hirzebruch surface $F_1 \to \mathbf{P}^1$. Since the section σ passes through a smooth point of $\pi^{-1}(t)$ by Lemma 2.6, the embedding $W^{+\!\!+} \hookrightarrow M^+$ is identified over t as the embedding of a non-exceptional section, and so $W^{+\!\!+} \to W^+$ is an isomorphism over t.

Consider now a node $t \in D$ lying under a node $x \in \widetilde{X}$. Lemmas 4.2 and 2.6 together with Corollary 2.5 imply that the fiber of $W^{+\!\!+} \to W^+$ above t is the smooth conic at the base of the quadric cone $\pi^{-1}(t)$, so it remains to see that the total space of $W^{+\!\!+}$ is smooth above t. The proof of Lemma 2.4(ii) implies that there exists étale coordinates (x_0, x_1, x_2) centered at $t \in \mathbf{P}\overline{V}$ such that $\widetilde{X} \to \mathbf{P}\overline{V}$ along with its section σ is, locally around t, pulled back from the quadric surface bundle in $\mathbf{A}^3 \times \mathbf{P}^3$ defined by

$$-y_0^2 + \beta_{11}y_1^2 + \beta_{12}y_1y_2 + \beta_{22}y_2^2 + (x_1y_1 + x_2y_2 + x_0^2y_3)y_3 = 0$$

with section $(x_0 : 0 : 0 : 1)$; here, $\beta_{11}, \beta_{12}, \beta_{22} \in \Gamma(\mathbf{A}^3, \mathcal{O}_{\mathbf{A}^3})$ are such that the binary quadratic form $\beta_{11}y_1^2 + \beta_{12}y_1y_2 + \beta_{22}y_2^2$ has rank 2 in a neighbourhood of $0 \in \mathbf{A}^3$.

Make the change in projective coordinates $y_0 \mapsto y_0 + x_0y_3$ to simplify the section to $(0 : 0 : 0 : 1)$. The equation of the quadric surface bundle then becomes

$$-y_0^2 + \beta_{11}y_1^2 + \beta_{12}y_1y_2 + \beta_{22}y_2^2 + (x_0y_0 + x_1y_1 + x_2y_2)y_3 = 0.$$

Projection away from the section $(0 : 0 : 0 : 1)$ has the effect of eliminating the coordinate y_3, and a standard computation shows that $W^{+\!\!+}$ is, locally around t, pulled back from the complete intersection in $\mathbf{A}^3 \times \mathbf{P}^2$ given by

$$-y_0^2 + \beta_{11}y_1^2 + \beta_{12}y_1y_2 + \beta_{22}y_2^2 = x_0y_0 + x_1y_1 + x_2y_2 = 0.$$

A direct computation with the Jacobian criterion now implies that the points above the origin of $\mathbf{A}^3$ are smooth. This implies that $W^{+\!\!+}$ is smooth above t, completing the proof. □

Combined with Lemma 4.3, this implies that $\widehat{X}$ is smooth, and so:

Corollary 4.6. *The morphism $\hat{b}_{\widetilde{X}} \colon \widehat{X} \to \widetilde{X}$ is a resolution of singularities.*

To finish the present discussion, consider the exceptional divisor $i' \colon Z' \hookrightarrow \widehat{X}$ of the blow up $\hat{b}_{\widetilde{X}}$. Write $\hat{\pi}_{Z'} := \hat{\pi} \circ i' \colon Z' \to \mathbf{P}\overline{V}$. The following shows that the structure sheaf $\mathcal{O}_{Z'}$ is a relatively exceptional object over $\mathbf{P}\overline{V}$, and that the conormal sheaf $\mathcal{O}_{Z'}(Z')$ has vanishing cohomology over $\mathbf{P}\overline{V}$:

Lemma 4.7. $\hat{\pi}_{Z',*}\mathcal{O}_{Z'} = \mathcal{O}_{\mathbf{P}\overline{V}}[0]$ *and* $\hat{\pi}_{Z',*}\mathcal{O}_{Z'}(Z') = 0$ *in* $\mathrm{D^b}(\mathbf{P}\overline{V})$.

Proof. Write the ideal sheaf sequence of Z' in $\widehat{X}$ using Lemma 4.1(iv) as

$$0 \to \mathcal{O}_{\widehat{X}}(-H + \xi) \to \mathcal{O}_{\widehat{X}} \to i'_*\mathcal{O}_{Z'} \to 0.$$

A straightforward computation using the ideal sheaf sequence of $\widehat{X}$ in $\mathbf{P}\mathcal{G}$ together with the facts of Lemma 4.1 shows that $(\bar{\pi} \circ \hat{a}_{\widehat{X}})_*\mathcal{O}_{\widehat{X}}(-H + \xi) = 0$, and so

$$\hat{\pi}_{Z',*}\mathcal{O}_{Z'} \cong (\bar{\pi} \circ \hat{a}_{\widehat{X}} \circ i')_*\mathcal{O}_{Z'} \cong (\bar{\pi} \circ \hat{a}_{\widehat{X}})_*\mathcal{O}_{\widehat{X}} \cong \hat{\pi}_*\mathcal{O}_{\widehat{X}} \cong \mathcal{O}_{\mathbf{P}\overline{V}}[0].$$

For the second claim, consider the sequence

$$0 \to \mathcal{O}_{\widehat{X}} \to \mathcal{O}_{\widehat{X}}(H-\xi) \to i'_*\mathcal{O}_{Z'}(Z') \to 0$$

obtained by twisting by $Z' = H - \xi$ the ideal sheaf sequence of Z' in $\widehat{X}$. Again, using the ideal sheaf sequence of $\widehat{X}$ in $\mathbf{P}\mathcal{G}$ together with the facts of Lemma 4.1, a straightforward computation shows that $(\bar{\pi} \circ \hat{a}_{\widehat{X}})_*\mathcal{O}_{\widehat{X}}(H-\xi) \cong \mathcal{O}_{\mathbf{P}\overline{V}}[0]$. Since also $(\bar{\pi} \circ \hat{a}_{\widehat{X}})_*\mathcal{O}_{\widehat{X}} \cong \mathcal{O}_{\mathbf{P}\overline{V}}[0]$, we find

$$\hat{\pi}_{Z',*}\mathcal{O}_{Z'}(Z') = (\bar{\pi} \circ \hat{a}_{\widehat{X}} \circ i')_*\mathcal{O}_{Z'}(Z') = 0. \qquad \square$$

4.3. **Geometric Kuznetsov component.** The geometric situation considered so far in this section is summarized by the commutative diagram

$$\begin{tikzcd}
Z' \arrow[rr, hook, "i'"'] \arrow[d] & & \widehat{X} \arrow[dl, "\hat{b}_{\widetilde{X}}"'] \arrow[dd, "\hat{\pi}"] \arrow[dr, "\hat{a}_{\widehat{X}}"] & & \mathbf{P}\mathcal{G}|_{W^{+\!\!\!+}} \arrow[ll, hook'] \arrow[d] \\
\mathbf{P}\mathcal{N} \arrow[r, hook] & \widetilde{X} \arrow[dr, "\pi"] & & \mathbf{P}\overline{\mathcal{E}} \arrow[dl, "\bar{\pi}"'] & W^{+\!\!\!+} \arrow[l, hook', "j"'] \arrow[d] \\
& & \mathbf{P}\overline{V} \arrow[ull, bend left, "\sigma", "\cong"'] & W \arrow[l, "\tau"'] & W^{+} \arrow[l]
\end{tikzcd}$$

where, notably, $\widehat{X}$ is identified as a blow up of schemes over $\mathbf{P}\overline{V}$ in two different ways. These two descriptions as blow ups over $\mathbf{P}\overline{V}$ distinguish two $\mathrm{D^b}(\mathbf{P}\overline{V})$-linear semiorthogonal decompositions of $\mathrm{D^b}(\widehat{X})$, where linearity means that each semiorthogonal component is stable under tensor products with objects in the image of $\hat{\pi}^*\colon \mathrm{D^b}(\mathbf{P}\overline{V}) \to \mathrm{D^b}(\widehat{X})$. They are:

Lemma 4.8. *There are* $\mathrm{D^b}(\mathbf{P}\overline{V})$*-linear semiorthogonal decompositions*

$$\begin{aligned}
\mathrm{D^b}(\widehat{X}) &= \langle \hat{a}^*_{\widehat{X}} j_* \mathrm{D^b}(W^{+\!\!\!+}), \hat{\pi}^*\mathrm{D^b}(\mathbf{P}\overline{V}) \otimes \mathcal{O}_{\widehat{X}}, \hat{\pi}^*\mathrm{D^b}(\mathbf{P}\overline{V}) \otimes \mathcal{O}_{\widehat{X}}(\xi), \hat{\pi}^*\mathrm{D^b}(\mathbf{P}\overline{V}) \otimes \mathcal{O}_{\widehat{X}}(2\xi) \rangle \\
&= \langle \hat{\pi}^*\mathrm{D^b}(\mathbf{P}\overline{V}) \otimes i'_*\mathcal{O}_{Z'}(Z'), \widehat{\mathcal{K}u}(X), \hat{\pi}^*\mathrm{D^b}(\mathbf{P}\overline{V}) \otimes \mathcal{O}_{\widehat{X}}, \hat{\pi}^*\mathrm{D^b}(\mathbf{P}\overline{V}) \otimes \mathcal{O}_{\widehat{X}}(H) \rangle
\end{aligned}$$

where $\widehat{\mathcal{K}u}(X)$ *is a crepant categorical resolution of singularities of* $\mathcal{K}u(X)$ *and* $\widetilde{\mathcal{K}u}(X)$*.*

Proof. The first semiorthogonal decomposition arises from the blow up and projective bundle formulas upon identifying $\hat{a}_{\widehat{X}}\colon \widehat{X} \to \mathbf{P}\overline{\mathcal{E}}$ as the blow up along $W^{+\!\!\!+}$ via Lemma 4.3.

For the second decomposition, we begin by noting that some of the arguments from §3 go through in the more singular setting we are considering now. First, examining the arguments of [Kuz08b, §4] shows that we may still define a subcategory $\widetilde{\mathcal{K}u}(X) \subset \mathrm{D^b}(\widetilde{X})$ by the semiorthogonal decomposition (3.1) of Lemma 3.3. It will no longer be a crepant categorical resolution, since $\widetilde{X}$ is singular, but pullback and pushforward along $b_X\colon \widetilde{X} \to X$ still restrict to functors

$$b_X^*\colon \mathcal{K}u(X)^{\mathrm{perf}} \to \widetilde{\mathcal{K}u}(X) \qquad \text{and} \qquad b_{X*}\colon \widetilde{\mathcal{K}u}(X) \to \mathcal{K}u(X)$$

which are mutually left and right adjoint. Second, Proposition 3.4 holds with verbatim proof, so that there is an equivalence $\widetilde{\mathcal{K}u}(X) \simeq \mathrm{D^b}(\mathbf{P}\overline{V}, \mathcal{B}_0)$.

Now apply [Kuz08b, Theorem 1] with the resolution of singularities $\hat{b}_{\widetilde{X}}\colon \widehat{X} \to \widetilde{X}$ from Corollary 4.6. A suitable semiorthogonal decomposition of $\mathrm{D^b}(Z')$ is provided by Lemma 4.7, which implies that there is a $\mathrm{D^b}(\mathbf{P}\overline{V})$-linear decomposition

$$\mathrm{D^b}(Z') = \langle \hat{\pi}^*_{Z'}\mathrm{D^b}(\mathbf{P}\overline{V}) \otimes \mathcal{O}_{Z'}(Z'), \mathcal{D}' \rangle$$

where $\mathcal{D}'$ is the left orthogonal to $\hat{\pi}^*_{Z'}\mathrm{D^b}(\mathbf{P}\overline{V}) \otimes \mathcal{O}_{Z'}(Z')$, and contains $\hat{\pi}^*_{Z'}\mathrm{D^b}(\mathbf{P}\overline{V}) \otimes \mathcal{O}_{Z'}$. Arguing as in Lemma 3.3 and using the decomposition (3.2) now gives the second promised semiorthogonal decomposition of $\mathrm{D^b}(\widehat{X})$ and the fact that $\widehat{\mathcal{K}u}(X)$ is a crepant categorical resolution of $\mathrm{D^b}(\mathbf{P}\overline{V}, \mathcal{B}_0)$. By the statements from the previous paragraph, this implies that $\widehat{\mathcal{K}u}(X)$ is also a crepant categorical resolution of $\mathcal{K}u(X)$ and $\widetilde{\mathcal{K}u}(X)$. □

To identify the crepant categorical resolution $\widehat{\mathcal{K}u}(X)$ with the geometric Calabi–Yau category $\mathrm{D^b}(W^{+\!\!+})$, we will use a mutation argument to relate the semiorthogonal decompositions of Lemma 4.8. For this purpose, we will make use of the following facts about mutation functors when working relatively to a base.

Lemma 4.9. *Let $f\colon Y \to B$ be a morphism of smooth proper varieties, and let*

$$\mathrm{D^b}(Y) = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$$

be a B-linear semiorthogonal decomposition with admissible components.

(i) *The left and right mutation functors throught any $\mathcal{A}_k$ are B-linear, i.e. commute with tensoring by pullbacks of objects from $\mathrm{D^b}(B)$.*

(ii) *If $\mathcal{A}_k$ is generated by a relatively exceptional object E over B, i.e. $f_*\mathcal{H}om_Y(E, E) \simeq \mathcal{O}_B$ (where $\mathcal{H}om_Y(-,-)$ is the derived sheaf* Hom *on Y) and $\mathcal{A}_k$ is the image of the fully faithful functor $f^*(-) \otimes E\colon \mathrm{D^b}(B) \to \mathrm{D^b}(Y)$, then the associated mutation functors are given by*

$$\mathbf{L}_{f^*\mathrm{D^b}(B)\otimes E}(F) = \mathrm{Cone}((f^*f_*\mathcal{H}om(E, F)) \otimes E \to F),$$
$$\mathbf{R}_{f^*\mathrm{D^b}(B)\otimes E}(F) = \mathrm{Cone}(F \to (f^*f_*\mathcal{H}om(F, E))^\vee \otimes E)[-1].$$

(iii) *We have*

$$\mathbf{L}_{\langle \mathcal{A}_1, \dots, \mathcal{A}_{n-1} \rangle}(\mathcal{A}_n) = \mathcal{A}_n \otimes \omega_{Y/B} \quad \textit{and} \quad \mathbf{R}_{\langle \mathcal{A}_2, \dots, \mathcal{A}_n \rangle}(\mathcal{A}_1) = \mathcal{A}_1 \otimes \omega_{Y/B}^{-1}.$$

Proof. These statements follow easily from the definitions, and in the cases of (i) and (iii) are analogous to their absolute versions in Lemma 3.5. For example, let us prove the first claim of (i). If $\alpha\colon \mathcal{A}_k \to \mathrm{D^b}(Y)$ is the inclusion functor, then the left mutation functor is given by

$$\mathbf{L}_{\mathcal{A}_k}(F) = \mathrm{Cone}(\alpha\alpha^!(F) \to F).$$

Since the right adjoint to the functor $f^*(-) \otimes E$ is $f_*(\mathcal{H}om_Y(E, -))\colon \mathrm{D^b}(Y) \to \mathrm{D^b}(B)$, the claimed formula for $\mathbf{L}_{f^*\mathrm{D^b}(B)\otimes E}$ follows. □

The following result completes the proof of Theorem 1.5.

Proposition 4.10. *There is a $\mathbf{P}\overline{V}$-linear equivalence of categories $\widehat{\mathcal{K}u}(X) \simeq \mathrm{D^b}(W^{+\!\!+})$.*

Proof. The following argument is similar to [Xie23, Theorem 4.2]. As in Proposition 3.4, the equivalence is obtained by comparing the two semiorthogonal decompositions of Lemma 4.8, starting from

$$\mathrm{D^b}(\widehat{X}) = \big\langle \hat{\pi}^*\mathrm{D^b}(\mathbf{P}\overline{V}) \otimes i'_*\mathcal{O}_{Z'}(Z'), \widehat{\mathcal{K}u}(X), \hat{\pi}^*\mathrm{D^b}(\mathbf{P}\overline{V}) \otimes \mathcal{O}_{\widehat{X}}, \hat{\pi}^*\mathrm{D^b}(\mathbf{P}\overline{V}) \otimes \mathcal{O}_{\widehat{X}}(H) \big\rangle.$$

Step 1. Mutate the first component to the far right. Note that by Lemma 4.1 we have $-K_{\widehat{X}} = 2H - Z' + 2h$, and that $H = h$ on the section $\mathbf{P}\mathcal{N}$ and hence also on Z'. Thus the result of the mutation is

$$\mathrm{D}^{\mathrm{b}}(\widehat{X}) = \langle \widehat{\mathcal{K}u}(X), \hat{\pi}^*\mathrm{D}^{\mathrm{b}}(\mathbf{P}\overline{V}) \otimes \mathcal{O}_{\widehat{X}}, \hat{\pi}^*\mathrm{D}^{\mathrm{b}}(\mathbf{P}\overline{V}) \otimes \mathcal{O}_{\widehat{X}}(H), \hat{\pi}^*\mathrm{D}^{\mathrm{b}}(\mathbf{P}\overline{V}) \otimes i'_*\mathcal{O}_{Z'} \rangle.$$

Step 2. Right mutate $\hat{\pi}^*\mathrm{D}^{\mathrm{b}}(\mathbf{P}\overline{V}) \otimes \mathcal{O}_{\widehat{X}}(H)$ through $\hat{\pi}^*\mathrm{D}^{\mathrm{b}}(\mathbf{P}\overline{V}) \otimes i'_*\mathcal{O}_{Z'}$. By the $\mathbf{P}\overline{V}$-linearity of the mutation functor, it suffices to understand the result for $\mathcal{O}_{\widehat{X}}(H)$. As noted above, $H = h$ on Z', and thus by Lemma 4.7 we have

$$\hat{\pi}_*\mathcal{H}om_{\widehat{X}}(\mathcal{O}_{\widehat{X}}(H), i'_*\mathcal{O}_{Z'}) \cong \hat{\pi}_* i'_*\mathcal{O}_{Z'}(-h) \cong \mathcal{O}_{\mathbf{P}\overline{V}}(-h).$$

Using the description of the right mutation functor in Lemma 4.9 and the equality $\xi = H - Z'$ from Lemma 4.1(iv), we find that the result of the mutation is

$$\mathrm{D}^{\mathrm{b}}(\widehat{X}) = \langle \widehat{\mathcal{K}u}(X), \hat{\pi}^*\mathrm{D}^{\mathrm{b}}(\mathbf{P}\overline{V}) \otimes \mathcal{O}_{\widehat{X}}, \hat{\pi}^*\mathrm{D}^{\mathrm{b}}(\mathbf{P}\overline{V}) \otimes i'_*\mathcal{O}_{Z'}, \hat{\pi}^*\mathrm{D}^{\mathrm{b}}(\mathbf{P}\overline{V}) \otimes \mathcal{O}_{\widehat{X}}(\xi) \rangle.$$

Step 3. Left mutate $\hat{\pi}^*\mathrm{D}^{\mathrm{b}}(\mathbf{P}\overline{V}) \otimes i'_*\mathcal{O}_{Z'}$ through $\hat{\pi}^*\mathrm{D}^{\mathrm{b}}(\mathbf{P}\overline{V}) \otimes \mathcal{O}_{\widehat{X}}$. Similarly to the previous step, the result is

$$\mathrm{D}^{\mathrm{b}}(\widehat{X}) = \langle \widehat{\mathcal{K}u}(X), \hat{\pi}^*\mathrm{D}^{\mathrm{b}}(\mathbf{P}\overline{V}) \otimes \mathcal{O}_{\widehat{X}}(-Z'), \hat{\pi}^*\mathrm{D}^{\mathrm{b}}(\mathbf{P}\overline{V}) \otimes \mathcal{O}_{\widehat{X}}, \hat{\pi}^*\mathrm{D}^{\mathrm{b}}(\mathbf{P}\overline{V}) \otimes \mathcal{O}_{\widehat{X}}(\xi) \rangle.$$

Step 4. Mutate $\widehat{\mathcal{K}u}(X)$ through $\hat{\pi}^*\mathrm{D}^{\mathrm{b}}(\mathbf{P}\overline{V}) \otimes \mathcal{O}_{\widehat{X}}(-Z')$ and then mutate $\hat{\pi}^*\mathrm{D}^{\mathrm{b}}(\mathbf{P}\overline{V}) \otimes \mathcal{O}_{\widehat{X}}(-Z')$ to the far right. Using again the formulas $-K_{\widehat{X}} = 2H - Z' + 2h$ and $\xi = H - Z'$, the result is

$$\begin{aligned} \mathrm{D}^{\mathrm{b}}(\widehat{X}) = \langle & \mathbf{R}_{\hat{\pi}^*\mathrm{D}^{\mathrm{b}}(\mathbf{P}\overline{V}) \otimes \mathcal{O}_{\widehat{X}}(-Z')} \widehat{\mathcal{K}u}(X), \hat{\pi}^*\mathrm{D}^{\mathrm{b}}(\mathbf{P}\overline{V}) \otimes \mathcal{O}_{\widehat{X}}, \\ & \hat{\pi}^*\mathrm{D}^{\mathrm{b}}(\mathbf{P}\overline{V}) \otimes \mathcal{O}_{\widehat{X}}(\xi), \hat{\pi}^*\mathrm{D}^{\mathrm{b}}(\mathbf{P}\overline{V}) \otimes \mathcal{O}_{\widehat{X}}(2\xi) \rangle. \end{aligned}$$

Comparing with the first semiorthogonal decomposition from Lemma 4.8, this shows that

$$\mathbf{R}_{\hat{\pi}^*\mathrm{D}^{\mathrm{b}}(\mathbf{P}\overline{V}) \otimes \mathcal{O}_{\widehat{X}}(-Z')} \widehat{\mathcal{K}u}(X) = \hat{a}^*_{\widehat{X}} j_* \mathrm{D}^{\mathrm{b}}(W^{+\!\!+}).$$

Since all of the functors $\mathbf{R}_{\hat{\pi}^*\mathrm{D}^{\mathrm{b}}(\mathbf{P}\overline{V}) \otimes \mathcal{O}_{\widehat{X}}(-Z')}$, $\hat{a}^*_{\widehat{X}}$, and j_* are $\mathbf{P}\overline{V}$-linear, it follows that there is a $\mathbf{P}\overline{V}$-linear equivalence $\widehat{\mathcal{K}u}(X) \simeq \mathrm{D}^{\mathrm{b}}(W^{+\!\!+})$; explicitly, the composition

$$\mathbf{L}_{\hat{\pi}^*\mathrm{D}^{\mathrm{b}}(\mathbf{P}\overline{V}) \otimes \mathcal{O}_{\widehat{X}}(-Z')} \circ \hat{a}^*_{\widehat{X}} \circ j_* \colon \mathrm{D}^{\mathrm{b}}(W^{+\!\!+}) \to \widehat{\mathcal{K}u}(X)$$

is an equivalence. □

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