DERIVED CATEGORIES OF QUADRIC BUNDLES AND MODULI STACKS OF SPINOR SHEAVES

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ABSTRACT. We prove that the Kuznetsov component of a flat family of even-dimensional quadrics of corank at most 2 is equivalent to the twisted derived category of an algebraic space whenever: (i) the open subset of the base over which the quadrics has corank at most 1 is scheme-theoretically dense; and (ii) a certain étale double cover of the closed complement admits a section. This provides the first general geometricity result for Kuznetsov components of higher dimensional quadrics, thereby generalizing works of Kapranov, Bondal, Orlov, Kuznetsov, Moschetti, Xie, and others. Our main tool is the moduli stack of spinor sheaves on a family of quadrics, which we define and study in detail. In the situation of our main result, we produce an open substack which is a $\mathbf{G}_m$-gerbe, and show that the associated twisted derived category is equivalent to the Kuznetsov component of the family of quadrics, thereby providing a geometric interpretation of the Brauer classes appearing in previous works.

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INTRODUCTION

Quadric bundles, being geometrically rich yet tractable, abound in algebraic geometry. Often, varieties of interest may be fibred in, or else otherwise related to quadrics, at which point invariants may be determined by studying a family of quadrics. Such a method has been particularly successful in the study of derived categories of varieties, especially in regards to rationality problems: see, for example, [Kuz16, AB17] for surveys. Results and techniques for quadric bundles are therefore foundational for applications. Our aim here is to generalize some of the basic results regarding derived categories of quadric bundles, making the theory applicable for higher dimensional quadrics and for more general base schemes; in particular, we make an effort to include points of even characteristic throughout. Our general treatment furthermore allows us to use moduli-theoretic techniques in our constructions.

Let $\rho\colon Q \to S$ be a quadric bundle, by which we mean a flat and proper family of degree 2 hypersurfaces in a $\mathbf{P}^n$-bundle $\pi\colon \mathbf{P}\mathscr{E} \to S$ over a base scheme S. Each fibre of ρ is a Fano variety of index $n-1$, so the derived category of the total space Q admits an S-linear semiorthogonal decomposition of the form

$$\mathrm{D}_{\mathrm{qc}}(Q) = \langle \mathrm{Ku}(Q), \rho^*\mathrm{D}_{\mathrm{qc}}(S) \otimes \mathscr{O}_\rho, \rho^*\mathrm{D}_{\mathrm{qc}}(S) \otimes \mathscr{O}_\rho(1), \ldots, \rho^*\mathrm{D}_{\mathrm{qc}}(S) \otimes \mathscr{O}_\rho(n-2)\rangle.$$

The *Kuznetsov component* $\mathrm{Ku}(Q)$ is the "interesting component" of $\mathrm{D}_{\mathrm{qc}}(Q)$ and it is of interest to identify it more explicitly. In general, $\mathrm{Ku}(Q)$ may be described algebraically as the derived category of the sheaf of even Clifford algebras of $\rho\colon Q \to S$, a certain finite locally free non-commutative $\mathscr{O}_S$-algebra, see [Kuz08, Theorem 4.2].

A more geometric description of $\mathrm{Ku}(Q)$ is often desirable and potentially more useful. Kapranov's decomposition from [Kap88, §4] for a quadric hypersurface $Q \subset \mathbf{P}^{2\ell+1}$ might be considered the first example: here, S is a point and $\mathrm{Ku}(Q)$ is equivalent to two copies of $\mathrm{D}_{\mathrm{qc}}(S)$. Bondal and Orlov, in their seminal work [BO95] on semiorthogonal decompositions of algebraic varieties, then give perhaps the first nontrivial example in the case $\rho\colon Q \to \mathbf{P}^1$ is a general pencil of smooth even dimensional quadrics, wherein they show that $\mathrm{Ku}(Q)$ is equivalent to the derived category of the classically associated hyperelliptic curve. General recent interest around these questions perhaps stems from Kuznetsov's work in [Kuz10, Theorem 4.3], where he proves that for the quadric surface fibration $\rho\colon Q \to \mathbf{P}^2$ associated with a smooth cubic fourfold containing a plane, if all fibres of ρ have at worst isolated singularities, then $\mathrm{Ku}(Q)$ is equivalent to the derived category of the K3 double cover of $\mathbf{P}^2$ branched along the discriminant divisor of $\rho\colon Q \to \mathbf{P}^2$, though perhaps twisted by a Brauer class; the hypothesis on the fibres were later removed by [Mos18]. His extensive study of this case led Kuznetsov to propose that a cubic fourfold should be rational if and only if its Kuznetsov component is equivalent to the untwisted derived category of a K3 surface. Spectacular recent progress was made on this conjecture by the recent work of Katzarkov, Kontsevich, Pantev, and Yu on Hodge atoms, see especially [KKPY25, Theorem 6.8 and Examples 6.17–6.21].

Returning to general quadric bundles, Xie shows in [Xie23, Theorem 5.10] that over an algebraically closed field of characteristic 0, for a flat family $\rho\colon Q \to S$ of quadric surfaces over a smooth projective surface, $\mathrm{Ku}(Q)$ is equivalent to the twisted derived category of a resolution of the discriminant double cover of S. She further conjectures that whenever a quadric surface bundle $Q \to S$ over an integral Noetherian scheme has simple degeneration generically and each fibre has corank ≤ 2, the Kuznetsov component of Q is equivalent to the twisted derived category of a scheme. Theorems 4.2 and 4.4 in *loc. cit.* show that such schemes, Brauer classes, and equivalences exist étale locally on S, suggesting an approach via glueing. One view of our work here is that we carry out this program.

Our main result identifies the Kuznetsov component of a quadric bundle with a twisted derived category of an algebraic space M, generically a double cover over S, whenever the degeneracy loci

$$S_c := \{s \in S : \operatorname{corank} Q_s \geq c\} = \{s \in S : \dim \operatorname{Sing} Q_s \geq c-1\}$$

stratifying S with respect to the singularities in the fibres of ρ satisfy three conditions. For example, the hypotheses below are satisfied whenever $\rho\colon Q \to S$ is a quadric bundle of even relative dimension over an algebraically closed field with $S_3 = \varnothing$ and S_2 a finite set of closed points.

Theorem A. — *Let $\rho\colon Q \to S$ be a quadric bundle of relative dimension 2ℓ. Assume that:*

(i) *$S_3 = \varnothing$;*

(ii) *S_2 contains no weakly associated points of S; and*

(iii) *the étale double cover $\tilde{S}_2 \to S_2$ parameterizing families of $(\ell+1)$-planes in $\rho\colon Q \to S$ splits.*

Then there exists an algebraic space M, a proper morphism $M \to S$ which is finite of degree 2 away from S_2, a Brauer class $\beta \in \mathrm{Br}(M)$, and an S-linear equivalence

$$\Phi\colon \mathrm{D}_{\mathrm{qc}}(M, \beta) \to \mathrm{Ku}(Q).$$

A slightly more precise version is given in 7.25. Although this and other results here are phrased in terms of the full quasi-coherent derived category, statements for, say, the bounded derived category in suitable situations may be deduced by adapting the arguments from [BS20, Theorem 6.2]. The hypotheses are discussed in more detail in 7.26, though briefly: When S is integral, (ii) simply means that $S_2 \neq S$. The covering $\tilde{S}_2 \to S_2$ in (iii) is discussed in 4.12. The algebraic space M is canonically an open subspace—depending on a choice of section of $\tilde{S}_2 \to S_2$—of the coarse moduli space of spinor sheaves for the quadric bundle $\rho\colon Q \to S$, β is the Brauer obstruction for the existence of a universal sheaf on $Q \times_S M$, and the equivalence is induced by the Fourier–Mukai transform with respect to the universal sheaf on the associated moduli stack. In this way, our result loosely says that the Kuznetsov component of an even-dimensional quadric bundle is generated by its spinor sheaves, making it apparent that this is a vast generalization of Kapranov's semiorthogonal decomposition from [Kap88, §4] of a smooth quadric Q of dimension 2ℓ over an algebraically closed field:

$$\mathrm{D}_{\mathrm{qc}}(Q) = \langle \mathscr{S}_+^\vee, \mathscr{S}_-^\vee, \mathscr{O}_Q, \mathscr{O}_Q(1), \ldots, \mathscr{O}_Q(2\ell-1)\rangle,$$

where $\mathscr{S}_+^\vee$ and $\mathscr{S}_-^\vee$ are the even and odd spinor bundles on Q. For example, when $\ell = 1$ so that $Q \cong \mathbf{P}^1 \times \mathbf{P}^1$, the spinors are $\mathscr{S}_+^\vee = \mathscr{O}_Q(-1,0)$ and $\mathscr{S}_-^\vee = \mathscr{O}_Q(0,-1)$.

Our approach to Theorem A begins with a careful study of the local situation on S, wherein $\rho\colon Q \to S$ admits a *regular section*; geometrically, this is a section $\sigma\colon S \to Q$ contained in the smooth locus of ρ. In this case, the scheme

$$Q' := \{[\ell] \in \mathbf{F}_1(Q/S) : \sigma \in \ell\}$$

parameterizing lines in the fibres of $\rho\colon Q \to S$ passing through the section σ is another family of quadrics $\rho'\colon Q' \to S$, possibly not flat and of 2 dimensions less, called the *hyperbolic reduction of* $\rho\colon Q \to S$ *along* σ. This construction is well-known, perhaps popularized in the setting of derived categories by the work [ABB14, §§1.3–1.4] of Auel–Bernardara–Bolognesi, and has since featured in many other works, such as [Xie23, KS18, JS25]; perhaps most notably for us, Kuznetsov shows in [Kuz24, Proposition 1.1(3)] that, often, the even Clifford algebras associated with Q' and Q are Morita equivalent. In particular, in view of Kuznetsov's description of the derived category of quadric bundles from [Kuz08, Theorem 4.2], this implies that $\mathrm{Ku}(Q')$ and $\mathrm{Ku}(Q)$ are equivalent.

The next result provides a generalization of this fact. Our result is applicable for more general base schemes S—in particular, for S on which 2 may not be invertible—as well as quadric bundles that may not be generically smooth. Moreover, we can explicitly identify the kernel underlying the equivalence between $\mathrm{Ku}(Q')$ and $\mathrm{Ku}(Q)$.

Theorem B. — *Let $\rho\colon Q \to S$ be a quadric bundle of relative dimension $n-1 \geq 2$, and let $\rho'\colon Q' \to S$ be its hyperbolic reduction along a regular section. Assume that S_{n-1} does not contain any weakly associated points of S. Then there exists an S-linear equivalence $\Phi\colon \mathrm{Ku}(Q') \to \mathrm{Ku}(Q)$.*

This is the content of 6.1. As above, when S is integral, the hypothesis regarding weakly associated points means that $S_{n-1} \neq S$; see 4.19 for its import here. The equivalence Φ is of Fourier–Mukai type, and 6.2 identifies its kernel as a twist of the ideal sheaf of the family over Q' of lines in Q through the chosen section. An important feature of Φ is that it preserves dual spinor sheaves: see 6.3.

Iterating this equivalence gives: If $\rho\colon Q \to S$ is a quadric bundle of relative dimension 2ℓ which carries a complete flag of linear spaces ending in a family of $(\ell-1)$-planes $\mathbf{P}\mathscr{W}$ contained in the smooth locus of ρ—so that fibres have corank at most 2!—and S_2 does not contain any weakly associated points of S, then there is an S-linear equivalence

$$\Psi\colon \mathrm{D}_{\mathrm{qc}}(M) \to \mathrm{Ku}(Q)$$

where $\mu\colon M \to S$ is the (maximal) hyperbolic reduction of $\rho\colon Q \to S$ along $\mathbf{P}\mathscr{W}$. The morphism μ is generically finite of degree 2 and has $\mathbf{P}^1$ as its geometric fibres over points of S_2. Crucially, Ψ is identified in 6.6 as the Fourier–Mukai transform induced by a family of dual spinor sheaves on $Q \times_S M$, essentially verifying the folklore fact that M is a moduli space of spinors for Q.

This suggests a natural approach to Theorem A in general: Consider a moduli stack $\overline{\mathscr{M}} \to S$ of dual spinor sheaves for $\rho\colon Q \to S$ and study the Fourier–Mukai transform associated with the universal sheaf on $Q \times_S \overline{\mathscr{M}}$. In pursuing this idea, however, one is confronted with the conundrum that the coarse moduli space of $\overline{\mathscr{M}}$ is not separated over S because its geometric fibres over S_2 consist of two copies of $\mathbf{P}^1$ and an extra point. This is most readily seen in case of a quadric surface Q: Spinor sheaves here are essentially ideal sheaves $\mathscr{I}_{\ell/Q}$ of lines $\ell \subset Q$, see 5.8. When $Q = \mathbf{P}^2 \cup_{\ell_0} \mathbf{P}^2$ has corank 2 with singular line ℓ_0, the isomorphism class of $\mathscr{I}_{\ell/Q}$ for $\ell \neq \ell_0$ is determined by the irreducible component it lies on and the intersection point $\ell \cap \ell_0$: see [Har94, Example 5.2]. To proceed, one must be able to consistently choose a family of spinor sheaves on corank 2 fibres, and this is encoded by condition (iii) in Theorem A. Fixing a choice of family determines an open substack $\mathscr{M} \subset \overline{\mathscr{M}}$, on which the strategy may be carried through.

Technical tools. — Implementing this argument required developing several general technical results which may be of independent interest. First is a result which allows us to verify whether or not an S-linear Fourier–Mukai functor is an equivalence fppf locally on S. When analyzing the functor $\mathrm{D}_{\mathrm{qc}}(\mathscr{M}) \to \mathrm{D}_{\mathrm{qc}}(S)$ in the setting of Theorem A, this allows us to replace the base S by a cover on which $\rho\colon Q \to S$ admits a regular section, at which point Theorem B applies. Our statement, set up and stated in 1.23, is a variant of Bergh's and Schnürer's conservative descent from [BS20], but modified to apply in a setting where the source may be a $\mathbf{G}_m$-gerbe over a proper and perfect algebraic stack and where the Fourier–Mukai kernel is only relatively perfect over the two factors.

Second is a precise formulation of the general philosophy, originating from ideas of Mukai, Orlov and Lieblich and Olsson, that Fourier–Mukai transforms correspond to morphisms of moduli spaces of complexes. Combined with the descent technique above, this allows us to relate derived category computations with geometric properties of the stack of spinors. Our statements here are phrased in terms of Lieblich's stack of complexes on a flat, proper, and finitely-presented morphism $X \to S$ as introduced in [Lie06]. We formulate a Yoneda-type lemma in 2.4 which canonically relates T-valued points of the stack of complexes with certain complexes in $\mathrm{D}_{\mathrm{qc}}(X \times_S T)$. We then show that complexes underlying fully faithful Fourier–Mukai transforms induce open immersions on stacks of complexes:

Theorem C. — *Let X and Y be flat, proper, and finitely-presented schemes over a scheme S, and let*

$$\Phi_K\colon \mathrm{D}_{\mathrm{qc}}(X) \to \mathrm{D}_{\mathrm{qc}}(Y)$$

be the Fourier–Mukai transform associated with an object $K \in \mathrm{D}_{\mathrm{qc}}(X \times_S Y)$ which is perfect relative to both X and Y. If Φ_K is fully faithful, the assignment $(T, E) \mapsto (T, \Phi_{K_T}(E))$ induces a morphism of stacks

$$\mathrm{FM}_K\colon \mathscr{C}omplexes_{X/S} \to \mathscr{C}omplexes_{Y/S}$$

which is an open immersion immersion.

This is 2.8, and a variant for objects in a semiorthogonal component is formulated in 3.12. Related results appear in [LO15, Lemma 5.2] and [HLT21, Proposition 3.2.3], for example, but have been restricted to embedding spaces of skyscraper sheaves of one into the stack of complexes of the other. We expect that this statement is more generally applicable.

Intersections of two quadrics. — Two applications of Theorem A are sketched in §8. The first concerns the derived category of an intersection X of $m \leq 4$ even-dimensional quadrics. For the Introduction, we focus on the most classical case, $m = 2$; results for $m = 3$ and $m = 4$ are given in

8.7(ii) and 8.7(iii) and further discussed in 8.8. In one of the first works regarding semiorthogonal decompositions in algebraic geometry, Bondal and Orlov show in [BO95, §2] that, over an algebraically closed field of characteristic different from 2, the bounded derived category $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ of coherent sheaves contains that of of the hyperelliptic curve C, as already considered by Weil and Reid in [Wei54, Rei72], arising from the associated pencil of quadrics. This was interpreted as a "categorical explanation" of classical results of [DR77] relating rank 2 vector bundles on C to a Fano scheme of linear spaces in X. Our results imply a generalization of the theorem of Bondal and Orlov to an arbitrary base field:

Theorem D. — *Let $X \subset \mathbf{P}^{2\ell+1}$ be a smooth complete intersection of two general quadrics over an arbitrary field $\mathbf{k}$. There is a semiorthogonal decomposition*

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) = \langle \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(C,\alpha), \mathcal{O}_X(1), \mathcal{O}_X(2), \ldots, \mathcal{O}_X(2\ell-2)\rangle$$

where C is a smooth projective curve and $\mathrm{Br}(C)$.

This is 8.7(i). For $\mathbf{k} \neq \bar{\mathbf{k}}$, this appears to be new even in characteristic 0; see, however, [ABB14, §5] for related results. It would be interesting to relate the Brauer class appearing here to $\mathbf{k}$-rationality of X as studied in [HT21, BW23]. For $\operatorname{char}\mathbf{k} = 2$, this is completely new and may be seen as a "categorical explanation" of the work [Bho90] of Bhosle.

Cubic fourfolds. — The second application concerns smooth cubic fourfolds. Their derived categories possess perhaps the most well-known semiorthogonal decomposition in algebraic geometry: see [Kuz10, Huy17, Huy23] for example. Specifically, since such $X \subset \mathbf{P}^5$ is a Fano variety of index 3, there is a semiorthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2)\rangle.$$

The category $\mathcal{A}_X$ looks like the derived category of a K3 surface: Its Serre functor is the shift-by-two functor by [Kuz19, Corollary 4.1] and the dimensions of its Hochschild homology coincide with the Betti numbers of a K3. This latter statement may be deduced from the weak form of the Hochschild–Kostant–Rosenberg theorem from [AV20, Example 1.7].

Whether or not $\mathcal{A}_X$ is equivalent to the (twisted) derived category of an actual K3 surface depends on the geometry of X—see [AT14, Theorem 1.1], [Huy17, Theorem 1.4], and [Huy23, Theorem 5.1]—and is conjectured in [Kuz10, Conjecture 1.1] to be related to the rationality of X. The most well-studied example for when $\mathcal{A}_X$ is twisted geometric is the case X contains a plane. Kuznetsov showed in [Kuz10, §4] that for a general such X over a field of characteristic different from 2, $\mathcal{A}_X \simeq \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(S,\alpha)$ where S is a double sextic K3 and $\alpha \in \mathrm{Br}(S)$. This was extended to all smooth cubic fourfolds containing a plane over an algebraically closed field of characteristic zero by Moschetti in [Mos18, Theorem 1.2]; Xie later gives in [Xie23, Example 6.2] a much more direct proof in this setting. Our results apply to prove this in any characteristic: see 8.10.

Rather than quote the general result, we highlight a particularly striking example: Consider the Fermat cubic fourfold $X \subset \mathbf{P}^5$ over an algebraically closed field $\mathbf{k}$. Of course, X is extremely special for a variety of reasons. For example, when $\mathbf{k} = \mathbf{C}$, it has the largest automorphism group by [LZ22, Corollary 6.14], it contains the most planes by [DIO23], and it is contained in every Hassett divisor by [YY23, Theorem 1.2]. When $\operatorname{char}\mathbf{k} = 2$, the geometry of X becomes in many ways even more special. Most notably in this context, every fibre of any quadric surface bundle obtained via projection from a plane in X is singular! Nevertheless, 8.10 associates with X a K3 surface S, and 8.13 furthermore determines S as the most special K3 over $\mathbf{k}$:

Theorem E. — *Let $X \subset \mathbf{P}^5_{\mathbf{k}}$ be the Fermat cubic fourfold over an algebraically closed field* $\mathbf{k}$ *of characteristic* 2*. Then there is a semiorthgonal decomposition*

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) = \langle \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(S), \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2)\rangle$$

where S is the supersingular K3 surface of Artin invariant 1.

In fact, the computations in 8.13 are valid over any field $\mathbf{k}$ and relate X to an explicit S given as a complete intersection in $\mathbf{P}^2 \times \mathbf{P}^2$. A precise relationship between X and S taken over a number field $\mathbf{k}$ appears to have first appeared in [HK07, §5], where the global Hasse–Weil zeta function of X is related to that of S. This may be viewed as a motivic shadow of Theorem E, in line with Orlov's conjecture from [Orl05] that semiorthogonal summands correspond to direct summands on the level of rational Chow motives.

Outline. — The first three sections develop the general technical tools used in the proof of the main results: §1 discusses derived categories of algebraic stacks, the main result being the descent statement 1.23 for Fourier–Mukai transforms. Moduli stacks of complexes are discussed in §2 with the open immersion induced by a fully faithful Fourier–Mukai transform in 2.8. Relative exceptional collections and their residual categories are then discussed in §3 where, notably, a generalization of 2.8 to the stack of complexes parameterizing objects in a residual category is formulated in 3.12.

Families of quadrics are discussed starting from §4; since our base scheme is rather general, extra care is required in discussing the corank stratification: see 4.4–4.8. Spinor sheaves for families of quadrics are introduced in §5, where we give new statements about how spinors are related along non-generic hyperplane sections and cones: see 5.11 and 5.15. In §6, we study how the Kuznetsov component of a quadric bundle behaves under hyperbolic reduction, cumulating in the proof of Theorem B. In §7, we construct the stack of spinor sheaves associated with a quadric bundle as the image of a smooth morphism from the Fano scheme of ℓ-planes: see 7.2. After discussing some properties of this stack, we prove Theorem A, see 7.25. Finally, we end off with §8 where we sketch applications of the main results to intersections of quadrics in 8.6 and cubic fourfolds in 8.9.

Conventions. — All stacks are algebraic stacks and we follow the conventions of the Stacks Project: see [Stacks, 026O]. We use the standard abbreviations *fppf*, *fpqc*, and *qcqs* for faithfully flat and finitely presented, faithfully flat and quasi-compact, and quasi-compact and quasi-separated, respectively. Given an algebraic space X and a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$, we write $\pi\colon \mathbf{P}\mathcal{E} \to X$ for the projective bundle of lines in $\mathcal{E}$, so that $\pi_*\mathcal{O}_\pi(1) \cong \mathcal{E}^\vee$. Unless otherwise stated, all tensor products are taken over the structure sheaf of the ambient space.

Acknowledgements. — A portion of this project was carried out during the Junior Trimester Program in Algebraic Geometry at the Hausdorff Research Institute for Mathematics during the autumn of 2023, funded by the Deutsche Forschungsgemeinschaft under Germany's Excellence Strategy - EXC-2047/1 - 390685813. During the completion of this work, RC was partially supported by a Humboldt Postdoctoral Research Fellowship and NO was partially supported by the National Science Foundation under grant DMS-2402087.

1. Derived categories of algebraic stacks

In this section, we collect and develop some facts regarding the derived categories of stacks. One of the primary aims is to formulate and prove a descent result regarding Fourier–Mukai functors from a single weight component of the derived category of a $\mathbf{G}_m$-gerbe to a scheme: see 1.23.

1.1. — We follow the conventions of [HR17] regarding derived categories of stacks. Namely, for a stack $\mathscr{X}$, we write $\mathrm{D}_{\mathrm{qc}}(\mathscr{X})$ for the full subcategory of $\mathrm{D}(\mathscr{X}_{\text{lis-ét}}, \mathscr{O}_{\mathscr{X}})$ consisting of objects with quasi-coherent cohomology sheaves. When $\mathscr{X} = X$ is a scheme, this coincides with the usual quasi-coherent derived category of X. In any case, $\mathrm{D}_{\mathrm{qc}}(\mathscr{X})$ admit the following operations:

(i) If $E, F \in \mathrm{D}_{\mathrm{qc}}(\mathscr{X})$, then $E \otimes^L F \in \mathrm{D}_{\mathrm{qc}}(\mathscr{X})$.
(ii) If $E, F \in \mathrm{D}_{\mathrm{qc}}(\mathscr{X})$ with E perfect, then $R\mathscr{H}om_{\mathscr{O}_{\mathscr{X}}}(E, F) \in \mathrm{D}_{\mathrm{qc}}(\mathscr{X})$.
(iii) If $f : \mathscr{X} \to \mathscr{Y}$ is a morphism of stacks, there is a pair of adjoint exact functors

$$Lf^*\colon \mathrm{D}_{\mathrm{qc}}(\mathscr{Y}) \to \mathrm{D}_{\mathrm{qc}}(\mathscr{X}) \text{ and } Rf_*\colon \mathrm{D}_{\mathrm{qc}}(\mathscr{X}) \to \mathrm{D}_{\mathrm{qc}}(\mathscr{Y})$$

We warn the reader that these are the functors denoted Lf_{qc}^* and $Rf_{qc,*}$ in [HR17].

Pullback and pushforward behave well when $f : \mathscr{X} \to \mathscr{Y}$ is a *concentrated morphism* of stacks, meaning that it is qcqs and, for every morphism $Y \to \mathscr{Y}$ from an affine scheme, the stack $\mathscr{X} \times_{\mathscr{Y}} Y$ has finite quasi-coherent cohomological dimension: see [HR17, Definition 2.4]. The latter condition is superfluous when $\mathscr{X}$ and $\mathscr{Y}$ are algebraic spaces by [Stacks, 073G]. Assuming that $f : \mathscr{X} \to \mathscr{Y}$ is a concentrated morphism of stacks:

(iv) Lf^* and Rf_* satisfy the projection formula and tor-independent base change;
(v) Rf_* commutes with coproducts; and
(vi) Rf_* admits an exact right adjoint $f^{\times}\colon \mathrm{D}_{\mathrm{qc}}(\mathscr{Y}) \to \mathrm{D}_{\mathrm{qc}}(\mathscr{X})$.

See [HR17, Theorem 2.6, Corollaries 4.12 and 4.13, and Theorem 4.14]. Finally, a stack is *concentrated* if its structure morphism to $\operatorname{Spec}\mathbf{Z}$ is. With this, [HR17, Lemma 4.4] shows that

(vii) if $\mathscr{X}$ is concentrated, perfect objects of $\mathrm{D}(\mathscr{X}_{\text{lis-ét}}, \mathscr{O}_{\mathscr{X}})$ are precisely compact objects of $\mathrm{D}_{\mathrm{qc}}(\mathscr{X})$.

1.2. Relative generators. — Recall that an object G of a triangulated category $\mathscr{D}$ is called a *generator* if an object $E \in \mathscr{D}$ is the zero object if and only if $\mathrm{Hom}_{\mathscr{D}}(G, E[i]) = 0$ for all $i \in \mathbf{Z}$: see [Stacks, 09SJ]. A relative version of this for our setting is as follows:

Given a $f : \mathscr{X} \to \mathscr{S}$ concentrated morphism of stacks, a perfect object $G \in \mathrm{D}_{\mathrm{qc}}(\mathscr{X})$ is said to be an *$\mathscr{S}$-linear generator* of $\mathrm{D}_{\mathrm{qc}}(\mathscr{X})$ if for every morphism $T \to \mathscr{S}$ from an affine scheme, the object G_T is a compact generator of $\mathrm{D}_{\mathrm{qc}}(\mathscr{X}_T)$. Basic properties of relative generators are:

(i) If $\mathscr{S}$ is an affine scheme, then G is an $\mathscr{S}$-linear generator if and only if it is a compact generator.
(ii) If G is an $\mathscr{S}$-linear generator of $\mathrm{D}_{\mathrm{qc}}(\mathscr{X})$ and $\mathscr{T} \to \mathscr{S}$ is a morphism of stacks, then $G_{\mathscr{T}}$ is a $\mathscr{T}$-linear generator of $\mathrm{D}_{\mathrm{qc}}(\mathscr{X}_{\mathscr{T}})$.
(iii) If $\{\mathscr{S}_i \to \mathscr{S}\}_i$ is an fpqc covering by stacks and each $G_{\mathscr{S}_i}$ is an $\mathscr{S}_i$-linear generator of $\mathrm{D}_{\mathrm{qc}}(\mathscr{X}_{\mathscr{S}_i})$, then G is an $\mathscr{S}$-linear generator of $\mathrm{D}_{\mathrm{qc}}(\mathscr{X})$.

Proof. Item (i) follows from the fact that perfect generators are preserved under pullback by affine morphisms: see [Stacks, 0BQT]. Item (ii) follows easily from definitions. For (iii), we may replace the base $\mathscr{S} = \operatorname{Spec} A$ by an affine scheme, whereupon to suffices to show that if $\operatorname{Spec} B \to \operatorname{Spec} A$ is a faithfully flat morphism such that $G_B \in \mathrm{D}_{\mathrm{qc}}(\mathscr{X}_B)$ is a compact generator, then so is $G \in \mathrm{D}_{\mathrm{qc}}(\mathscr{X})$. So consider an object $E \in \mathrm{D}_{\mathrm{qc}}(\mathscr{X})$ with $\mathrm{Ext}^i_{\mathscr{X}}(G, E) = 0$ for all $i \in \mathbf{Z}$. Flat base change gives

$$\mathrm{Ext}^i_{\mathscr{X}_B}(G_B, E_B) \cong \mathrm{Ext}^i_{\mathscr{X}}(G, E) \otimes_A B = 0$$

so, since G_B is a generator, $E_B = 0$. Faithful flatness thus implies $E = 0$, and so G is a generator. ■

We may now give a more familiar characterization of relative generators in terms of relative Hom:

1.3. Lemma. — *Let $f : \mathscr{X} \to \mathscr{S}$ be a concentrated morphism of stacks. A perfect object $G \in \mathrm{D}_{\mathrm{qc}}(\mathscr{X})$ is an $\mathscr{S}$-linear generator if and only if for every $0 \neq E \in \mathrm{D}_{\mathrm{qc}}(\mathscr{X})$,*

$$0 \neq Rf_* R\mathscr{H}om_{\mathscr{O}_{\mathscr{X}}}(G, E) \in \mathrm{D}_{\mathrm{qc}}(\mathscr{S}).$$

Proof. Flat base change for $Rf_* R\mathscr{H}om_{\mathscr{O}_{\mathscr{X}}}(G, E)$ combined with **1.2**(iii) reduces this to the case $\mathscr{S}$ is affine, in which case the result follows from **1.2**(i), the definition of a generator, and the fact that the pushforward may be identified with the object $R\mathrm{Hom}_{\mathscr{X}}(G, E)$. ■

1.4. Relative generators of subcategories. — The characterization of **1.3** allows us to generalize the definition of a relative generator to linear triangulated subcategories. We will only need the situation when $f : \mathscr{X} \to S$ is a concentrated morphism from a stack to a qcqs algebraic space. Let $\mathscr{A} \subseteq \mathrm{D}_{\mathrm{qc}}(\mathscr{X})$ be a full triangulated subcategory which is $\mathscr{S}$-*linear*, meaning that for $E \in \mathscr{A}$ and $F \in \mathrm{D}_{\mathrm{qc}}(\mathscr{S})$, $E \otimes^L Lf^*F \in \mathscr{A}$. A perfect object $G \in \mathscr{A}$ is called an $\mathscr{S}$-*linear generator* of $\mathscr{A}$ if

$$E \neq 0 \in \mathscr{A} \iff Rf_* R\mathscr{H}om_{\mathscr{O}_{\mathscr{X}}}(G, E) \neq 0 \in \mathrm{D}_{\mathrm{qc}}(\mathscr{S}).$$

Note that a generator G of $\mathscr{A}$ is also an $\mathscr{S}$-linear generator. More interestingly, given a perfect object $G \in \mathscr{A}$ which is an S-linear generator, we have the following two statements:

(i) If $F \in \mathrm{D}_{\mathrm{qc}}(S)$ is a perfect generator, then $G \otimes^L Lf^*F$ is a compact generator of $\mathscr{A}$.

(ii) The smallest strictly full triangulated S-linear subcategory of $\mathrm{D}_{\mathrm{qc}}(\mathscr{X})$ containing G is $\mathscr{A}$.

Proof. For (i), the stack $\mathscr{X}$ is concentrated, so G is compact in $\mathrm{D}_{\mathrm{qc}}(\mathscr{X})$ and hence also in $\mathscr{A}$—this makes sense as $\mathscr{A}$ is closed under direct sums by virtue of being S-linear. Now, for any $E \in \mathscr{A}$,

$$R\mathrm{Hom}_{\mathscr{X}}(G \otimes^L Lf^*F, E) \cong R\mathrm{Hom}_{\mathscr{X}}(Lf^*F, R\mathscr{H}om_{\mathscr{O}_{\mathscr{X}}}(G, E)) \cong R\mathrm{Hom}_S(F, Rf_* R\mathscr{H}om_{\mathscr{O}_{\mathscr{X}}}(G, E)).$$

The object of $\mathrm{D}_{\mathrm{qc}}(S)$ on the right is nonzero whenever E is nonzero since G is an $\mathscr{S}$-linear perfect generator of $\mathscr{A}$ and F is a perfect generator of $\mathrm{D}_{\mathrm{qc}}(S)$. Looking at the object on the left then shows that $G \otimes^L Lf^*F$ is a compact generator of $\mathscr{A}$. This implies that the category in (ii) contains a compact generator. Since it is also strictly full, triangulated, and closed under direct sums, it follows from [Stacks, 09SN] that it must be equal to $\mathscr{A}$. ■

1.5. Fourier–Mukai transforms. — Given morphisms $f : \mathscr{X} \to \mathscr{S}$ and $g : \mathscr{Y} \to \mathscr{S}$ of stacks, and an object $K \in \mathrm{D}_{\mathrm{qc}}(\mathscr{X} \times_{\mathscr{S}} \mathscr{Y})$, there is an associated *Fourier–Mukai transform* with *kernel* K:

$$\Phi_K : \mathrm{D}_{\mathrm{qc}}(\mathscr{X}) \to \mathrm{D}_{\mathrm{qc}}(\mathscr{Y}) \qquad E \mapsto R\mathrm{pr}_{2,*}(L\mathrm{pr}_1^* E \otimes^L K).$$

When f and g are concentrated morphisms of stacks, the functor Φ_K satisfies many familiar properties:

(i) Φ_K is $\mathscr{S}$-linear: For $E \in \mathrm{D}_{\mathrm{qc}}(\mathscr{X})$ and $F \in \mathrm{D}_{\mathrm{qc}}(\mathscr{S})$, there is a natural isomorphism

$$\Phi_K(E \otimes^L_{\mathscr{O}_{\mathscr{X}}} Lf^*F) \cong \Phi_K(E) \otimes^L_{\mathscr{O}_{\mathscr{Y}}} Lg^*F \in \mathrm{D}_{\mathrm{qc}}(\mathscr{Y}).$$

(ii) Φ_K is an enriched functor: For $E, P \in \mathrm{D}_{\mathrm{qc}}(\mathscr{X})$, if P and $\Phi_K(P)$ are perfect, then there is a natural morphism in $\mathrm{D}_{\mathrm{qc}}(\mathscr{S})$

$$Rf_* R\mathscr{H}om_{\mathscr{O}_{\mathscr{X}}}(P, E) \to Rg_* R\mathscr{H}om_{\mathscr{O}_{\mathscr{Y}}}(\Phi_K(P), \Phi_K(E))$$

which, after applying the functor $\mathrm{H}^0(\mathscr{S}, -)$, becomes the homomorphism

$$\Phi_K : \mathrm{Hom}_{\mathscr{X}}(P, E) \to \mathrm{Hom}_{\mathscr{Y}}(\Phi_K(P), \Phi_K(E)).$$

(iii) Φ_K is compatible with flat base change: For a morphism $h : \mathscr{T} \to \mathscr{S}$ of stacks, if either h is flat or both f and g are flat, then for any $E \in \mathrm{D}_{\mathrm{qc}}(\mathscr{X})$, there is a canonical isomorphism

$$\Phi_K(E)_{\mathscr{T}} \cong \Phi_{K_{\mathscr{T}}}(E_{\mathscr{T}}) \in \mathrm{D}_{\mathrm{qc}}(\mathscr{Y} \times_{\mathscr{S}} \mathscr{T}).$$

Proof. Part (i) is a calculation using the projection formula. Construct the morphism in (ii) via the Yoneda lemma: for every $F \in \mathrm{D}_{\mathrm{qc}}(\mathscr{S})$, there is a map

$$\begin{aligned}\mathrm{Hom}_{\mathscr{S}}(F, Rf_*R\mathscr{H}om_{\mathscr{O}_{\mathscr{X}}}(P,E)) &\cong \mathrm{Hom}_{\mathscr{X}}(P \otimes^L_{\mathscr{O}_{\mathscr{X}}} Lf^*F, E)\\ &\xrightarrow{\Phi_K} \mathrm{Hom}_{\mathscr{Y}}(\Phi_K(P \otimes^L_{\mathscr{O}_{\mathscr{X}}} Lf^*F), \Phi_K(E)) \cong \mathrm{Hom}_{\mathscr{S}}(F, Rg_*R\mathscr{H}om_{\mathscr{O}_{\mathscr{Y}}}(\Phi_K(P), \Phi_K(E)))\end{aligned}$$

where the two identifications use perfectness of P and $\Phi_K(P)$, and the bottom identification additionally uses (i). Finally, (iii) follows from tor-independent base change applied to the square

$$\begin{array}{ccc}\mathscr{X}_{\mathscr{T}} \times_{\mathscr{T}} \mathscr{Y}_{\mathscr{T}} & \longrightarrow & \mathscr{Y}_{\mathscr{T}}\\ \downarrow & & \downarrow\\ \mathscr{X} \times_{\mathscr{S}} \mathscr{Y} & \longrightarrow & \mathscr{Y}.\end{array}$$
■

The Fourier–Mukai transform often has a right adjoint:

1.6. Lemma. — *Let $f: \mathscr{X} \to \mathscr{S}$ and $g: \mathscr{Y} \to \mathscr{S}$ be morphisms of stacks and $K \in \mathrm{D}_{\mathrm{qc}}(\mathscr{X} \times_{\mathscr{S}} \mathscr{Y})$. If f is concentrated, then the functor $\Phi_K: \mathrm{D}_{\mathrm{qc}}(\mathscr{X}) \to \mathrm{D}_{\mathrm{qc}}(\mathscr{Y})$ has an exact right adjoint.*

Proof. Φ_K commutes with coproducts since it is a composition of three functors which do, with $R\mathrm{pr}_{2,*}$ doing so by the assumption that f is concentrated and **1.1**(vi). Since the category $\mathrm{D}_{\mathrm{qc}}(\mathscr{X})$ is well-generated by [HNR19, Theorem B.1], the result therefore follows from Neeman's version of Brown Representability Theorem for well-generated categories: see [Nee01, Theorem 8.4.4]. ■

We now give a criterion for when Φ_K is fully faithful in terms of a perfect generator. First, the following is a well-known criterion for when an exact functor preserves compact objects.

1.7. Lemma. — *Let $\Psi: \mathscr{D} \to \mathscr{D}'$ be an exact functor between triangulated categories with arbitrary direct sums. Assume that Ψ has a right adjoint R and that $\mathscr{D}$ is compactly generated. Then Ψ takes compact objects to compact objects if and only if R commutes with direct sums.* ■

Next, the following is a general criterion for fully faithfulness in terms of a compact generator:

1.8. Lemma. — *Let $\Psi: \mathscr{D} \to \mathscr{D}'$ be an exact functor between triangulated categories with arbitrary direct sums. Assume that*

- *$\mathscr{D}$ has a compact generator G;*
- *Ψ has a right adjoint R; and*
- *Ψ takes compact objects to compact objects.*

Then Ψ is fully faithful if and only if the following map is an isomorphism for all $i \in \mathbf{Z}$:

$$\Psi: \mathrm{Ext}^i_{\mathscr{D}}(G,G) \to \mathrm{Ext}^i_{\mathscr{D}'}(\Psi(G), \Psi(G)).$$

Proof. The "if" direction is clear. Conversely, suppose that the map between self extensions of G and $\Psi(G)$ are all isomorphisms. Consider the set of objects $M \in \mathscr{D}$ such that the unit $\eta_M: M \to R\Psi(M)$ of adjunction is an isomorphism. This set is closed under cones since R and Ψ are exact. It is closed under direct sums since both R and Ψ preserves direct sums by **1.7** and left adjointness, respectively. Finally, this set contains the generator G since, for every $i \in \mathbf{Z}$,

$$\mathrm{Ext}^i_{\mathscr{D}}(G,G) \cong \mathrm{Ext}^i_{\mathscr{D}'}(\Psi(G), \Psi(G)) \cong \mathrm{Ext}^i_{\mathscr{D}}(G, R\Psi(G))$$

and G is a generator. Since the generator G is compact, it now follows from [Stacks, 09SR] that the set in question contains every object of $\mathscr{D}$. Therefore Ψ is fully faithful. ■

In our setting of Fourier–Mukai transforms, the condition on self extensions of a generator may be verified on the level of enriched Homs, leading to the following full faithfulness criterion:

1.9. Lemma. — *Let $f : \mathscr{X} \to \mathscr{S}$ and $g : \mathscr{Y} \to \mathscr{S}$ be morphisms of concentrated stacks, and $K \in \mathrm{D}_{\mathrm{qc}}(\mathscr{X} \times_{\mathscr{S}} \mathscr{Y})$. Assume that*

- *$\mathrm{D}_{\mathrm{qc}}(\mathscr{S})$ has a perfect generator F;*
- *$\mathrm{D}_{\mathrm{qc}}(\mathscr{X})$ has an $\mathscr{S}$-linear perfect generator G; and*
- *$\Phi_K : \mathrm{D}_{\mathrm{qc}}(\mathscr{X}) \to \mathrm{D}_{\mathrm{qc}}(\mathscr{Y})$ takes perfect complexes to perfect complexes.*

Then Φ_K is fully faithful if and only if the morphism in $\mathrm{D}_{\mathrm{qc}}(\mathscr{S})$

$$\phi : Rf_* R\mathscr{H}om_{\mathscr{O}_{\mathscr{X}}}(G, G) \to Rg_* R\mathscr{H}om_{\mathscr{O}_{\mathscr{Y}}}(\Phi_K(G), \Phi_K(G))$$

of **1.5**(ii) *is an isomorphism.*

Proof. Since $F \in \mathrm{D}_{\mathrm{qc}}(\mathscr{S})$ is a perfect generator, ϕ is an isomorphism if and only if it becomes an isomorphism after applying the functor $\mathrm{Ext}^i_{\mathscr{S}}(F, -)$ for every $i \in \mathbf{Z}$. By its construction in **1.5**(ii), this gives the map

$$\Phi_K\colon\ \mathrm{Ext}^i_{\mathscr{X}}(Lf^*F \otimes^L G, G) \to \mathrm{Ext}^i_{\mathscr{Y}}(\Phi_K(Lg^*F \otimes^L G), \Phi_K(G)).$$

If Φ_K is fully faithful, then this is, of course, an isomorphism for all $i \in \mathbf{Z}$, whence ϕ is an isomorphism. Conversely, if ϕ is an isomorphism, then so is the map

$$\phi' : Rf_* R\mathscr{H}om_{\mathscr{O}_{\mathscr{X}}}(G, G \otimes^L Lf^*F) \to Rg_* R\mathscr{H}om_{\mathscr{O}_{\mathscr{Y}}}(\Phi_K(G), \Phi_K(G \otimes^L Lf^*F))$$

obtained from $\phi \otimes^L F$ via the projection formula and $\mathscr{S}$-linearity from **1.5**(i). Applying $\mathrm{Ext}^i_{\mathscr{S}}(F, -)$ to ϕ' then shows that the map

$$\Phi_K\colon\ \mathrm{Ext}^i_{\mathscr{X}}(G \otimes^L Lf^*F, G \otimes^L Lf^*F) \to \mathrm{Ext}^i_{\mathscr{Y}}(\Phi_K(G \otimes^L Lf^*F), \Phi_K(G \otimes^L Lf^*F))$$

is an isomorphism for every $i \in \mathbf{Z}$. Since Φ_K has a right adjoint by **1.6** and $G \otimes^L Lf^*F$ is a compact generator of $\mathrm{D}_{\mathrm{qc}}(\mathscr{X})$ by **1.4**(i), full faithfulness of Φ_K now follows from the criterion **1.8**. ∎

Combined with tor-independent base change and **1.5**(iii), the criterion **1.9** implies that full faithfulness of Φ_K is often preserved under base change, and also that it may be checked fpqc locally:

1.10. Lemma. — *In the setting of* **1.9**, *furthermore suppose given a morphism $h\colon \mathscr{T} \to \mathscr{S}$ of concentrated stacks. If either h is flat or both f and g are flat, then:*

(i) *If $\Phi_K : \mathrm{D}_{\mathrm{qc}}(\mathscr{X}) \to \mathrm{D}_{\mathrm{qc}}(\mathscr{Y})$ is fully faithful, then so is $\Phi_{K_{\mathscr{T}}} : \mathrm{D}_{\mathrm{qc}}(\mathscr{X}_{\mathscr{T}}) \to \mathrm{D}_{\mathrm{qc}}(\mathscr{Y}_{\mathscr{T}})$.*

(ii) *If $h\colon \mathscr{T} \to \mathscr{S}$ is faithfully flat and quasi-compact, then the converse of* (i) *also holds.* ∎

1.11. Gerbes. — In this work, a $\mathbf{G}_m$*-gerbe* refers to a gerbe banded by $\mathbf{G}_m$; that is, the data of

- a morphism $\pi\colon \mathscr{X} \to X$ from a stack to an algebraic space; and
- an isomorphism $\xi\colon \mathbf{G}_{m,\mathscr{X}} \to \mathscr{I}_{\mathscr{X}}$ of sheaves on $\mathscr{X}_{\mathrm{fppf}}$ from $\mathbf{G}_m$ to the inertia stack of $\mathscr{X}$.

The identification ξ provides each quasi-coherent $\mathscr{O}_{\mathscr{X}}$-module $\mathscr{F}$ with a right action by $\mathbf{G}_m$, whence a weight decomposition $\mathscr{F} = \bigoplus_{i \in \mathbf{Z}} \mathscr{F}_i$ where the weight i component is characterized by

$$\mathscr{F}_i(x) = \{s \in \mathscr{F}(x) : s \cdot \gamma = \gamma^i s \text{ for all } \gamma \in \mathscr{A}ut_T(x) = \mathbf{G}_m(T)\}$$

for each morphism $T \to X$ and $x \in \mathscr{X}_T$. If $\mathscr{F} = \mathscr{F}_i$, then the sheaf $\mathscr{F}$ is said to *have weight* i. Sheaves of different weights do not map to one another, so we have a product decomposition $\mathrm{QCoh}(\mathscr{X}) = \prod_{i \in \mathbf{Z}} \mathrm{QCoh}_i(\mathscr{X})$ where $\mathrm{QCoh}_i(\mathscr{X}) \subset \mathrm{QCoh}(\mathscr{X})$ denotes the full subcategory of quasi-coherent sheaves of weight i. Pullback along π induces an equivalence

$$\pi^*\colon\ \mathrm{QCoh}(\mathscr{O}_X) \to \mathrm{QCoh}_0(\mathscr{O}_{\mathscr{X}})$$

between the categories of quasi-coherent $\mathcal{O}_X$-modules and quasi-coherent $\mathcal{O}_{\mathscr{X}}$-modules of weight 0. If $\pi\colon \mathscr{X} \to X$ admits a section $\sigma\colon X \to \mathscr{X}$, then the restriction of pullback along σ induces, for any $i \in \mathbf{Z}$, an equivalence

$$\sigma^*\colon \mathrm{QCoh}_i(\mathcal{O}_{\mathscr{X}}) \to \mathrm{QCoh}(\mathcal{O}_X)$$

between the categories of quasi-coherent $\mathcal{O}_{\mathscr{X}}$-modules of weight i and quasi-coherent $\mathcal{O}_X$-modules.

1.12. — For each $i \in \mathbf{Z}$, let $\mathrm{D}_{\mathrm{qc},i}(\mathscr{X}) \subset \mathrm{D}_{\mathrm{qc}}(\mathscr{X})$ be the full subcategory consisting of objects whose cohomology sheaves all have weight i. Furthermore, [BS21, Theorem 5.4] shows that there is an orthogonal decomposition

$$\mathrm{D}_{\mathrm{qc}}(\mathscr{X}) \cong \prod\nolimits_{i\in\mathbf{Z}} \mathrm{D}_{\mathrm{qc},i}(\mathscr{X}).$$

Namely, the $\mathrm{D}_{\mathrm{qc},i}(\mathscr{X})$ are completely orthogonal to one another for different $i \in \mathbf{Z}$, and every object $E \in \mathrm{D}_{\mathrm{qc}}(\mathscr{X})$ may be uniquely expressed as a direct sum $E = \bigoplus_{i\in\mathbf{Z}} E_i$ with $E_i \in \mathrm{D}_{\mathrm{qc},i}(\mathscr{X})$. It is shown in [BS21, §5] that these decompositions enjoy the following compatibilities:

(i) If $f : \mathscr{X} \to \mathscr{Y}$ is a morphism of $\mathbf{G}_m$-gerbes, then Lf^* and Rf_* preserve the decomposition.
(ii) If $E \in \mathrm{D}_{\mathrm{qc},i}(\mathscr{X})$ and $F \in \mathrm{D}_{\mathrm{qc},j}(\mathscr{X})$, then $E \otimes^L F \in \mathrm{D}_{\mathrm{qc},i+j}(\mathscr{X})$.
(iii) Pullback along π induces an equivalence $L\pi^*\colon \mathrm{D}_{\mathrm{qc}}(X) \to \mathrm{D}_{\mathrm{qc},0}(\mathscr{X})$.
(iv) The functor $R\pi_*\colon \mathrm{D}_{\mathrm{qc}}(\mathscr{X}) \to \mathrm{D}_{\mathrm{qc}}(X)$ may be identified as $E \mapsto E_0$.
(v) If $\sigma\colon X \to \mathscr{X}$ is a section, then $L\sigma^*\colon \mathrm{D}_{\mathrm{qc},i}(\mathscr{X}) \to \mathrm{D}_{\mathrm{qc}}(X)$ is an equivalence for any $i \in \mathbf{Z}$.

Note that $\pi\colon \mathscr{X} \to X$ is a concentrated morphism of stacks: this is because, in addition to π being qcqs, the functor $\pi_*\colon \mathrm{QCoh}(\mathcal{O}_{\mathscr{X}}) \to \mathrm{QCoh}(\mathcal{O}_X)$ is exact. When the algebraic space X is qcqs, each of the categories $\mathrm{D}_{\mathrm{qc},i}(\mathscr{X})$ admits a perfect generator by [HR17, Example 9.3].

1.13. Trivial gerbes. — The *trivial* $\mathbf{G}_m$*-gerbe* over X is the relative classifying stack $B\mathbf{G}_{m,X} \to X$. The tautological line bundle on $B\mathbf{G}_{m,X}$ is of weight 1 and provides a section $X \to B\mathbf{G}_{m,X}$. Conversely, suppose $\pi\colon \mathscr{X} \to X$ is a $\mathbf{G}_m$-gerbe and that $\mathscr{X}$ carries a line bundle $\mathscr{L}$ of weight 1. Then

(i) $\pi\colon \mathscr{X} \to X$ is isomorphic to the trivial $\mathbf{G}_m$-gerbe; and
(ii) there is a canonical isomorphism $X \cong \mathbf{A}(\mathscr{L}) \setminus \{0\}$ between X and the complement of the zero section in the affine bundle $\mathbf{A}(\mathscr{L}) := \operatorname{Spec}\operatorname{Sym}(\mathscr{L}^\vee)$ on $\mathscr{L}$ over $\mathscr{X}$.

Item (i) is because $\mathscr{L}$ is classified by a morphism of $\mathbf{G}_m$-gerbes $\mathscr{X} \to B\mathbf{G}_{m,X}$ over X, and this map is necessarily an isomorphism. As for (ii), this comes from identifying the universal $\mathbf{G}_m$-torsor over $B\mathbf{G}_{m,X}$ as, on the one hand, the canonical morphism $X \to B\mathbf{G}_{m,X}$ and, on the other hand, as the affine bundle $\underline{Isom}(\mathcal{O}_{B\mathbf{G}_{m,X}}, \mathscr{L})$ of local trivializations for the universal line bundle $\mathscr{L}$. Finally, the latter is easily seen to be isomorphic to $\mathbf{A}(\mathscr{L}) \setminus \{0\}$.

More generally, if $\mathscr{X}$ carries a rank r vector bundle of weight 1, then its associated cohomology class $[\mathscr{X}] \in \mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_m)$ is r-torsion. This is because the determinant line bundle has weight r, and this gives a line bundle of weight 1 on the $\mathbf{G}_m$-gerbe on X with class $r[\mathscr{X}]$.

1.14. Relatively perfect objects. — Given a morphism $f : X \to S$ of schemes which is locally of finite type an object $E \in \mathrm{D}_{\mathrm{qc}}(X)$ is said to be *perfect relative to* S or, briefly, S*-perfect* if it is

– pseudo-coherent relative to S, in the sense of [Stacks, 09UI], and
– locally has finite tor-dimension as an object of $\mathrm{D}(f^{-1}\mathcal{O}_S)$, see [Stacks, 08CG].

We adopt here the generalized definition suggested in [Stacks, 0DI9]. In contrast, the Stacks Project defines relatively perfect objects only for morphisms which are flat and locally of finite presentation, in which case an S-perfect object in $\mathrm{D}_{\mathrm{qc}}(X)$ is the same as one that can be represented, locally on X, by a bounded complex of S-flat finitely presented quasi-coherent sheaves on X: see [Stacks, 0DI0 and

0DI2]. An analogous characterization of relatively perfect objects in this setting can be obtained by combining **1.16** and **1.17** below. When $f: X \to S$ is smooth, relatively perfect is the same as perfect:

1.15. Lemma. — *If $f: X \to S$ is a smooth morphism of schemes, then an object $E \in \mathrm{D}_{\mathrm{qc}}(X)$ is perfect relative to S if and only if it is perfect.*

Proof. That an S-perfect object is perfect follows from [Stacks, 068X]. Conversely, if E is perfect, then locally on X, it can be represented by a bounded complex of vector bundles. Since vector bundles are S-flat and finitely presented quasi-coherent sheaves, it follows that E is S-perfect. ∎

When X and S are both affine, relatively perfect may be reduced to perfect in an affine space:

1.16. Lemma. — *Let $f: X \to S$ be a finite type morphism of affine schemes. If $f = \mathrm{pr} \circ i : X \to \mathbf{A}^n_S \to S$ is any factorization through a closed immersion $i: X \to \mathbf{A}^n_S$, then an object $E \in \mathrm{D}_{\mathrm{qc}}(X)$ is perfect relative to S if and only if Ri_*E is a perfect object of $\mathrm{D}_{\mathrm{qc}}(\mathbf{A}^n_S)$.*

Proof. First, essentially by definition, E is pseudo-coherent relative to S if and only if Ri_*E is: compare [Stacks, 09UI and 09VC]. Next, writing $S = \operatorname{Spec} R$, E has finite tor-dimension as an object of $\mathrm{D}(f^{-1}\mathcal{O}_S)$ if and only if $R\Gamma(X,E) \cong R\Gamma(\mathbf{A}^n_S, Ri_*E)$ has finite tor-dimension as an object of $\mathrm{D}(R)$, and so these conditions are equivalent to Ri_*E having finite tor-dimension as an object of $\mathrm{D}(\mathrm{pr}^{-1}\mathcal{O}_S)$. Put together, this means that $E \in \mathrm{D}_{\mathrm{qc}}(X)$ is S-perfect if and only if $Ri_*E \in \mathrm{D}_{\mathrm{qc}}(\mathbf{A}^n_S)$ is S-perfect, and now the result follows from applying **1.15** to $\mathrm{pr}: \mathbf{A}^n_S \to S$. ∎

In general, being relatively perfect is fppf local on both the source and target:

1.17. Lemma. — *Let $f: X \to S$ be a morphism of schemes which is locally of finite type and $E \in \mathrm{D}_{\mathrm{qc}}(X)$.*

(i) *If $\{g_i : U_i \to X\}_i$ is an fppf covering, then E is perfect relative to S if and only if each $Lg_i^*E \in \mathrm{D}_{\mathrm{qc}}(U_i)$ is perfect relative to S.*
(ii) *Given a factorization $f: X \to V \to S$ with $V \to S$ flat and finitely presented, E is perfect relative to S if and only if it is perfect relative to V.*

Proof. The analogues of the statements (i) and (ii) for relative pseudo-coherence hold by [Stacks, 0CSN and 0CSP]. Similarly, for the statements about tor-dimension, note that E has tor-amplitude in $[a,b]$ as an object of $\mathrm{D}(f^{-1}\mathcal{O}_X)$ if and only if E_x has tor-amplitude in $[a,b]$ as an object of $\mathrm{D}(\mathcal{O}_{S,f(x)})$ for every $x \in X$ by [Stacks, 09U9]. With this, the analogue of (i) follows from [Stacks, 0DJF] whereas (ii) is [Stacks, 066L and 068S]. Together, this gives the result. ∎

1.18. — This allows us to extend the definition of relative perfectness to stacks: Let $f: \mathscr{X} \to \mathscr{S}$ be a morphism of stacks which is locally finite type, and choose a commutative square

$$\begin{array}{ccc} U & \longrightarrow & V \\ {\scriptstyle\varphi}\downarrow & & \downarrow \\ \mathscr{X} & \longrightarrow & \mathscr{S} \end{array}$$

where U and V are schemes, $V \to \mathscr{S}$ and $U \to \mathscr{X} \times_{\mathscr{S}} V$ are smooth surjections, and $U \to V$ is locally of finite type. An object $E \in \mathrm{D}_{\mathrm{qc}}(\mathscr{X})$ is then said to be *perfect relative to $\mathscr{S}$* or *$\mathscr{S}$-perfect* if $\varphi^*E \in \mathrm{D}_{\mathrm{qc}}(U)$ is perfect relative to V in the sense of **1.14**. By **1.17** and a standard argument, the definition is independent of the choice of the square.

When $f: \mathscr{X} \to \mathscr{S}$ is flat and locally of finite presentation, relative perfectness is preserved under arbitrary base change: see [Stacks, 0DI5]. In our generality, however, that is not necessarily the case. Nonetheless, relative perfectness is preserved under flat base change:

1.19. Lemma. — *Let $f\colon \mathscr{X} \to \mathscr{S}$ and $g\colon \mathscr{T} \to \mathscr{S}$ be morphisms of stacks, with f locally of finite type. Assume either g is flat or f is flat and locally of finite presentation. If $E \in \mathrm{D}_{\mathrm{qc}}(\mathscr{X})$ is perfect relative to $\mathscr{S}$, then $E_{\mathscr{T}} \in \mathrm{D}_{\mathrm{qc}}(\mathscr{X}_{\mathscr{T}})$ is perfect relative to $\mathscr{T}$.*

Proof. Since relative perfectness is fppf local on both the source and target by 1.17, we may reduce to the case that $f\colon X \to S$ and $g\colon T \to S$ are morphisms of affine schemes. It remains to consider the case g is flat. Choose a factorization $f = \mathrm{pr} \circ i\colon X \to \mathbf{A}^n_S \to S$ where i is a closed immersion. Since $E \in \mathrm{D}_{\mathrm{qc}}(X)$ is S-perfect, $Ri_*E \in \mathrm{D}_{\mathrm{qc}}(\mathbf{A}^n_S)$ by 1.16. Flat base change applied to the square

$$\begin{array}{ccc} X_T & \xrightarrow{\;i_T\;} & \mathbf{A}^n_T \\ \downarrow & & \downarrow \\ X & \xrightarrow{\;i\;} & \mathbf{A}^n_S \end{array}$$

implies that $Ri_{T,*}E_T \in \mathrm{D}_{\mathrm{qc}}(\mathbf{A}^n_T)$ is perfect: see [Stacks, 08IB and 066X]. Applying 1.16 once again then shows that E_T is T-perfect. ■

A key feature of relatively perfect objects is that they push forward to perfect objects under reasonable proper morphisms. When $f\colon X \to S$ is flat and locally of finite presentation, a general version is given in [Stacks, 0DJT]. In our more general setting, we prove a weaker version that suffices for our purposes. By way of terminology, call a morphism $f\colon X \to S$ of algebraic spaces *fpqc locally H-projective* if there exists a faithfully flat and quasi-compact cover $T \to S$ such that the base change $f_T\colon X_T \to T$ is H-projective, in the sense that it factors as $f_T = \mathrm{pr} \circ i\colon X_T \to \mathbf{P}^n_T \to T$ where i is a closed immersion. In this setting, Rf_* takes relatively perfect complexes to perfect ones:

1.20. Lemma. — *Let $f\colon X \to S$ be a morphism of algebraic spaces that is fpqc locally H-projective. If $E \in \mathrm{D}_{\mathrm{qc}}(X)$ is perfect relative to S, then $Rf_*E \in \mathrm{D}_{\mathrm{qc}}(S)$ is perfect.*

Proof. Since relative perfectness is preserved under flat base change by 1.19 and perfectness may be checked fpqc locally by [Stacks, 09UG], we may reduce to the case $f\colon X \to S$ itself admits a factorization $f = \mathrm{pr} \circ i\colon X \to \mathbf{P}^n_S \to S$ with $i\colon X \to \mathbf{P}^n_S$ a closed immersion. Since relative perfectness is Zariski local on the source, the argument of 1.16 may be used to show that $Ri_*E \in \mathrm{D}_{\mathrm{qc}}(\mathbf{P}^n_S)$ is perfect. Since the projection $\mathrm{pr}\colon \mathbf{P}^n_S \to S$ is smooth and proper, $R\mathrm{pr}_*$ preserves perfect complexes, yielding the result. ■

A final useful fact is that, at least for a morphism $f\colon \mathscr{X} \to \mathscr{S}$ of stacks which is flat and locally of finite presentation, formation of internal Hom with relatively perfect objects often preserves $\mathrm{D}_{\mathrm{qc}}(\mathscr{X})$ and commutes with arbitrary base change:

1.21. Lemma. — *Let $f\colon \mathscr{X} \to \mathscr{S}$ be a morphism of stacks which is flat and locally of finite presentation. If $E, F \in \mathrm{D}_{\mathrm{qc}}(\mathscr{X})$ are such that E is pseudo-coherent and F is perfect relative to $\mathscr{S}$, then*

$$R\mathscr{H}\!om_{\mathcal{O}_{\mathscr{X}}}(E,F) \in \mathrm{D}_{\mathrm{qc}}(\mathscr{X})$$

and its formation commutes with arbitrary base change in that, for $\mathscr{T} \to \mathscr{S}$ a morphism of stacks,

$$R\mathscr{H}\!om_{\mathcal{O}_{\mathscr{X}}}(E,F)_{\mathscr{T}} \cong R\mathscr{H}\!om_{\mathcal{O}_{\mathscr{X}_{\mathscr{T}}}}(E_{\mathscr{T}},F_{\mathscr{T}}) \in \mathrm{D}_{\mathrm{qc}}(\mathscr{X}_{\mathscr{T}}).$$

Proof. The first statement holds more generally for E pseudo-coherent and F locally in $\mathrm{D}^+_{\mathrm{qc}}(\mathscr{X})$: this may be checked smooth locally on $\mathscr{X}$ and so it reduces to the affine case, where [Stacks, 0A6H] applies. The second statement may now be reduced to the case where $f\colon X \to \mathrm{Spec}\,A$ is a morphism of affine schemes, and the base change is of the form $\mathrm{Spec}\,B \to \mathrm{Spec}\,A$. In this setting, E may be represented by a bounded above complex $\mathscr{E}^\bullet$ of finite free $\mathscr{O}_X$-modules, and F by a bounded complex $\mathscr{F}^\bullet$ of A-flat finitely presented $\mathscr{O}_X$-modules.

On the one hand, as in the proof of *ibid.*, the internal Hom in question may be represented as

$$R\mathscr{H}om_{\mathscr{O}_X}(E,F) \simeq \mathscr{H}om^\bullet_{\mathscr{O}_X}(\mathscr{E}^\bullet,\mathscr{F}^\bullet).$$

The boundedness properties of $\mathscr{E}^\bullet$ and $\mathscr{F}^\bullet$ imply that every term of this complex is a finite direct sum of $\mathscr{O}_X$-modules of the form $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}^i,\mathscr{F}^j)$. On the other hand, since the terms of $\mathscr{E}^\bullet$ are free, and $\mathscr{F}^\bullet$ are A-flat, the internal Hom on the base change may be represented as

$$R\mathscr{H}om_{\mathscr{O}_{X_B}}(E_B,F_B) \simeq \mathscr{H}om^\bullet_{\mathscr{O}_{X_B}}(\mathscr{E}^\bullet \otimes_A B,\mathscr{F}^\bullet \otimes_A B)$$

and the complex satisfies the same finiteness condition. This implies the result since

$$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}^i,\mathscr{F}^j)\otimes_A B \cong \mathscr{H}om_{\mathscr{O}_{X_B}}(\mathscr{E}^i \otimes_A B,\mathscr{F}^j \otimes_A B). \qquad \blacksquare$$

1.22. Relative orthogonal categories. — Let S be a qcqs scheme and $g\colon Y \to S$ a morphism of schemes which is flat, proper, and of finite presentation. Given a perfect object $P \in \mathrm{D}_{\mathrm{qc}}(Y)$, its *$S$-linear right orthogonal category* is the full subcategory $\langle P\rangle^\perp \subset \mathrm{D}_{\mathrm{qc}}(Y)$ with objects

$$\langle P\rangle^\perp := \{E \in \mathrm{D}_{\mathrm{qc}}(Y) : Rg_*R\mathscr{H}om_{\mathscr{O}_Y}(P,E) = 0\}.$$

Two basic properties of this construction are that:

(i) $\langle P\rangle^\perp$ is triangulated and S-linear, and so by our convenctions closed under direct sums; and
(ii) membership in $\langle P\rangle^\perp$ is flat local: for a faithfully flat morphism $U \to S$ from a qcqs scheme, if $E \in \mathrm{D}_{\mathrm{qc}}(Y)$ is such that $E_U \in \langle P_U\rangle^\perp$, then $E \in \langle P\rangle^\perp$.

Proof. For (i), that the subcategory $\langle P\rangle^\perp$ is triangulated follows from the fact that the functor $Rg_*R\mathscr{H}om_{\mathscr{O}_Y}(P,-)$ is exact. For S-linearity, let $E \in \langle P\rangle^\perp$ and $F \in \mathrm{D}_{\mathrm{qc}}(S)$, then

$$Rg_*R\mathscr{H}om_{\mathscr{O}_Y}(P,E\otimes^L Lg^*F) \cong Rg_*R\mathscr{H}om_{\mathscr{O}_Y}(P,E)\otimes^L F = 0$$

using the projection formula, so $E\otimes^L Lg^*F \in \langle P\rangle^\perp$. Item (ii) follows directly from flat base change and faithfully flat descent. ■

We now arrive at the main result of this section, which essentially says that we may check whether or not a Fourier–Mukai transform from a single homogeneous component of the derived category of a $\mathbf{G}_m$-gerbe to the derived category of a scheme is an equivalence after passing to an fppf cover. This may be seen as a variant of the descent result [BS20, Theorem B] of Bergh and Schnürer; unfortunately, our statement does not directly follow from theirs because our kernel is not perfect, only relatively so, and the source of our functor is not the derived category of a proper algebraic stack. Note also that this result, although phrased in terms of a $\mathbf{G}_m$-gerbe, also applies to schemes by taking a weight 0 kernel on any gerbe and using **1.12**(iii). The result is:

1.23. Theorem. — *Let $g\colon Y \to S$ be a flat, proper, and finitely presented morphism of qcqs schemes, $f\colon X \to S$ a fpqc locally H-projective morphism from an algebraic space, and $\pi\colon \mathscr{X} \to X$ a $\mathbf{G}_m$-gerbe. Let*

$$\Phi\colon \mathrm{D}_{\mathrm{qc},-i}(\mathscr{X}) \subset \mathrm{D}_{\mathrm{qc}}(\mathscr{X}) \xrightarrow{\Phi_K} \mathrm{D}_{\mathrm{qc}}(Y)$$

be the functor induced by the Fourier–Mukai transform with a Y-perfect kernel $K \in \mathrm{D}_{\mathrm{qc},i}(\mathscr{X}\times_S Y)$, and let $U \to S$ be a faithfully flat morphism from a qcqs scheme.

(i) *If the functor $\Phi_U\colon \mathrm{D}_{\mathrm{qc},-i}(\mathscr{X}_U) \to \mathrm{D}_{\mathrm{qc}}(Y_U)$ induced by the Fourier–Mukai transform with the Y_U-perfect kernel $K_U \in \mathrm{D}_{\mathrm{qc},i}(\mathscr{X}_U\times_U Y_U)$ is fully faithful, then Φ is fully faithful.*

Let $P \in \mathrm{D}_{\mathrm{qc}}(Y)$ be a perfect complex and let $\mathscr{A} := \langle P\rangle^\perp \subset \mathrm{D}_{\mathrm{qc}}(Y)$ be its S-linear right orthogonal.

(ii) *If the essential image of Φ_U lies in $\mathscr{A}_U$, then the essential image of Φ is lies in $\mathscr{A}$.*

(iii) *If Φ' is fully faithful with essential image $\mathscr{A}_U$, then Φ induces an equivalence $\mathrm{D}_{\mathrm{qc},-i}(\mathscr{X}) \cong \mathscr{A}$.*

Toward the proof, observe that Φ is a composition of an inclusion of the weight $-i$ component of the derived category of $\mathscr{X}$ and the Fourier–Mukai transform Φ_K. The hypotheses on the situation guarantee that Φ has some good properties:

1.24. Lemma. — *The functor $\Phi\colon \mathrm{D}_{\mathrm{qc},-i}(\mathscr{X}) \to \mathrm{D}_{\mathrm{qc}}(Y)$*

(i) *has an exact right adjoint; and*

(ii) *takes perfect complexes to perfect complexes.*

Proof. For (i), observe that Φ has an exact right adjoint since it is a composition of two functors which do: the Fourier–Mukai transform Φ_K does by **1.6**, and the right adjoint to the inclusion $\mathrm{D}_{\mathrm{qc},-i}(\mathscr{X}) \subset \mathrm{D}_{\mathrm{qc}}(\mathscr{X})$ is the projection onto the weight i component, see **1.12**.

For (ii), let $P \in \mathrm{D}_{\mathrm{qc},-i}(\mathscr{X})$ be a perfect complex. Then $L\mathrm{pr}_1^*P \in \mathrm{D}_{\mathrm{qc},-i}(\mathscr{X} \times_S Y)$ is perfect, so that

$$L\mathrm{pr}_1^*P \otimes^L K \in \mathrm{D}_{\mathrm{qc},0}(\mathscr{X} \times_S Y) \cong \mathrm{D}_{\mathrm{qc}}(X \times_S Y)$$

is Y-perfect, where the equivalence is induced by the surjective and smooth structure map $\pi \times \mathrm{id}\colon \mathscr{X} \times_S Y \to X \times_S Y$ as in **1.12**(iii). Under this identification, $\Phi(P)$ is now obtained via pushforward along $f_Y : X \times_S Y \to Y$. The hypotheses on $g : Y \to S$ ensure that f_Y is also fpqc locally H-projective, and so **1.20** implies that $Rf_{Y,*}$ takes Y-perfect complexes to perfect complexes, yielding the result. ■

We now phrase full faithfulness and essential surjectivity of the functor Φ in terms of a compact generator of $\mathrm{D}_{\mathrm{qc},-i}(\mathscr{X})$—note that such an object exists, as discussed in **1.12**.

1.25. Lemma. — *Let $G \in \mathrm{D}_{\mathrm{qc},-i}(\mathscr{X})$ be a perfect S-linear generator.*

(i) *Φ is fully faithful if and only if the morphism in $\mathrm{D}_{\mathrm{qc}}(S)$*

$$Rf_*R\mathscr{H}om_{\mathscr{O}_\mathscr{X}}(G,G) \to Rg_*R\mathscr{H}om_{\mathscr{O}_Y}(\Phi(G),\Phi(G))$$

*from **1.5**(ii) is an isomorphism.*

(ii) *If Φ is fully faithful, then the essential image of Φ is $\mathscr{A} := \langle P\rangle^\perp$ if and only if the essential image of Φ is contained in $\mathscr{A}$ and $\Phi(G) \oplus P$ is an S-linear perfect generator of $\mathrm{D}_{\mathrm{qc}}(Y)$.*

Proof. Item (i) may be proven in the same way as **1.9** since $\mathrm{D}_{\mathrm{qc}}(S)$ has a perfect generator, $\mathrm{D}_{\mathrm{qc},-i}(\mathscr{X})$ a perfect S-linear generator, and Φ an exact right adjoint by **1.24**(i), and Φ takes perfect complexes to perfect complexes by **1.24**(ii).

For (ii), suppose that the essential image of Φ is $\mathscr{A}$. Since $\Phi(G)$ is perfect by **1.24**(ii), it remains to see that $\Phi(G) \oplus P$ is an S-linear generator of $\mathrm{D}_{\mathrm{qc}}(Y)$. Let $E \in \mathrm{D}_{\mathrm{qc}}(Y)$ be a nonzero object. If $E \notin \mathscr{A}$, then by definition of the orthogonal category,

$$Rg_*R\mathscr{H}om_{\mathscr{O}_Y}(P,E) \neq 0.$$

If $E \in \mathscr{A}$, choose $E' \in \mathrm{D}_{\mathrm{qc},-i}(\mathscr{X})$ with $\Phi(E') \cong E$. Then, using (i) and the hypothesis that G is an S-linear generator of $\mathrm{D}_{\mathrm{qc},-i}(\mathscr{X})$,

$$Rg_*R\mathscr{H}om_{\mathscr{O}_Y}(\Phi(G),E) \cong Rf_*R\mathscr{H}om_{\mathscr{O}_\mathscr{X}}(G,E') \neq 0.$$

Namely, the collection of F for which

$$Rf_*R\mathscr{H}om_{\mathscr{O}_\mathscr{X}}(G,F) \to Rg_*R\mathscr{H}om_{\mathscr{O}_Y}(\Phi(G),\Phi(F))$$

is an isomorphism is an S-linear full triangulated subcategory of $D_{qc,-i}(\mathscr{X})$ by the projection formula. Put together, this implies that $\Phi(G) \oplus P$ is an S-linear generator of $\mathrm{D}_{\mathrm{qc}}(Y)$.

Conversely, assume that the essential image of Φ is contained in $\mathscr{A}$ and $\Phi(G) \oplus P$ is an S-linear generator of $\mathrm{D}_{\mathrm{qc}}(Y)$. Given a nonzero object $E \in \mathscr{A}$, this gives

$$Rg_* R\mathscr{H}om_{\mathscr{O}_Y}(\Phi(G), E) = Rg_* R\mathscr{H}om_{\mathscr{O}_Y}(\Phi(G) \oplus P, E) \neq 0.$$

This means that $\Phi(G)$ is an S-linear generator of $\mathscr{A}$. Since $\mathscr{A}$ is triangulated by 1.22(i) and Φ is S-linear by 1.5(i) and fully faithful, hence the essential image of Φ is triangulated and S-linear, 1.4(ii) now implies that Φ is essentially surjective onto $\mathscr{A}$. ■

Proof of 1.23. Let $G \in \mathrm{D}_{\mathrm{qc},-i}(\mathscr{X})$ be a perfect S-linear generator. Descent implies that $G_U \in \mathrm{D}_{\mathrm{qc},-i}(\mathscr{X}_U)$ is a perfect U-linear generator. Statement (i) now follows from the full faithfulness criterion of 1.25(i) since the condition there is compatible with fppf base change. Part (ii) follows from 1.22(ii) since, for any flat qcqs S-scheme T,

$$\Phi(\mathrm{D}_{\mathrm{qc},-i}(\mathscr{X}_T)) \subseteq \mathscr{A}_T \iff \Phi(G)_T \cong \Phi_T(G_T) \in \mathscr{A}_T \iff Rg_* R\mathscr{H}om_{\mathscr{O}_{Y_T}}(P_T, G_T) = 0$$

since $G_T \in \mathrm{D}_{\mathrm{qc},-i}(\mathscr{X}_T)$ is a T-linear generator and the image of Φ is a T-linear triangulated category. Finally, (iii) follows from the essential surjectivity criterion 1.25(ii) since the condition there, once again, is compatible with flat base change. ■

2. Moduli of complexes

In this section, we discuss Lieblich's stack of complexes on a flat, proper, and finitely presented morphism $f : X \to S$. The main result of this section is 2.8, which says that fully faithful Fourier–Mukai transforms whose kernels are perfect relative to both source and target induce open immersions between stacks of complexes.

2.1. Glueability and simplicity. — Let $f : \mathscr{X} \to \mathscr{S}$ be a flat, proper, finitely presented, and concentrated morphism of stacks. An $\mathscr{S}$-perfect object $E \in \mathrm{D}_{\mathrm{qc}}(\mathscr{X})$ is said to be

- *glueable* if $R^i f_* R\mathscr{H}om_{\mathscr{O}_{\mathscr{X}}}(E, E) = 0$ for all integers $i < 0$;
- *simplistic* if $\mathscr{O}_{\mathscr{S}} \to Rf_* R\mathscr{H}om_{\mathscr{O}_{\mathscr{X}}}(E, E)$ is an isomorphism on 0-th cohomology sheaves;
- *universally glueable* if for every morphism $\mathscr{T} \to \mathscr{S}$ of stacks, $E_{\mathscr{T}} \in \mathrm{D}_{\mathrm{qc}}(\mathscr{X}_{\mathscr{T}})$ is glueable; and
- *simple* if for every morphism $\mathscr{T} \to \mathscr{S}$ of stacks, $E_{\mathscr{T}} \in \mathrm{D}_{\mathrm{qc}}(\mathscr{X}_{\mathscr{T}})$ is simplistic.

These notions depend on the morphism f, so we may sometimes emphasize that the properties are *relative to* $\mathscr{S}$. In particular, the condition on the base changed object $E_{\mathscr{T}}$ in universal glueability and simplicity are relative to $\mathscr{T}$.

Being glueable or simplistic behave well under base change. Namely, given a flat and quasi-compact morphism $\mathscr{T} \to \mathscr{S}$ of stacks, flat base change together with 1.21 imply that:

(i) If $E \in \mathrm{D}_{\mathrm{qc}}(\mathscr{X})$ is either glueable or simplistic, the same is true for $E_{\mathscr{T}} \in \mathrm{D}_{\mathrm{qc}}(\mathscr{X}_{\mathscr{T}})$.
(ii) If $\mathscr{T} \to \mathscr{S}$ is surjective and $E_{\mathscr{T}}$ is glueable or simplistic, the same is true for E.

In other words, the properties of being glueable or simplistic satisfy fppf descent. Since stacks have fppf coverings by a disjoint union of affine schemes, being universally glueable or simple may be characterized more concretely as follows:

(iii) E is universally glueable if and only if for every morphism $T \to \mathscr{S}$ from an affine scheme, $E_T \in \mathrm{D}_{\mathrm{qc}}(\mathscr{X}_T)$ satisfies $\mathrm{Ext}^i_{\mathscr{X}_T}(E_T, E_T) = 0$ for all integers $i < 0$.
(iv) E is simple if and only if for every morphism $T \to \mathscr{S}$ from an affine scheme, the canonical map $\mathscr{O}_{\mathscr{X}}(T) \to \mathrm{Hom}_{\mathscr{X}_T}(E_T, E_T)$ is an isomorphism.

In the case that the morphism f is between locally Noetherian algebraic spaces, [Lie06, Proposition 2.1.9] gives a fibrewise criterion for universal glueability:

(v) Let $f: X \to S$ be a flat, proper, and finitely presented morphism between locally Noetherian algebraic spaces. An S-perfect object $E \in \mathrm{D}_{\mathrm{qc}}(X)$ is universally glueable if and only if $E_{\bar{s}} \in \mathrm{D}_{\mathrm{qc}}(X_{\bar{s}})$ is glueable for each geometric point $\bar{s} \to S$.

Universally glueable complexes have the pleasant property that their presheaves of endomorphisms are actually sheaves. More generally:

2.2. Lemma. — *Let $f: \mathscr{X} \to \mathscr{S}$ be a flat, proper, finitely presented, and concentrated morphism of stacks. Let $E, F \in \mathrm{D}_{\mathrm{qc}}(\mathscr{X})$ be $\mathscr{S}$-perfect objects that satisfy, for every object $T \to \mathscr{S}$, the vanishing*

$$R^i f_{T,*} R\mathscr{H}om_{\mathscr{O}_{\mathscr{X}_T}}(E_T, F_T) = 0 \text{ for all integers } i < 0.$$

Then the assignment $(T \to \mathscr{S}) \mapsto \mathrm{Hom}_{\mathscr{X}_T}(E_T, F_T)$ defines an fpqc sheaf on $\mathscr{S}$.

Proof. This follows from flat base change together with **1.21**, which implies that the assignment

$$(T \to \mathscr{S}) \mapsto \Gamma(T, R^0 f_{T,*} R\mathscr{H}om_{\mathscr{O}_{\mathscr{X}_T}}(E_T, F_T))$$

defines an fpqc sheaf on $\mathscr{S}$, and the fact that the vanishing of negative pushforwards implies that the group on the right is $\mathrm{Hom}_{\mathscr{X}_T}(E_T, F_T)$. ■

2.3. Stack of complexes. — Lieblich constructs in [Lie06]—though see also [Stacks, 0DLB]—a stack of complexes on a flat, proper, finitely presented morphism $f: X \to S$ of schemes of the form

$$\mathscr{C}omplexes_{X/S} := \big\{(T, E) : T \in \mathrm{Sch}_S \text{ and } E \in \mathrm{D}_{\mathrm{qc}}(X_T) \text{ is } T\text{-perfect and universally glueable}\big\}.$$

This is algebraic and locally of finite presentation over S. Its objects are pairs (T, E) consist of an S-scheme T and an object $E \in \mathrm{D}_{\mathrm{qc}}(X_T)$ which is universally glueable and perfect relative to T, and a morphism $(T, E) \to (T', E')$ between objects in consist of a morphism $h\colon T \to T'$ of schemes over S, and an isomorphism $Lh^* E' \to E$. It is shown in [Lie06, Lemma 4.3.2 and Corollary 4.3.3] that there is an open substack

$$s\mathscr{C}omplexes_{X/S} \subset \mathscr{C}omplexes_{X/S}$$

consisting of those objects (T, E) where E is simple relative to T, and that it naturally has the structure of a $\mathbf{G}_m$-gerbe over an algebraic space $\mathrm{sComplexes}_{X/S}$.

Universal objects on these stacks may be constructed, as usual, with a suitable version of the Yoneda lemma. We have been unable to locate a reference, so we give a proof of the version we need below in **2.4**. Applying the result to the identity morphisms thus provides objects

$$E_{\mathrm{univ}} \in \mathrm{D}_{\mathrm{qc}}(X \times_S \mathscr{C}omplexes_{X/S}) \text{ and } sE_{\mathrm{univ}} \in \mathrm{D}_{\mathrm{qc},1}(X \times_S s\mathscr{C}omplexes_{X/S})$$

which is perfect and universally glueable relative to $\mathscr{C}omplexes_{X/S}$ and $s\mathscr{C}omplexes_{X/S}$, respectively. Moreover, sE_{univ} is also simple and may be identified as a weight 1 object with respect to the $\mathbf{G}_m$-gerbe structure, see **1.12**.

In the following, the objects on the right are the groupoids with the displayed set of objects:

2.4. Lemma. — *Let $f: X \to S$ be a flat, proper, and finitely presented morphism of schemes, and let $\mathscr{T}$ be a stack over S. There are canonical equivalences of groupoids between*

$$\mathrm{Mor}_S(\mathscr{T}, \mathscr{C}omplexes_{X/S}) \simeq \{E \in \mathrm{D}_{\mathrm{qc}}(X \times_S \mathscr{T}) : E \text{ is } \mathscr{T}\text{-perfect and universally glueable}\}, \text{ and}$$

$$\mathrm{Mor}_S(\mathscr{T}, s\mathscr{C}omplexes_{X/S}) \simeq \{E \in \mathrm{D}_{\mathrm{qc}}(X \times_S \mathscr{T}) : E \text{ is } \mathscr{T}\text{-perfect, universally glueable, and simple}\}.$$

Proof. We explain the first equivalence, the second being analogous. Given $E \in \mathrm{D}_{\mathrm{qc}}(X \times_S \mathscr{T})$ which is $\mathscr{T}$-perfect and universally glueable, define a morphism $\varphi_E : \mathscr{T} \to \mathscr{C}omplexes_{X/S}$ by

$$\varphi_E(T \to \mathscr{T}) := E_T \in \mathscr{C}omplexes_{X/S}(T).$$

To see that the functor $E \mapsto \varphi_E$ is fully faithful, let $E, F \in \mathrm{D}_{\mathrm{qc}}(X \times_S \mathscr{T})$ be $\mathscr{T}$-perfect and universally glueable objects. Then a 2-morphism $\varphi_E \to \varphi_F$ is the data of, for every object $T \to \mathscr{T}$, an isomorphism $E_T \to F_T$ in $\mathrm{D}_{\mathrm{qc}}(X_T)$ such that, for every morphism in $\mathscr{T}$, the obvious square commutes. This amounts to a section of the sheaf of isomorphisms

$$(T \to \mathscr{T}) \mapsto \mathrm{Isom}_{X_T}(E_T, F_T).$$

Note that this is a sheaf because it is a subsheaf of $\mathrm{Hom}_{X_T}(E_T, F_T)$ from **2.2**, and the vanishing hypothesis there holds beacuse the isomorphisms $E_T \to F_T$ provides an isomorphism

$$R^i f_{T,*} R\mathscr{H}om_{\mathscr{O}_{X_T}}(E_T, F_T) \cong R^i f_{T,*} R\mathscr{H}om_{\mathscr{O}_{X_T}}(E_T, E_T)$$

and the latter vanishes for all $i < 0$ since E_T is universally glueable. This, in turn, is equivalent to an isomorphism $E \to F$ in $\mathrm{D}_{\mathrm{qc}}(X \times_S \mathscr{T})$.

Essential surjectivity of $E \mapsto \varphi_E$ is a consequence of the Beilinson–Bernstein–Deligne glueing lemma [BBD82, Theorem 3.2.4] in the form stated in [Stacks, 0DCB]. Namely, consider the functor

$$u \colon \mathscr{D} := \mathscr{T}_{\text{lis-ét}} \to \mathscr{C} := (X \times_S \mathscr{T})_{\text{lis-ét}} \qquad T \mapsto X_T.$$

A morphism $\varphi \colon \mathscr{T} \to \mathscr{C}omplexes_{X/S}$ of stacks gives, for each $T \in \mathscr{D}$ an object $E_T \in \mathrm{D}_{\mathrm{qc}}(X_T)$. For T in the full subcategory $\mathscr{B} \subset \mathscr{D}$ consisting of affine schemes, the objects E_T are universally bounded and have vanishing negative self-Exts, and so *ibid.* applies to give a unique object $E \in \mathrm{D}_{\mathrm{qc}}(X \times_S \mathscr{T})$ whose restriction to $\mathrm{D}_{\mathrm{qc}}(X_T)$ is E_T for each $T \in \mathscr{T}$. In other words, $\varphi_E \simeq \varphi$. ∎

2.5. Autoequivalences. — The stack of complexes on a flat, proper, and finitely presented morphism $f : X \to S$ of schemes carries several natural autoequivalences. Let $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module and let $\omega^\bullet_{X/S}$ be a relative dualizing complex for $f : X \to S$; this exists in this generality by [Stacks, 0E2X]. Then the functors $- \otimes^L_{\mathscr{O}_X} \mathscr{L}$ and $R\mathscr{H}om_{\mathscr{O}_X}(-, \omega^\bullet_{X/S})$ induce equivalences of stacks

$$\tau_{\mathscr{L}} \colon \mathscr{C}omplexes_{X/S} \to \mathscr{C}omplexes_{X/S} \quad \text{and} \quad D \colon \mathscr{C}omplexes_{X/S} \to \mathscr{C}omplexes_{X/S}$$

satisfying $\tau_{\mathscr{L}^\vee} \circ \tau_{\mathscr{L}} \simeq \mathrm{id} \simeq D \circ D$, and which preserve the substack of simple complexes.

That $\tau_{\mathscr{L}}$ exists and has quasi-inverse $\tau_{\mathscr{L}^\vee}$ is straightforward. That D makes sense and is an equivalence on the stack of complexes, however, does not appear to be well-documented, so we include a proof here. The main task is to verify that $R\mathscr{H}om_{\mathscr{O}_X}(-, \omega^\bullet_{X/S})$ preserves relative perfectness, universal glueability, and simplicity:

2.6. Lemma. — *Let $f : X \to S$ be a flat, proper, and finitely presented morphism of schemes. Let $E \in \mathrm{D}_{\mathrm{qc}}(X)$ and $D(E) := R\mathscr{H}om_{\mathscr{O}_X}(E, \omega^\bullet_{X/S})$.*

(i) *If E is S-perfect, then so is $D(E)$.*

(ii) *The canonical morphism $E \to D(D(E))$ is an isomorphism.*

(iii) *Assuming E is S-perfect, if E is universally glueable or simple, then the same is true for $D(E)$.*

Proof. The dualizing complex $\omega^\bullet_{X/S}$ is S-perfect by its definition in [Stacks, 0E2T], so **1.21** applies to show that $D(E) = R\mathscr{H}om_{\mathscr{O}_X}(E, \omega^\bullet_{X/S})$ lies in $\mathrm{D}_{\mathrm{qc}}(X)$ and that its formation commutes with arbitrary base change. Since each of the statements are local on the base by **1.17** and **2.1**(iii)–(iv), we may assume for the remainder that S is affine.

To see (i), reduce to the case S is affine and use the criterion [Stacks, 0GET], wherein we must show that $Rf_*(D(E) \otimes^L F) \in \mathrm{D}_{\mathrm{qc}}(S)$ is perfect for every perfect object $F \in \mathrm{D}_{\mathrm{qc}}(X)$. But now properties of $\omega^\bullet_{X/S}$, as in [Stacks, 0A9Q], give

$$Rf_*(D(E) \otimes^L F) \cong Rf_* R\mathscr{H}om_{\mathscr{O}_X}(E \otimes^L F, \omega^\bullet_{X/S}) \cong R\mathscr{H}om_{\mathscr{O}_S}(Rf_*(E \otimes^L F), \mathscr{O}_S)$$

and this is perfect since $Rf_*(E \otimes^L F)$. Item (ii) now follows from [Stacks, 0A89]. So far, this means that D defines an anti-equivalence of categories on $\mathrm{D}_{\mathrm{qc}}(X)$, and this implies (iii) since the criteria from **2.1**(iii)–(iv), are therefore preserved by D. ∎

2.7. Fourier–Mukai open immersion. — Let $f: X \to S$ and $g: Y \to S$ be flat, proper, and finitely presented morphisms of schemes and let $\Phi_K : \mathrm{D}_{\mathrm{qc}}(X) \to \mathrm{D}_{\mathrm{qc}}(Y)$ be the Fourier–Mukai transform associated with an object $K \in \mathrm{D}_{\mathrm{qc}}(X \times_S Y)$ that is perfect relative to both X and Y. Compatibility of Φ_K with base change, as in **1.5**(iii), allows one to define a functor FM_K from the stack of complexes on $f: X \to S$ to certain complexes on $g: Y \to S$ as follows:

On objects, set $\mathrm{FM}_K(T,E) := (T, \Phi_{K_T}(E_T))$, and on a morphism $(h,\alpha): (T,E) \to (T',E')$, meaning as in **2.3** a S-morphism $h: T \to T'$ and an isomorphism $\alpha: E'_T \to E$ in $\mathrm{D}_{\mathrm{qc}}(X_T)$, set

$$\mathrm{FM}_K(h,\alpha) := (h, \mathrm{FM}_K(\alpha)) : (T, \Phi_{K_T}(E)) \to (T', \Phi_{K_{T'}}(E'))$$

where $\mathrm{FM}_K(\alpha)$ is the isomorphism obtained by composing the compatibility of the Fourier–Mukai transforms with base change together with $\Phi_{K_T}(\alpha)$. Lemmas **2.9** and **2.10** below imply that when Φ_K is fully faithful, FM_K takes values in the stack of complexes on $g: Y \to S$ and that it preserves simple complexes. More notably, the functor is even an open immersion of stacks:

2.8. Theorem. — *In the setting of* **2.7**, *if $\Phi_K : \mathrm{D}_{\mathrm{qc}}(X) \to \mathrm{D}_{\mathrm{qc}}(Y)$ is fully faithful, then the assignment $(T,E) \mapsto (T, \Phi_{K_T}(E_T))$ induces an open immersion of S-stacks*

$$\mathrm{FM}_K : \mathscr{C}omplexes_{X/S} \to \mathscr{C}omplexes_{Y/S}$$

which preserves the open substack of simple complexes.

We first verify that the Fourier–Mukai transform preserves S-perfectness:

2.9. Lemma. — *In the setting of* **2.7**, *$\Phi_K : \mathrm{D}_{\mathrm{qc}}(X) \to \mathrm{D}_{\mathrm{qc}}(Y)$ takes*

(i) *perfect complexes to perfect complexes;*

(ii) *pseudo-coherent complexes to pseudo-coherent complexes; and*

(iii) *S-perfect complexes to S-perfect complexes.*

Proof. For (i), if $E \in \mathrm{D}_{\mathrm{qc}}(X)$ perfect, then $L\mathrm{pr}_1^* E \otimes^L K$ is Y-perfect and so its direct image along $R\mathrm{pr}_{2,*}$, which is $\Phi_K(E)$, is perfect: see [Stacks, 0DI4 and 0DJT]. Item (ii) follows upon observing that Φ_K is the composition of three functors which individually preserve pseudo-coherence, the main point being that $R\mathrm{pr}_{2,*}$ does by Kiehl's theorem [Kie72, Theorem 2.9]; see also [Stacks, 0CSD]. For (iii), let $E \in \mathrm{D}_{\mathrm{qc}}(X)$ be S-perfect. By (ii), $\Phi_K(E) \in \mathrm{D}_{\mathrm{qc}}(Y)$ is pseudo-coherent, so since $g: Y \to S$ is flat and finitely presented, [Stacks, 0GEH] shows that $\Phi_K(E)$ is S-perfect if and only if its fibre over every $s \in S$ is bounded below. Since $\Phi_K(E)_s \cong \Phi_{K_s}(E_s)$ by **1.5**(iii), we may reduce to the case where $S = \operatorname{Spec} \mathbf{k}$ is the spectrum of a field, in which case the result follows from [Stacks, 0FYU]. ∎

We now observe that when the Fourier–Mukai transform is fully faithful, Φ_K also preserves universal glueability and simplicity, proving that FM_K takes values in the stack of complexes of $g: Y \to S$:

2.10. Lemma. — *In the setting of* **2.7**, *assume $\Phi_K : \mathrm{D}_{\mathrm{qc}}(X) \to \mathrm{D}_{\mathrm{qc}}(Y)$ is fully faithful. If an S-perfect object $E \in \mathrm{D}_{\mathrm{qc}}(X)$ is universally glueable or simple, then the same is true for $\Phi_K(E) \in \mathrm{D}_{\mathrm{qc}}(Y)$.*

Proof. Let $E \in \mathrm{D}_{\mathrm{qc}}(X)$ be an S-perfect object which is universally glueable. By **2.1**(iii), this means that for any affine S-scheme T, we have $\mathrm{Ext}^i_{X_T}(E_T, E_T) = 0$ for all $i < 0$. Since $\Phi_K(E)_T \cong \Phi_{K_T}(E_T)$ by **1.5**(iii), and the base changed Fourier–Mukai functor $\Phi_{K_T} : \mathrm{D}_{\mathrm{qc}}(X_T) \to \mathrm{D}_{\mathrm{qc}}(Y_T)$ is also fully faithful by

1.10, this implies

$$\mathrm{Ext}^i_{Y_T}(\Phi_K(E)_T, \Phi_K(E)_T) = \mathrm{Ext}^i_{Y_T}(\Phi_{K_T}(E_T), \Phi_{K_T}(E_T)) = 0 \text{ for all } i < 0.$$

Applying 2.1(iii) once again shows that $\Phi_K(E)$ is universally glueable. The same argument works for E simple, upon using the criterion from 2.1(iv). ■

The next statement shows that the right adjoint to the Fourier–Mukai transform, which exists in this setting by 1.6, is compatible with base change and preserves relative perfectness:

2.11. Lemma. — *In the setting of 2.7, let $R\colon \mathrm{D}_{\mathrm{qc}}(Y) \to \mathrm{D}_{\mathrm{qc}}(X)$ be the right adjoint of Φ_K.*

(i) *R is compatible with base change in that if $T \to S$ is a morphism of schemes and $R_T : \mathrm{D}_{\mathrm{qc}}(Y_T) \to \mathrm{D}_{\mathrm{qc}}(X_T)$ is the right adjoint to Φ_{K_T}, then there are canonical isomorphisms*

$$R(E)_T \cong R_T(E_T) \text{ for all } E \in \mathrm{D}_{\mathrm{qc}}(Y).$$

(ii) *R takes relatively perfect objects to relatively perfect objects.*

Proof. For (i), since $\Phi_K = R\mathrm{pr}_{2,*} \circ (K \otimes^L -) \circ L\mathrm{pr}_1^* \colon \mathrm{D}_{\mathrm{qc}}(X) \to \mathrm{D}_{\mathrm{qc}}(Y)$, its right adjoint R may be written as the composition of three functors

$$R = R\mathrm{pr}_{1,*} \circ R\mathscr{H}om_{\mathscr{O}_{X\times_S Y}}(K, -) \circ a$$

where a is the right adjoint to $R\mathrm{pr}_{2,*}\colon \mathrm{D}_{\mathrm{qc}}(X \times_S Y) \to \mathrm{D}_{\mathrm{qc}}(Y)$. Since each of a, $R\mathscr{H}om_{\mathscr{O}_{X\times_S Y}}(K, -)$, and $R\mathrm{pr}_{1,*}$ commute with base change by [Stacks, 0AA8], 1.21, and tor-independent base change, respectively, R also commutes with base change.

For (ii), compatibility with base change in (i) together the local nature of relative perfectness from 1.17 reduces the statement to the case S is affine, wherein it suffices to verify the criterion [Stacks, 0GET]: Given $E \in \mathrm{D}_{\mathrm{qc}}(X)$ perfect and $F \in \mathrm{D}_{\mathrm{qc}}(Y)$ S-perfect, we must show that

$$R\Gamma(X, E \otimes^L R(F)) \cong R\mathrm{Hom}_X(E^\vee, R(F)) \cong R\mathrm{Hom}_Y(\Phi_K(E^\vee), F) \cong R\Gamma(Y, \Phi_K(E^\vee)^\vee \otimes^L F)$$

is perfect; note that the second identification uses the fact that Φ_K is an enriched functor, as in 1.5(ii). Since Φ_K preserves perfect objects by 2.9(i) and F itself is S-perfect, so is the displayed complex. ■

Proof of 2.8. It remains to show that FM_K is an open immersion. Full faithfulness of Φ_K implies that it is a monomorphism of stacks: indeed, given $E_1, E_2 \in \mathscr{C}omplexes_{X/S}(T)$, 1.10 implies that the base changed Fourier–Mukai functor Φ_{K_T} still provides an isomorphism

$$\Phi_{K_T} \colon \mathrm{Hom}_{X_T}(E_1, E_2) \cong \mathrm{Hom}_{Y_T}(\Phi_{K_T}(E_1), \Phi_{K_T}(E_2)).$$

We may therefore conclude by showing that FM_K is a smooth morphism of stacks. Since the stacks of complexes are locally of finite presentation over S, we may do so via the infinitesimal criterion for smoothness: Given a square zero thickening $T' \to T$ and a solid commutative square

$$\begin{array}{ccc} T' & \longrightarrow & \mathscr{C}omplexes_{X/S} \\ \big\downarrow & \nearrow & \big\downarrow \mathrm{FM}_K \\ T & \longrightarrow & \mathscr{C}omplexes_{Y/S} \end{array}$$

construct a dashed arrow making the diagram commute. In other words, given

- objects $(T', E') \in \mathscr{C}omplexes_{X/S}$ and $(T, F) \in \mathscr{C}omplexes_{Y/S}$, and
- an isomorphism $\alpha\colon F_{T'} \cong \Phi_{K_{T'}}(E')$,

the task is to construct an object $(T, E) \in \mathscr{C}omplexes_{X/S}$ and isomorphisms $E_{T'} \cong E'$ and $\Phi_{K_T}(E) \cong F$.

Let $R\colon \mathrm{D}_{\mathrm{qc}}(Y) \to \mathrm{D}_{\mathrm{qc}}(X)$ be the right adjoint to $\Phi_K : \mathrm{D}_{\mathrm{qc}}(X) \to \mathrm{D}_{\mathrm{qc}}(Y)$. Then **2.11** shows that R is compatible with base change and preserves S-perfectness. Writing R_T for its base change to T, the object $E := R_T(F) \in \mathrm{D}_{\mathrm{qc}}(X)$ is thus S-perfect. Base change, α, and full faithfulness of $\Phi_{K_{T'}}$, now give an isomorphism

$$E_{T'} = R_T(F)_{T'} \cong R_{T'}(F_{T'}) \cong R_{T'}(\Phi_{K_{T'}}(E')) \cong E'.$$

Similarly, the counit $\varepsilon\colon \Phi_{K_T}(R(F)) \to F$ becomes an isomorphism after restriction over T':

$$(\Phi_{K_{T'}} \circ R_{T'})(F_{T'}) \cong (\Phi_{K_{T'}} \circ R_{T'} \circ \Phi_{K_{T'}})(E') \cong \Phi_{K_{T'}}(E') \cong F_{T'}.$$

Since the inclusion $X_{T'} \to X_T$ is a homeomorphism, applying **2.12** then implies that the cone of ε vanishes, and so ε itself is an isomorphism; in other words, $\Phi_{K_T}(E) \cong F$. Universal glueability of F together with full faithfulness of Φ_{K_T} now implies that E is also universally glueable. It is now straightforward to see that (T, E) provides the sought-after dashed arrow. ■

2.12. Lemma. — *Let $(R, \mathfrak{m})$ be a local ring and $E \in \mathrm{D}(R)$ pseudo-coherent. Then*

$$E \otimes^L_R R/\mathfrak{m} = 0 \iff E = 0.$$

Proof. Suppose that $E \neq 0$ and let $i \in \mathbf{Z}$ be maximal such that $\mathrm{H}^i(E) \neq 0$. Then $\mathrm{H}^i(E)$ is a finite R-module by [Stacks, 0645], so $\mathrm{H}^i(E \otimes^L_R R/\mathfrak{m}) \cong \mathrm{H}^i(E) \otimes_R R/\mathfrak{m} \neq 0$ by Nakayama's lemma. ■

3. Residual category to an exceptional collection

The main object of concern in this paper is the *Kuznetsov component* of a flat family $\rho : Q \to S$ of quadric hypersurfaces in a $\mathbf{P}^n$-bundle: this is the full subcategory of $\mathrm{D}_{\mathrm{qc}}(Q)$ with objects

$$\mathrm{Ku}(Q) := \{F \in \mathrm{D}_{\mathrm{qc}}(Q) : R\rho_*(F \otimes^L \mathscr{O}_\rho(-i)) = 0 \text{ for } i = 0, \ldots, n-2\}$$

that are S-linearly right orthogonal to the S-exceptional collection $\mathscr{O}_\rho, \mathscr{O}_\rho(1), \ldots, \mathscr{O}_\rho(n-2)$: see **3.1** and **3.3**. Thus there is a semiorthogonal decomposition

$$\mathrm{D}_{\mathrm{qc}}(Q) = \langle \mathrm{Ku}(Q),\ L\rho^*\mathrm{D}_{\mathrm{qc}}(S),\ L\rho^*\mathrm{D}_{\mathrm{qc}}(S) \otimes^L \mathscr{O}_\rho(1), \ldots,\ L\rho^*\mathrm{D}_{\mathrm{qc}}(S) \otimes^L \mathscr{O}_\rho(n-2)\rangle.$$

In this section, we generalize **2.8** to allow the source category to be an admissible subcategory of the quasi-coherent derived category of a scheme: see **3.12**; here, *admissible* is always taken to mean two-sided admissible. Since we work with the entire quasi-coherent derived category of a scheme—rather than the bounded derived category of coherent sheaves—we also use this section to document some facts regarding mutations and semiorthogonal decompositions in this generality.

3.1. Relatively exceptional objects. — Let $f : X \to S$ be a flat, proper, and finitely presented morphism of schemes. Assume that f has Gorenstein fibres, so that its relative dualizing complex $\omega_{X/S} := \omega^\bullet_{X/S}$, moreover, is a line bundle. A perfect complex $E \in \mathrm{D}_{\mathrm{qc}}(X)$ is called *exceptional relative to S* or, briefly, *S-exceptional* if the natural morphism

$$\mathscr{O}_S \to Rf_*R\mathscr{H}om_{\mathscr{O}_X}(E, E) \in \mathrm{D}_{\mathrm{qc}}(X)$$

is an isomorphism. Basic properties are that relative exceptionality is preserved under arbitrary base change and that such objects provide admissible subcategories of $\mathrm{D}_{\mathrm{qc}}(X)$. Precisely:

(i) If $T \to S$ is any morphism of schemes, then $E_T \in \mathrm{D}_{\mathrm{qc}}(X_T)$ is T-exceptional.

(ii) The functor $Lf^*(-) \otimes^L E : \mathrm{D}_{\mathrm{qc}}(S) \to \mathrm{D}_{\mathrm{qc}}(X)$ is fully faithful and has both adjoints.

Proof. Item (i) follows from **1.21** together with tor-independent base change. For (ii), given $F, G \in \mathrm{D}_{\mathrm{qc}}(S)$, the projection formula and exceptionality of E shows

$$\begin{aligned}\mathrm{Hom}_X(Lf^*F \otimes^L E, Lf^*G \otimes^L E) &\cong \mathrm{Hom}_X(Lf^*F, Lf^*G \otimes^L R\mathscr{H}om_{\mathscr{O}_X}(E,E)) \\ &\cong \mathrm{Hom}_S(F, G \otimes^L Rf_*R\mathscr{H}om_{\mathscr{O}_X}(E,E)) \cong \mathrm{Hom}_S(F,G)\end{aligned}$$

implying full faithfulness. The right adjoint is given by $Rf_*R\mathscr{H}om_{\mathscr{O}_X}(E,-)\colon \mathrm{D}_{\mathrm{qc}}(X) \to \mathrm{D}_{\mathrm{qc}}(S)$. For the left adjoint, compute:

$$\begin{aligned}\mathrm{Hom}_X(F, Lf^*G \otimes^L E) &\cong \mathrm{Hom}_X(R\mathscr{H}om_{\mathscr{O}_X}(E,F), Lf^*G) \\ &\cong \mathrm{Hom}_X(R\mathscr{H}om_{\mathscr{O}_X}(E,F) \otimes^L \omega_{X/S}, Lf^*G \otimes^L \omega_{X/S}) \\ &\cong \mathrm{Hom}_X(Rf_*R\mathscr{H}om_{\mathscr{O}_X}(E, F \otimes^L \omega_{X/S}), G)\end{aligned}$$

where the second isomorphism is because $\omega_{X/S}$ is a line bundle, and the third is because Rf_* is left adjoint to $Lf^*(-) \otimes^L \omega_{X/S}$. Thus $Rf_*R\mathscr{H}om_{\mathscr{O}_X}(E, - \otimes^L \omega_{X/S})$ is the left adjoint. ■

Let $\mathscr{A} := Lf^*\mathrm{D}_{\mathrm{qc}}(S) \otimes^L E \subset \mathrm{D}_{\mathrm{qc}}(X)$ be the essential image of the functor in **3.1**(ii). This is an admissible subcategory, so it induces two semiorthogonal decompositions

$$\mathrm{D}_{\mathrm{qc}}(X) = \langle \mathscr{A}, {}^\perp\mathscr{A} \rangle = \langle \mathscr{A}^\perp, \mathscr{A} \rangle.$$

By definition, the left orthogonal ${}^\perp\mathscr{A}$ and right orthogonal $\mathscr{A}^\perp$ categories to $\mathscr{A}$ are the full subcategories of $\mathrm{D}_{\mathrm{qc}}(X)$ whose objects have no global maps to and from objects in $\mathscr{A}$, respectively. Since E is exceptional relative to S, these categories also admit a relative description, as follows:

3.2. Lemma. — *In the setting of* **3.1**, *the orthogonals to* $\mathscr{A} := Lf^*\mathrm{D}_{\mathrm{qc}}(S) \otimes^L E$ *are given by*

$$\begin{aligned}\mathscr{A}^\perp &= \{F \in \mathrm{D}_{\mathrm{qc}}(X) : Rf_*R\mathscr{H}om_{\mathscr{O}_X}(E,F) = 0\}, \text{ and} \\ {}^\perp\mathscr{A} &= \{F \in \mathrm{D}_{\mathrm{qc}}(X) : Rf_*R\mathscr{H}om_{\mathscr{O}_X}(E, F \otimes^L \omega_{X/S}) = 0\}.\end{aligned}$$

In particular, ${}^\perp\mathscr{A}$ *and* $\mathscr{A}^\perp$ *are* S*-linear subcategories of* $\mathrm{D}_{\mathrm{qc}}(X)$ *and are equivalent via* $- \otimes^L \omega_{X/S}$

Proof. By definition, $F \in \mathscr{A}^\perp$ if and only if for every $G \in \mathrm{D}_{\mathrm{qc}}(S)$,

$$0 = \mathrm{Hom}_X(Lf^*G \otimes^L E, F) \cong \mathrm{Hom}_S(G, Rf_*R\mathscr{H}om_X(E,F)),$$

and this is equivalent to $Rf_*R\mathscr{H}om_{\mathscr{O}_X}(E,F) = 0$ by Yoneda. Similarly, $F \in {}^\perp\mathscr{A}$ if and only if

$$0 = \mathrm{Hom}_X(F, Lf^*G \otimes^L E) \cong \mathrm{Hom}_S(Rf_*R\mathscr{H}om_{\mathscr{O}_X}(E, F \otimes^L \omega_{X/S}), G),$$

and the conclusion follows again by the Yoneda lemma. These descriptions now imply S-linearity, as was explained in **1.22**(i), and the equivalence via tensor by $\omega_{X/S}$. ■

3.3. Relatively exceptional collections. — Continuing with the setting of **3.1**, a sequence of perfect complexes $E_1, \ldots, E_n \in \mathrm{D}_{\mathrm{qc}}(X)$ is called an *exceptional collection relative to* S, or briefly, an S*-exceptional collection* if each E_i is S-exceptional and $Rf_*R\mathscr{H}om_{\mathscr{O}_X}(E_i, E_j) = 0$ for $i > j$. By **3.1**(ii), each $\mathscr{A}_i := Lf^*\mathrm{D}_{\mathrm{qc}}(S) \otimes^L E_i$ is an admissible subcategory of $\mathrm{D}_{\mathrm{qc}}(X)$, and repeatedly applying **3.2** and argueing as [Xie23, Lemma 4.1(2)] produces a semiorthogonal decomposition

$$\mathrm{D}_{\mathrm{qc}}(X) = \langle \mathscr{A}_0, \mathscr{A}_1, \ldots, \mathscr{A}_n \rangle$$

where each $\mathscr{A}_i$ is S-linear and admissible, and the objects of $\mathscr{A}_0$ may be described as

$$\mathscr{A}_0 = \{F \in \mathrm{D}_{\mathrm{qc}}(X) : Rf_*R\mathscr{H}om_{\mathscr{O}_X}(E_i, F) = 0 \text{ for } i = 1, \ldots, n\}.$$

We refer to $\mathscr{A}_0$ as the *residual component* to the S-exceptional collection $E_1, \ldots, E_n$.

Since relative exceptionality is compatible with arbitrary base change as in 3.1(i), such a semiorthogonal decomposition is also compatible with arbitrary base change. Namely, if $T \to S$ be any morphism of schemes, then there is a semiorthogonal decomposition

$$\mathrm{D}_{\mathrm{qc}}(X_T) = \langle \mathscr{A}_{0,T}, \mathscr{A}_{1,T}, \dots, \mathscr{A}_{n,T} \rangle \text{ with } \mathscr{A}_{i,T} := Lf_T^* \mathrm{D}_{\mathrm{qc}}(T) \otimes^L E_{i,T} \text{ for } i = 1, \dots, n$$

and the T-exceptional collection $E_{1,T}, \dots, E_{n,T}$, and $\mathscr{A}_{0,T}$ is their T-linear right orthgonal.

3.4. Cyclic shifts and mutations. — Starting from any semiorthogonal decomposition

$$\mathrm{D}_{\mathrm{qc}}(X) = \langle \mathscr{B}_0, \mathscr{B}_1, \dots, \mathscr{B}_n \rangle,$$

induced from an S-exceptional collection—meaning that all but possibly one component is equivalent to a subcategory of the form $Lf^*\mathrm{D}_{\mathrm{qc}}(S) \otimes^L E_j$ for a perfect S-exceptional $E_j \in \mathrm{D}_{\mathrm{qc}}(X)$—there are two standard ways to construct a new one:

First, the descriptions of the two orthogonals of a subcategory generated by a relatively exceptional object in 3.2 shows that it is possible to cyclically shift the semiorthogonal decomposition, at least after twisting the shifted component by the relative dualizing line bundle:

$$\mathrm{D}_{\mathrm{qc}}(X) = \langle \mathscr{B}_n \otimes^L \omega_{X/S}, \mathscr{B}_0, \dots, \mathscr{B}_{n-1} \rangle = \langle \mathscr{B}_1, \dots, \mathscr{B}_n, \mathscr{B}_0 \otimes^L \omega_{X/S}^\vee \rangle.$$

Second, is left or right mutation across a semiorthogonal component $\mathscr{B}_i$. When $\mathscr{B}_i$ is equivalent to a subcategory of the form $Lf^*\mathrm{D}_{\mathrm{qc}}(S) \otimes^L E$ for an S-exceptional object $E \in \mathrm{D}_{\mathrm{qc}}(X)$, the mutation functors $\mathbf{L}_E, \mathbf{R}_E \colon \mathrm{D}_{\mathrm{qc}}(X) \to \mathrm{D}_{\mathrm{qc}}(X)$ were already introduced in [Bon89] and are characterized as follows: They vanish on $\mathscr{B}_i$, induce equivalences $\mathbf{L}_E \colon {}^\perp\mathscr{B}_i \to \mathscr{B}_i^\perp$ and $\mathbf{R}_E \colon \mathscr{B}_i^\perp \to {}^\perp\mathscr{B}_i$, and are described explicitly on $F \in \mathrm{D}_{\mathrm{qc}}(X)$ by

$$\begin{aligned}
\mathbf{L}_E(F) &:= \mathrm{cone}(\iota R(F) \to F) \cong \mathrm{cone}(Lf^*Rf_*R\mathscr{H}om_{\mathscr{O}_X}(E,F) \otimes^L E \to F), \text{ and} \\
\mathbf{R}_E(F) &:= \mathrm{cone}(F \to \iota L(F))[-1] \cong \mathrm{cone}(F \to Lf^*Rf_*R\mathscr{H}om_{\mathscr{O}_X}(E, F \otimes^L \omega_{X/S}) \otimes^L E)[-1]
\end{aligned}$$

where $\iota \colon \mathscr{B}_i \to \mathrm{D}_{\mathrm{qc}}(X)$ is the inclusion, and $L, R \colon \mathrm{D}_{\mathrm{qc}}(X) \to \mathscr{B}_i$ are its left and right adjoints; note that the cones are functorial by semiorthogonality, and the second isomorphism comes from the identification of the adjoints from the proof of 3.1(ii).

3.5. Lemma. — *In the setting of 3.3, let $L \colon \mathrm{D}_{\mathrm{qc}}(X) \to \mathscr{A}_0$ and $R \colon \mathrm{D}_{\mathrm{qc}}(X) \to \mathscr{A}_0$ be the left and right adjoints to the inclusion $\mathscr{A}_0 \subset \mathrm{D}_{\mathrm{qc}}(X)$ of the residual component.*

(i) *If T is an S-scheme and L_T and R_T are the adjoints to the inclusion $\mathscr{A}_{0,T} \subset \mathrm{D}_{\mathrm{qc}}(X_T)$, then*

$$L_T(F_T) \cong L(F)_T \text{ and } R_T(F_T) \cong R(F)_T \text{ for any } F \in \mathrm{D}_{\mathrm{qc}}(X).$$

(ii) *The right adjoint R sends S-perfect objects to S-perfect objects.*

Proof. For (i), note that the left and right adjoints to the inclusion $\mathscr{A}_0 \subset \mathrm{D}_{\mathrm{qc}}(X)$ may be described as a composition of the mutation functors:

$$L = \mathbf{L}_{E_1} \circ \cdots \circ \mathbf{L}_{E_n} \text{ and } R = ((-) \otimes^L \omega_{X/S}) \circ \mathbf{R}_{E_n} \circ \cdots \circ \mathbf{R}_{E_1}.$$

Combining the description of the mutation functors from 3.4 together with flat base change and 1.21 shows that each of the mutation functors commute with base change, in that

$$\mathbf{L}_{E_i}(F)_T \cong \mathbf{L}_{E_{i,T}}(F_T) \text{ and } \mathbf{R}_{E_i}(F)_T \cong \mathbf{R}_{E_{i,T}}(F_T).$$

Since formation of the relative dualizing bundle $\omega_{X/S}$ also commutes with base change, this implies that L and R commute with base change.

For (ii), it suffices to show that the right mutation functors $\mathbf{R}_{E_i}$ preserve S-perfect objects. Given an S-perfect object $F \in \mathrm{D}_{\mathrm{qc}}(X)$, consider the distinguished triangle

$$\mathbf{R}_{E_i}(F) \longrightarrow F \longrightarrow Lf^*(Rf_*R\mathcal{H}om_{\mathcal{O}_X}(E_i,F)\otimes^L \omega_{X/S})\otimes^L E_i \xrightarrow{+1}$$

from the definition of $\mathbf{R}_{E_i}$. Since $f : X \to S$ is proper, Rf_* takes S-perfect complexes to perfect complexes, so the term on the right is perfect. Since perfect complexes on X are also X-perfect, and since the category of S-perfect complexes is triangulated, this shows that $\mathbf{R}_{E_i}(F)$ is S-perfect. ■

3.6. Generator for residual component. — In order to adapt the proofs of §§1–2 to prove analogous results for the residual category in 3.3, we prove that, at least when the base scheme S is additionally qcqs, $\mathscr{A}_0$ also has a compact generator that is a perfect complex on X: see 3.10.

Given a triangulated category $\mathscr{D}$ and two full triangulated subcategories $\mathscr{A},\mathscr{B} \subseteq \mathscr{D}$, we write $\langle \mathscr{A},\mathscr{B}\rangle \subseteq \mathscr{D}$ for the smallest strictly full triangulated subcategory containing both $\mathscr{A}$ and $\mathscr{B}$. When $\mathscr{A}$ and $\mathscr{B}$ satisfy semiorthogonality conditions, objects of this category can be explicitly described:

3.7. Lemma. — *Let $\mathscr{D}$ be a triangulated category and $\mathscr{A},\mathscr{B} \subseteq \mathscr{D}$ full triangulated subcategories. If $\mathscr{A} \subseteq \mathscr{B}^\perp$, then an object $C \in \mathscr{D}$ lies in $\langle \mathscr{A},\mathscr{B}\rangle$ if and only if there is a distinguished triangle*

$$B \longrightarrow C \longrightarrow A \xrightarrow{+1} \text{ with } A \in \mathscr{A} \text{ and } B \in \mathscr{B}.$$

In particular, if $\mathscr{D}$ has arbitrary direct sums and $\mathscr{A}$ and $\mathscr{B}$ are closed under them, then so is $\langle \mathscr{A},\mathscr{B}\rangle$.

Proof. Every such C must lie in $\langle \mathscr{A},\mathscr{B}\rangle$ because it is strictly full and triangulated. Conversely, it suffices to show that the full subcategory $\mathscr{C} \subseteq \mathscr{D}$ consisting of such C is itself triangulated. It is clear that $\mathscr{C}$ closed under taking shifts, so it remains to show it is closed under cones: Given a morphism $C_1 \to C_2$ between objects fitting in distinguished triangles $B_i \to C_i \to A_i \to B_i[1]$ with $A_i \in \mathscr{A}$ and $B_i \in \mathscr{B}$ for $i = 1,2$, semiorthogonality of $\mathscr{A}$ and $\mathscr{B}$ implies that there is a unique morphism $B_1 \to B_2$ making the diagram

$$\begin{array}{ccc} B_1 & \longrightarrow & C_1 \\ \downarrow & & \downarrow \\ B_2 & \longrightarrow & C_2 \end{array}$$

commute. By [BBD82, Proposition 1.1.11], this square can be completed to a 4×4 commutative-up-to-sign diagram whose rows and columns are distinguished triangles. This provides a distinguished triangle $B_3 \to C_3 \to A_3 \to B_3[1]$ where B_3 is a cone of $B_1 \to B_2$, C_3 is a cone of $C_1 \to C_2$, and there is a morphism $A_1 \to A_2$ whose cone is A_3. But then $A_3 \in \mathscr{A}$ and $B_3 \in \mathscr{B}$, so the cone C_3 of $C_1 \to C_2$ is itself an extension of an object of $\mathscr{A}$ by an object of $\mathscr{B}$, showing that $\mathscr{C}$ is triangulated.

Regarding direct sums, let $\{C_i\}_{i\in I} \subset \langle \mathscr{A},\mathscr{B}\rangle$ be a set of objects where, for each $i \in I$, there is a distinguished triangle $B_i \to C_i \to A_i \to B_i[1]$. Then there is a distinguished triangle

$$\bigoplus_{i\in I} B_i \longrightarrow \bigoplus_{i\in I} C_i \longrightarrow \bigoplus_{i\in I} A_i \xrightarrow{+1}$$

see [Stacks, 0CRG]. Since $\mathscr{A}$ and $\mathscr{B}$ are closed under sums, this implies that $\bigoplus_{i\in I} C_i \in \langle \mathscr{A},\mathscr{B}\rangle$. ■

3.8. Lemma. — *Let $\mathscr{D}$ be a triangulated category with arbitrary direct sums and $\iota : \mathscr{A} \to \mathscr{D}$ an admissible subcategory. If $\mathscr{A}$ is closed under sums and $\mathscr{D}$ has a compact generator G, then the left adjoint $L : \mathscr{D} \to \mathscr{A}$ to ι preserves compact objects and $L(G) \in \mathscr{A}$ is a compact generator.*

Proof. Since ι also has a right adjoint, it commutes with arbitrary direct sums. Thus L has a right adjoint that commutes with arbitrary direct sums, so L preserves compact objects by 1.7. To see that

$L(G) \in \mathscr{A}$ is a generator, let $A \in \mathscr{A}$ be nonzero and choose a nonzero morphism $G \to \iota(A)[i] \in \mathscr{D}$ for some $i \in \mathbf{Z}$. Adjunction then provides a nonzero morphism $L(G) \to A[i] \in \mathscr{A}$. ■

3.9. Lemma. — *Let $\mathscr{D}$ be a triangulated category with arbitrary direct sums and let $\mathscr{D} = \langle \mathscr{A}, \mathscr{B} \rangle$ be a semiorthogonal decomposition into components which are full triangulated subcategories closed under direct sums. Then the right adjoint $R \colon \mathscr{D} \to \mathscr{B}$ to the inclusion $j \colon \mathscr{B} \to \mathscr{D}$ commutes with direct sums.*

Proof. Write $\iota \colon \mathscr{A} \to \mathscr{D}$ be the other inclusion and $L \colon \mathscr{D} \to \mathscr{A}$ for its left adjoint. It is standard R and L exist, and that every object $D \in \mathscr{D}$ fits into a functorial distinguished triangle

$$jRD \longrightarrow D \longrightarrow \iota LD \xrightarrow{+1}$$

Since $\mathscr{A}$ and $\mathscr{B}$ are closed under direct sums, the inclusion functors ι and j commute with direct sums, and L commutes with direct sums being a left adjoint. Thus, for any set of objects $\{D_i\}_{i \in I} \subset \mathscr{D}$, there is a morphism of distinguished triangles

$$\begin{array}{ccccccc} \bigoplus_{i \in I} jRD_i & \longrightarrow & \bigoplus_{i \in I} D_i & \longrightarrow & \bigoplus_{i \in I} \iota LD_i & \xrightarrow{+1} & \\ \downarrow & & \downarrow & & \downarrow & & \\ jR(\bigoplus_{i \in I} D_i) & \longrightarrow & \bigoplus_{i \in I} D_i & \longrightarrow & \iota L(\bigoplus_{i \in I} D_i) & \xrightarrow{+1} & \end{array}$$

The middle two vertical arrows are isomorphisms by the discussion above, hence so is the first vertical arrow. Therefore $j(\bigoplus_{i \in I} RD_i) \cong jR(\bigoplus_{i \in I} D_i)$ so, since j is fully faithful, $\bigoplus_{i \in I} RD_i \cong R(\bigoplus_{i \in I} D_i)$. ■

3.10. Proposition. — *In the setting of 3.1, the residual category $\mathscr{A}_0$ satisfies:*

(i) *writing $L \colon \mathrm{D}_{\mathrm{qc}}(X) \to \mathscr{A}_0$ for the left adjoint to inclusion, if G is a compact generator of $\mathrm{D}_{\mathrm{qc}}(X)$, then $L(G)$ is a compact generator of $\mathscr{A}_0$; and*

(ii) *$E \in \mathscr{A}_0$ is compact as an object of $\mathscr{A}_0$ if and only if E is compact as an object of $\mathrm{D}_{\mathrm{qc}}(X)$.*

Proof. Item (i) follows direct from 3.8. For (ii), let $E \in \mathscr{A}_0$ be compact when viewed as an object in $\mathrm{D}_{\mathrm{qc}}(X)$. Since $E = L(E)$ and L preserves compact objects by 3.8, E is also compact as an object in $\mathscr{A}_0$. Conversely, we must show that the inclusion $\mathscr{A}_0 \to \mathrm{D}_{\mathrm{qc}}(X)$ preserves compact objects. Since $\mathscr{A}_0$ is compactly generated by (i), it suffices by 1.7 to show that the right adjoint to inclusion commutes with direct sums. So consider the semiorthogonal decomposition

$$\mathrm{D}_{\mathrm{qc}}(X) = \langle \mathscr{A}_0^{\perp}, \mathscr{A}_0 \rangle.$$

Since $\mathscr{A}_0$ is an S-linear triangulated subcategory as explained in 3.3, it is closed under direct sums by our conventions and so by 3.9, it suffices to see that $\mathscr{A}_0^{\perp}$ is also closed under sums. Applying cyclic shifts as in 3.4 to the given semiorthogonal decomposition shows that

$$\mathscr{A}_0^{\perp} = \langle \mathscr{A}_1 \otimes^L \omega_{X/S}, \ldots, \mathscr{A}_n \otimes^L \omega_{X/S} \rangle$$

is the smallest strictly full triangulated subcategory of $\mathrm{D}_{\mathrm{qc}}(X)$ containing each of the $\mathscr{A}_i \otimes^L \omega_{X/S}$ for $i = 1, \ldots, n$. Since each of these subcategories are also S-linear, whence closed under direct sums, it follows from 3.7 that $\mathscr{A}_0^{\perp}$ is also closed under sums, as required. ■

3.11. Fourier–Mukai on residual component. — Let $f : X \to S$ and $g : X \to S$ be flat, proper, and finitely presented morphisms of qcqs schemes. Assume additionally that the fibres of f are Gorenstein and that there is a semiorthogonal decomposition

$$\mathrm{D}_{\mathrm{qc}}(X) = \langle \mathscr{A}_0, \mathscr{A}_1, \ldots, \mathscr{A}_n \rangle \text{ with } \mathscr{A}_i := Lf^*\mathrm{D}_{\mathrm{qc}}(S) \otimes^L E_i \text{ for } i = 1, \ldots, n$$

and an S-exceptional collection $E_1, \dots, E_n$. Let $K \in \mathrm{D}_{\mathrm{qc}}(X \times_S Y)$ be a object that is perfect relative to both X and Y, $\Phi_K \colon \mathrm{D}_{\mathrm{qc}}(X) \to \mathrm{D}_{\mathrm{qc}}(Y)$ the associated Fourier–Mukai transform, and

$$\Phi_{K,0} := \Phi_K \circ \iota \colon \mathscr{A}_0 \subset \mathrm{D}_{\mathrm{qc}}(X) \to \mathrm{D}_{\mathrm{qc}}(Y)$$

its restriction to the residual category. Generalizing **2.9**, **1.10**, and **2.10**, this satisfies:

(i) the functor $\Phi_{K,0} \colon \mathscr{A}_0 \to \mathrm{D}_{\mathrm{qc}}(Y)$ takes perfect complexes to perfect complexes;
(ii) if $\Phi_{K,0} \colon \mathscr{A}_0 \to \mathrm{D}_{\mathrm{qc}}(Y)$ is fully faithful, then for any morphism $T \to S$ from a qcqs scheme, the base changed functor $\Phi_{K_T,0} \colon \mathscr{A}_{0,T} \to \mathrm{D}_{\mathrm{qc}}(Y_T)$ is also fully faithful;
(iii) if the morphism $T \to S$ in (ii) is moreover faithfully flat, then the converse holds; and
(iv) if $\Phi_{K,0} \colon \mathscr{A}_0 \to \mathrm{D}_{\mathrm{qc}}(Y)$ is fully faithful, then it preserves universal glueability and simplicity.

Proof. For (i), write $\Phi_{K,0} = \Phi_K \circ \iota$ and note that both Φ_K and the inclusion ι preserve compact objects by **2.9** and the proof of **3.10**(ii), respectively. For (ii) and (iii), let $G \in \mathscr{A}_0$ be a compact generator obtained by applying the left adjoint of inclusion to a compact generator of $\mathrm{D}_{\mathrm{qc}}(X)$ as in **3.10**(i), and argue as in **1.9** to show that $\Phi_{K,0}$ is fully faithful if and only if the morphism

$$\phi \colon Rf_*R\mathscr{H}om_{\mathscr{O}_X}(G,G) \to Rg_*R\mathscr{H}om_{\mathscr{O}_Y}(\Phi_K(G),\Phi_K(G))$$

from **1.5**(ii) is an isomorphism; here, the third hypothesis of the criterion **1.8** is satisfied by (i). Since the base change of G to an S-scheme T remains a compact generator of $\mathscr{A}_{0,T}$ by **3.5**(i) and **3.8**, flat base change together with **1.5**(iii) shows that this full faithfulness criterion is invariant under base change, implying the first two items. This also gives (iv) upon argueing exactly as in **2.10** using this base change property in place of **1.10**. ∎

Consider the full subcategory in the stack of complexes for the morphism $f \colon X \to S$ from **2.3** parameterizing objects in the residual component $\mathscr{A}_0$:

$$\mathscr{C}omplexes_{\mathscr{A}_0/S} := \{(T,F) : T \in \mathrm{Sch}_S \text{ and } F \in \mathscr{A}_{0,T} \text{ is } T\text{-perfect and universally glueable}\}.$$

This is an open substack since membership in this subcategory is defined by

$$Rf_{T,*}R\mathscr{H}om_{\mathscr{O}_{X_T}}(E_{i,T},F) = 0 \;\text{ for } i = 1,\dots,n,$$

and this condition is closed under pullbacks by **1.21** and flat base change, and is an open condition on T by 2.12. Further write

$$s\mathscr{C}omplexes_{\mathscr{A}_0/S} \subset \mathscr{C}omplexes_{\mathscr{A}_0/S}$$

for the open substack parameterizing simple objects $F \in \mathscr{A}_{0,T}$. The analogue of **2.8** in this setting is:

3.12. Theorem. — *In the setting of* **3.11***, if $\Phi_{K,0} \colon \mathscr{A}_0 \to \mathrm{D}_{\mathrm{qc}}(Y)$ is fully faithful, then the assignment $(T,F) \mapsto (T,\Phi_{K_T}(F_T))$ induces an open immersion of S-stacks*

$$\mathrm{FM}_{K,0} \colon \mathscr{C}omplexes_{\mathscr{A}_0/S} \to \mathscr{C}omplexes_{Y/S}$$

which preserves the open substack of simple complexes.

Proof. The argument is the same as that for **2.8**, except that one uses **3.11**(ii) to see that $\mathrm{FM}_{K,0}$ is a monomorphism; and for the smoothness argument, additionally use **3.5**(i) to see that the right adjoint R of $\Phi_{K,0}$ is compatible with base change; and **3.5**(ii) to see that R preserves S-perfect complexes. ∎

4. Geometry of quadric bundles

In this section, we set our notation and conventions regarding families of quadrics $\rho\colon Q \to S$ over an arbitrary base scheme S. Much of the material is well-known, but perhaps not well-documented in this generality. In particular, since 2 is not necessarily invertible on S nor is S assumed to be reduced, some care needs to be taken when discussing matters like the corank stratification. Throughout, S is a scheme, $\mathscr{E}$ is a locally free $\mathscr{O}_S$-module of rank $n+1$, and $\mathscr{L}$ is an invertible $\mathscr{O}_S$-module.

4.1. Quadric bundles. — A *quadratic form* on $\mathscr{E}$ with values in $\mathscr{L}$ is a morphism of sheaves of sets $q\colon \mathscr{E} \to \mathscr{L}$ such that $q(fv) = f^2 q(v)$ for local sections f of $\mathscr{O}_S$ and v of $\mathscr{E}$, and such that the associated *polar form* $b_q\colon \mathscr{E} \times \mathscr{E} \to \mathscr{L}$, defined on local sections v and w of $\mathscr{E}$ by

$$b_q(v,w) := q(v+w) - q(v) - q(w),$$

is a symmetric bilinear form. Let $\pi\colon \mathbf{P}\mathscr{E} \to S$ be the projective bundle of lines associated with $\mathscr{E}$, so that a nonzero quadratic form $q\colon \mathscr{E} \to \mathscr{L}$ corresponds to a nonzero section

$$s_q \in \Gamma(\mathbf{P}\mathscr{E}, \mathscr{O}_\pi(2) \otimes \pi^*\mathscr{L}) \cong \Gamma(S, \mathrm{Sym}^2(\mathscr{E}^\vee) \otimes \mathscr{L}).$$

Its vanishing locus $\iota\colon Q \hookrightarrow \mathbf{P}\mathscr{E}$ is a family of quadrics over S. The morphism $\rho := \pi \circ \iota\colon Q \to S$ is flat of relative dimension $n-1$ if and only if the form q is *primitive* in that it is nonzero over every residue field of S. In this case, we refer to the family $\rho\colon Q \to S$ as the *quadric* $(n-1)$*-fold bundle* associated with q.

4.2. Orthogonals. — Given a submodule $\mathscr{F} \subseteq \mathscr{E}$, its *orthogonal* is

$$\mathscr{F}^\perp := \ker\big(b_q(-,\ )|_{\mathscr{F}}\colon \mathscr{E} \to \mathscr{E}^\vee \otimes \mathscr{L} \to \mathscr{F}^\vee \otimes \mathscr{L}\big)$$

where the first map is the adjoint $b_q\colon \mathscr{E} \to \mathscr{E}^\vee \otimes \mathscr{L}$ to the polar form, and the second map is the restriction. A special case is when $\mathscr{F} = \mathscr{E}$, wherein the orthogonal

$$\mathrm{rad}\, b_q := \mathscr{E}^\perp := \ker(b_q\colon \mathscr{E} \to \mathscr{E}^\vee \otimes \mathscr{L})$$

is referred to as the *bilinear radical* of q.

4.3. Singular locus. — The *singular locus* $\mathrm{Sing}\,\rho$ of a family of quadrics $\rho\colon Q \to S$ refers to the locus in Q where the morphism ρ is not smooth of relative dimension $n-1$. This carries a scheme structure given by the $(n-1)$-st Fitting ideal of the sheaf $\Omega^1_{Q/S}$ of relative differentials: see [Stacks, 0C3K]. By the conormal sequence

$$\mathscr{C}_{Q/S} \to \Omega^1_{\mathbf{P}\mathscr{E}/S}|_Q \to \Omega^1_{Q/S} \to 0,$$

this is simply the zero scheme of the map of finite locally free $\mathscr{O}_Q$-modules $\mathscr{C}_{Q/S} \to \Omega^1_{\mathbf{P}\mathscr{E}/S}$ on Q. The vector bundle $\Omega^1_{\mathbf{P}\mathscr{E}/S}|_Q$ is a subbundle of $\rho^*\mathscr{E}(-1)$ via the relative Euler sequence for $\pi\colon \mathbf{P}\mathscr{E} \to S$, and we have $\mathscr{C}_{Q/S} \cong \mathscr{O}_\rho(-2) \otimes \rho^*\mathscr{L}$ via the section s_q. Thus twisting up by $\mathscr{O}_\rho(1)$ identifies $\mathrm{Sing}\,\rho$ with the zero scheme in Q of a map of finite locally free $\mathscr{O}_Q$-modules

$$\mathrm{Sing}\,\rho = \mathrm{V}(\mathscr{O}_\rho(-1) \otimes \rho^*\mathscr{L}^\vee \to \rho^*\mathscr{E}^\vee).$$

In local coordinates, this section gives the vector of partial derivatives of the quadratic polynomial s_q. A straightforward computation therefore shows that this is the composition

$$\rho^* b_q(\mathrm{eu}_\pi|_Q,\ )\colon \mathscr{O}_\rho(-1) \otimes \rho^*\mathscr{L}^\vee \to \rho^*(\mathscr{E} \otimes \mathscr{L}^\vee) \to \rho^*(\mathscr{E}^\vee \otimes \mathscr{L} \otimes \mathscr{L}^\vee) \cong \rho^*\mathscr{E}^\vee$$

where $\mathrm{eu}_\pi\colon \mathscr{O}_\pi(-1) \to \pi^*\mathscr{E}$ is the tautological section, and $b_q\colon \mathscr{E} \to \mathscr{E}^\vee \otimes \mathscr{L}$ is adjoint to the polar form of q. The formation of the closed subscheme $\mathrm{Sing}\,\rho \subset Q$ clearly commutes with base change, and in particular for $s \in S$ we have

$$(\mathrm{Sing}\,\rho)_s = \mathrm{Sing}\,Q_s = Q_s \cap \mathbf{P}(\mathrm{rad}\, b_{q,s})$$

where $b_{q,s}\colon \mathscr{E}_s \to (\mathscr{E}^\vee \otimes \mathscr{L})_s$ is the restriction of the polar form to the residue field at s, and where the bilinear radical is the kernel of $b_{q,s}$ as defined in 4.2.

4.4. Coranks. — Over a field $\mathbf{k}$, the *corank* of a quadric $Q \subseteq \mathbf{P}^n$ is one more than the dimension of the locus $\operatorname{Sing} Q$ of points not smooth of dimension $n-1$; for example, a smooth quadric is of corank 0, and that defined by the zero quadratic form is of corank $n+1$. Since the singular locus is the intersection $Q \cap \mathbf{P}(\operatorname{rad} b_q)$ by 4.3, setting $\operatorname{corank} b_q := \dim_{\mathbf{k}} \operatorname{rad} b_q$, gives inequalities

$$\operatorname{corank} b_q - 1 \leq \operatorname{corank} Q \leq \operatorname{corank} b_q.$$

When $\operatorname{corank} Q = \operatorname{corank} b_q$, then the singular locus of Q is, scheme-theoretically, the linear space on $\operatorname{rad} b_q$. This is essentially because $\mathbf{P}(\operatorname{rad} b_q)$ is reduced. We shall see, however, that when $\operatorname{corank} Q < \operatorname{corank} b_q$, the singular locus of Q is never geometrically reduced over $\mathbf{k}$.

When $\operatorname{char}(\mathbf{k}) \neq 2$, the equation $q(x) = \frac{1}{2} b_q(x,x)$ means that $\mathbf{P}(\operatorname{rad} b_q) \subset Q$, and so the corank of Q coincides with the corank of b_q. When $\operatorname{char}(\mathbf{k}) = 2$, a choice of basis for $\operatorname{rad} b_q$ gives an expression

$$q|_{\operatorname{rad} b_q} = \sum\nolimits_{i=1}^{\operatorname{corank} b_q} a_i x_i^2$$

for some $a_i \in \mathbf{k}$. If this vanishes, then Q contains all of $\mathbf{P}(\operatorname{rad} b_q)$ and so $\operatorname{corank} Q = \operatorname{corank} b_q$; otherwise, the singular locus of Q is geometrically a double plane—which may be defined only over an inseparable extension of $\mathbf{k}$!—inside $\mathbf{P}(\operatorname{rad} b_q)$. This gives the first statement of:

4.5. Lemma. — *Let Q be a quadric hypersurface in $\mathbf{P}^n$ over a field $\mathbf{k}$. Then*

$$\operatorname{corank} Q = \begin{cases} \operatorname{corank} b_q - 1 & \textit{if } \operatorname{char}(\mathbf{k}) = 2 \textit{ and } q|_{\operatorname{rad} b_q} \neq 0 \textit{ and} \\ \operatorname{corank} b_q & \textit{otherwise.} \end{cases}$$

If $n + 1 - \operatorname{corank} Q$ is even, then $\operatorname{corank} Q = \operatorname{corank} b_q$.

Proof. The second part concerns only $\operatorname{char}(\mathbf{k}) = 2$, in which case the polar form b_q is alternating, so its rank is even. Since $\operatorname{corank} Q$ is either $\operatorname{corank} b_q$ or $\operatorname{corank} b_q - 1$, the former must hold. ■

4.6. Corank stratification. — A family of quadrics $\rho\colon Q \to S$ defines a decreasing filtration of S by closed subsets

$$S_c := \{\, s \in S : \operatorname{corank} Q_s \geq c \,\} = \{\, s \in S : \dim \operatorname{Sing} Q_s \geq c - 1 \,\}.$$

When 2 is invertible on S, the corank of a quadric is equal to that of its polar form, so the sets S_c coincide set-theoretically with the sets

$$S_{\mathrm{bilin},c} := \{\, s \in S : \operatorname{corank} b_q \geq c \,\} = \{\, s \in S : \operatorname{rank} b_q \leq n + 1 - c \,\}.$$

These carry the scheme structure given locally by the vanishing of the size $n + 2 - c$ minors of b_q, and this is what is given to the S_c.

When 2 is not necessarily invertible on S, it is a subtle issue to endow the S_c with a suitable scheme structure: see, for example, [ABBGvB21, Definition 2.8] and [Tan24] for some possibilities in low-dimensions. For the purposes of this article, we will take that which is locally pulled back from the parameter space of the universal quadric in $\mathbf{P}^n$ over $\mathbf{Z}$, wherein they are equipped with their reduced closed structure. Since universal degeneracy loci are reduced, as follows from [Kut74, Theorem 1], this is the usual structure away from points of characteristic 2. Together with the corank inequalities of 4.4, this means that, for all integers $c \geq 0$, there are scheme-theoretic inclusions

$$S_{\mathrm{bilin},c+1} \subseteq S_c \subseteq S_{\mathrm{bilin},c}.$$

The content of these definitions is perhaps best illustrated with an example:

4.7. Example. — Let $S = \operatorname{Spec} \mathbf{k}[\epsilon]$ be the spectrum of the ring of dual numbers over a field $\mathbf{k}$, and consider the quadric surface bundle over S given by

$$Q := \mathrm{V}(x_0x_1 + \epsilon x_2^2) \subset \mathbf{P}^3_S.$$

The moduli map of $Q \to S$ identifies S with a tangent vector from a closed point in the universal corank 2 locus into the corank 1 locus. This means that the corank 2 stratum associated with the family $Q \to S$ is given by the reduced closed subscheme $S_2 = S_{\mathrm{red}}$, and $S_1 = S$ scheme-theoretically; importantly, $S_2 \neq S$ scheme-theoretically. Furthermore, note that if $\operatorname{char} \mathbf{k} = 2$, then the bilinear corank 2 locus is all of S scheme-theoretically, so $S_2 \neq S_{\mathrm{bilin},2}$ scheme-theoretically.

It is often useful to restrict a quadric bundle to pieces of the corank stratification, so that the singular locus may be identified as the projective bundle on the bilinear radical. In general, as in cases such as 4.7, some care is required. The following gives a useful situation in which this is possible, and may be viewed as a relative version of the discussion in 4.4 and 4.5:

4.8. Lemma. — *Let $\rho : Q \to S$ be a quadric $(n-1)$-fold bundle. Assume that $S = S_c$ scheme-theoretically, $S_{c+1} = \varnothing$, and that $r := n + 1 - c > 0$ is even. Then $\operatorname{rad} b_q \subset \mathcal{E}$ is a subbundle and $\operatorname{Sing} \rho$ is given by the projective bundle $\mathbf{P}(\operatorname{rad} b_q)$.*

Proof. Since formation of the corank stratification and singular locus commute with base change, it suffices to treat the case $\rho : Q \to S$ is the universal corank c quadric over $\mathbf{Z}$, so that we may take S to be reduced. The polar form $b_q : \mathcal{E} \to \mathcal{E}^\vee \otimes \mathcal{L}$ has constant rank $n+1-c$ so, since S is reduced, its kernel $\operatorname{rad} b_q$ is a subbundle of $\mathcal{E}$ of corank c. Now, on the one hand, $\operatorname{Sing} \rho$ is the scheme-theoretic intersection of Q with $\mathbf{P}(\operatorname{rad} b_q)$. On the other hand, since $n+1-c$ is even, 4.5 gives a set-theoretic inclusion $\mathbf{P}(\operatorname{rad} b_q) \subseteq Q$; but since the former is reduced, being a projective bundle over a reduced base, the inclusion holds scheme-theoretically, from which it follows that $\operatorname{Sing} \rho = \mathbf{P}(\operatorname{rad} b_q)$. ■

4.9. Fano schemes. — A submodule $\mathcal{W} \subseteq \mathcal{E}$ is *isotropic* for the quadratic form $q : \mathcal{E} \to \mathcal{L}$ if the restriction $q|_{\mathcal{W}}$ is the zero map. The subscheme of the Grassmannian $\mathbf{G}(r+1, \mathcal{E})$ of rank $r+1$ subbundles in $\mathcal{E}$ parameterizing isotropic subbundles is identified with the *Fano scheme*

$$\rho_r : \mathbf{F}_r(Q/S) \to S$$

of r-planes in $\rho : Q \to S$, which parameterizes $\mathbf{P}^r$-bundles contained in Q. A basic fact is that when $\rho : Q \to S$ is smooth of relative dimension $n-1$, then the Fano schemes are smooth over S, with the last nonempty one satisfying

$$\dim(\rho_\ell : \mathbf{F}_\ell(Q/S) \to S) = \binom{\ell+1}{2} \text{ and } \operatorname{rank}_{\mathcal{O}_S}\left(\rho_{\ell,*}\mathcal{O}_{\mathbf{F}_\ell(Q/S)}\right) = \begin{cases} 2 & \text{if } n-1 = 2\ell, \text{ and} \\ 1 & \text{if } n-1 = 2\ell+1. \end{cases}$$

In particular, this means that when $\rho : Q \to S$ is of even relative dimension 2ℓ, each fibre of $\rho_\ell : \mathbf{F}_\ell(Q/S) \to S$ has two geometric connected components.

4.10. — Let $\mathcal{F} \subseteq \mathcal{E}$ be an isotropic subbundle of rank r such that its orthogonal $\mathcal{F}^\perp \subseteq \mathcal{E}$ is also a subbundle. Isotropicity implies that $\mathcal{F} \subseteq \mathcal{F}^\perp$. The quotient $\mathcal{F}^\perp/\mathcal{F}$ is then a finite locally free $\mathcal{O}_S$-module and $q : \mathcal{E} \to \mathcal{L}$ induces a quadratic form $q' : \mathcal{F}^\perp/\mathcal{F} \to \mathcal{L}$, defined on a local section v' by $q'(v') := q(v)$ for any lift v to $\mathcal{F}^\perp \subseteq \mathcal{E}$. This is well-defined because

$$q(v+w) = q(v) + q(w) + b_q(v,w) = q(v)$$

for local sections v and w of $\mathcal{F}^\perp$ and $\mathcal{F}$, respectively. Let $\rho' : Q' \to S$ be the corresponding family of quadrics. Comparing functors of points shows that families of linear spaces in $\rho : Q' \to S$ naturally correspond to linear spaces in $\rho : Q \to S$ containing $\mathbf{P}\mathcal{F}$:

4.11. Lemma. — *For every integer $k \geq 0$, the Fano scheme $\mathbf{F}_k(Q'/S)$ embeds into $\mathbf{F}_{k+r}(Q/S)$ as the subscheme of $(k+r)$-planes containing $\mathbf{P}\mathscr{F}$.*

Proof. In detail, for every S-scheme T, the map $\mathscr{G} \mapsto \mathscr{G}/\mathscr{F}_T$ provides a bijection between subbundles $\mathscr{G} \subseteq \mathscr{E}_T$ such that $\mathscr{F}_T \subseteq \mathscr{G} \subseteq \mathscr{F}_T^\perp$ and subbundles of $\mathscr{F}_T^\perp/\mathscr{F}_T$, sending q-isotropic subbundles to q'-isotropic ones. Furthermore, taking v to be a local section of $\mathscr{G}$ and t a local section of $\mathscr{F}_T$ in the computation of 4.10 shows that an isotropic subbundle of $\mathscr{G} \subseteq \mathscr{E}_T$ containing $\mathscr{F}_T$ is automatically contained in $\mathscr{F}_T^\perp$, so that the map above restricts to a bijection between isotropic subbundles of $\mathscr{E}_T$ containing $\mathscr{F}_T$ and isotropic subbundles of $\mathscr{F}_T^\perp/\mathscr{F}_T$. ∎

Combining this with 4.8 relates Fano schemes of singular quadric bundles with those of smooth ones. An important case for us is when $\rho\colon Q \to S$ is a quadric 2ℓ-fold bundle where all fibres are of corank 2, so that each fibre is a cone with vertex $\mathbf{P}^1$ over a smooth quadric of dimension $2\ell-2$. The following shows that such quadric bundles also have two families of maximal isotropic subspaces. A related construction seems to have appeared in [DK20].

4.12. Lemma. — *Let $\rho\colon Q \to S$ be a quadric 2ℓ-fold bundle such that $S = S_2$ scheme-theoretically and $S_3 = \varnothing$. The Stein factorization of $\rho_{\ell+1}\colon \mathbf{F}_{\ell+1}(Q/S) \to S$ provides an étale double cover $\tilde{S} \to S$.*

Proof. By 4.8, $\operatorname{rad} b_q \subset \mathscr{E}$ is a rank 2 isotropic subbundle, and the quadric bundle $\bar\rho\colon \overline{Q} \to S$ defined by the induced quadratic form on $\mathscr{E}/\operatorname{rad} b_q$, as in 4.10, is smooth of relative dimension $2\ell-2$. Then 4.11 embeds $\mathbf{F}_{\ell-1}(\overline{Q}/S)$ in $\mathbf{F}_{\ell+1}(Q/S)$ as the locus of $(\ell+1)$-planes containing $\mathbf{P}(\operatorname{rad} b_q)$. We claim this is in fact all of $\mathbf{F}_{\ell+1}(Q/S)$, whereupon we can conclude by the discussion of 4.9. We note that it is a routine exercise in quadratic forms to check that this is true on the level of sets. Let T be an S-scheme and $\mathscr{F} \in \mathbf{F}_{\ell+1}(Q/S)_T$. Then by 4.13 below, $\mathscr{F} \cap \operatorname{rad}(b_q)_T \subset \mathscr{F}$ is a subbundle, and $\mathscr{F}/\mathscr{F} \cap \operatorname{rad}(b_q)_T$ is an isotropic subbundle of $(\mathscr{E}/\operatorname{rad} b_q)_T$. Since $(\mathscr{E}/\operatorname{rad} b_q)_T$ is non-degenerate of rank 2ℓ, the maximal rank of an isotropic subbundle of it is ℓ, so the rank of $\mathscr{F} \cap \operatorname{rad}(b_q)_T$ must be at least 2. But this is a subbundle of the rank two vector bundle $\operatorname{rad}(b_q)_T$ (as they are each subbundles of $\mathscr{E}_T$), so in fact we must have $\mathscr{F} \cap \operatorname{rad}(b_q)_T = \operatorname{rad}(b_q)_T$ or $\mathscr{F} \supset \operatorname{rad}(b_q)_T$, as needed. ∎

Linear spaces of dimension ℓ and $\ell+1$ must intersect the singular locus of a corank 2 quadric 2ℓ-fold. This behaves well in families, even when the base S may be nonreduced:

4.13. Lemma. — *Let $\rho\colon Q \to S$ be a quadric 2ℓ-fold bundle associated with a quadratic form $q\colon \mathscr{E} \to \mathscr{L}$ such that $S = S_2$ scheme-theoretically and $S_3 = \varnothing$. If $\mathscr{F} \subset \mathscr{E}$ is an isotropic subbundle satisfying*

$$\operatorname{rank}(b_q|_{\mathscr{F}}\colon \mathscr{F} \subset \mathscr{E} \to \mathscr{E}^\vee \otimes \mathscr{L}) \geq \ell$$

at every point $s \in S$. Then $\mathscr{F} \cap \operatorname{rad} b_q$ is a subbundle of $\mathscr{F}$ and $\mathscr{F}/\mathscr{F} \cap \operatorname{rad} b_q$ is a subbundle of $\mathscr{E}/\operatorname{rad} b_q$.

Proof. Note that $\ker b_q|_{\mathscr{F}} = \mathscr{F} \cap \operatorname{rad} b_q \subset \mathscr{E}$, so writing $b_q(\mathscr{F})$ for the image of $\mathscr{F}$, there is a short exact sequence of $\mathscr{O}_S$-modules

$$0 \to \mathscr{F} \cap \operatorname{rad} b_q \to \mathscr{F} \to b_q(\mathscr{F}) \to 0.$$

We show that $b_q(\mathscr{F})$ is a local direct summand of $\mathscr{F}$ of rank ℓ, from which the conclusion follows from the short exact sequence. The question is thus local on S, so assume for the remainder that $S = \operatorname{Spec} R$ is the spectrum of a local ring.

On the one hand, the hypothesis on $\operatorname{rank} b_q|_{\mathscr{F}}$ means that some $\ell \times \ell$ minor of the matrix representing $b_q|_{\mathscr{F}}$ is invertible, so $b_q(\mathscr{F})$ contains a subsheaf $\mathscr{F}'$ which is a rank ℓ direct summand of $\mathscr{E}^\vee$. On the other hand, the modules $b_q(\mathscr{F}) \subset b_q(\mathscr{E})$ are isomorphic to the quotient of $\mathscr{F} \subset \mathscr{E}$ by $\operatorname{rad} b_q$. In this way, q induces a regular quadratic form on the rank 2ℓ module $b_q(\mathscr{E})$ with respect to which $b_q(\mathscr{F})$ is a totally isotropic submodule; in particular, the rank ℓ summand $\mathscr{F}'$ is totally isotropic. In a regular quadratic module M of rank 2ℓ, an isotropic summand N of rank ℓ is equal to its own orthogonal and hence maximal, since if $N \subset N'$ is isotropic then $N \subset N' \subset N^{\perp} = N$, and so this implies $b_q(\mathscr{F}) = \mathscr{F}'$ is a direct summand of $\mathscr{F}$. ■

4.14. Hyperbolic reduction. — Assume now that $\rho\colon Q \to S$ is a quadric bundle, meaning that it is flat of relative dimension $n-1$. Let $\mathscr{F} \subseteq \mathscr{E}$ be an isotropic subbundle of rank r. Following [Kuz24, §2.1], $\mathscr{F}$ is furthermore called *regular isotropic* if the map

$$b_q(-,\ \ )|_{\mathscr{F}}\colon \mathscr{E} \to \mathscr{E}^{\vee} \otimes \mathscr{L} \to \mathscr{F}^{\vee} \otimes \mathscr{L}$$

is surjective. A straightforward argument using the discussion of 4.3 shows that this is equivalent to the property that $\mathbf{P}\mathscr{F}$ is contained in the smooth locus of $\rho\colon Q \to S$. The orthogonal $\mathscr{F}^{\perp}$ is then a corank r subbundle of $\mathscr{E}$ containing $\mathscr{F}$, so q induces via 4.10 a quadratic form $q'\colon \mathscr{F}^{\perp}/\mathscr{F} \to \mathscr{L}$. The associated family of quadrics $\rho'\colon Q' \to S$ is called the *hyperbolic reduction* of Q along $\mathscr{F}$. This has a modular interpretation by 4.11 as the scheme of r-planes along $\rho\colon Q \to S$ containing $\mathbf{P}\mathscr{F}$.

4.15. — Hyperbolic reduction can be realized geometrically via linear projection centred at $\mathbf{P}\mathscr{F}$. We describe this here, though see also [KS18, Proposition 2.5]. Let $\mathscr{E}' := \mathscr{E}/\mathscr{F}$ be the quotient bundle, $\widetilde{\mathbf{P}}\mathscr{E} \to \mathbf{P}\mathscr{E}$ the blowup along $\mathbf{P}\mathscr{F}$, and $\widetilde{Q}$ the strict transform of Q. Then there is a diagram

$$\begin{array}{ccccccccc}
 & & \widetilde{Q} & \hookrightarrow & \widetilde{\mathbf{P}}\mathscr{E} & & & & \\
 & \swarrow & & \overset{b}{\swarrow} & & \overset{a}{\searrow} & & & \\
Q & \hookrightarrow & \mathbf{P}\mathscr{E} & \dashrightarrow & & & \mathbf{P}\mathscr{E}' & \hookleftarrow \mathbf{P}(\mathscr{F}^{\perp}/\mathscr{F}) \hookleftarrow & Q'.
\end{array}$$

The morphism $\widetilde{Q} \to \mathbf{P}\mathscr{E}'$ is generically a $\mathbf{P}^{r-1}$-bundle, and restricts to a $\mathbf{P}^{r}$-bundle along the locus in $\mathbf{P}\mathscr{E}'$ parameterizing r-planes in Q which contain $\mathbf{P}\mathscr{F}$: this is Q' by 4.11.

This situation furthermore gives natural equations for Q'. To describe this, set some notation: Let $\pi'\colon \mathbf{P}\mathscr{E}' \to S$ be the structure morphism and form the following commutative exact diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi'^{*}\mathscr{F} & \longrightarrow & \mathscr{G} & \longrightarrow & \mathscr{O}_{\pi'}(-1) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi'^{*}\mathscr{F} & \longrightarrow & \pi'^{*}\mathscr{E} & \longrightarrow & \pi'^{*}\mathscr{E}' & \longrightarrow & 0
\end{array}$$

where the locally free $\mathscr{O}_{\mathbf{P}\mathscr{E}'}$-module $\mathscr{G}$ sits as the pullback of the right hand square. Write E for the exceptional divisor of $b\colon \widetilde{\mathbf{P}}\mathscr{E} \to \mathbf{P}\mathscr{E}$. The following are standard facts:

4.16. Lemma. — *In the setting of 4.14, the following are true:*

(i) *$a\colon \widetilde{\mathbf{P}}\mathscr{E} \to \mathbf{P}\mathscr{E}'$ is the $\mathbf{P}^{r}$-bundle $\mathbf{P}\mathscr{G} \to \mathbf{P}\mathscr{E}'$;*

(ii) *the exceptional divisor E of $b\colon \widetilde{\mathbf{P}}\mathscr{E} \to \mathbf{P}\mathscr{E}$ is the subbundle $\mathbf{P}(\pi'^{*}\mathscr{F}) \subset \mathbf{P}\mathscr{G}$;*

(iii) *$b^{*}\mathscr{O}_{\pi}(1) = \mathscr{O}_a(1)$ and $b^{*}\mathscr{O}_{\pi}(1) = a^{*}\mathscr{O}_{\pi'}(1) \otimes \mathscr{O}_{\widetilde{\mathbf{P}}\mathscr{E}}(E)$; and*

(iv) *$\widetilde{Q}$ an effective Cartier divisor in $\widetilde{\mathbf{P}}\mathscr{E}$ defined by a section*

$$\tilde{q} \in \mathrm{H}^0(\widetilde{\mathbf{P}}\mathscr{E}, \mathscr{O}_a(1) \otimes a^{*}(\mathscr{O}_{\pi'}(1) \otimes \pi'^{*}\mathscr{L})).$$ ■

Pushing the section $\tilde{q}$ from 4.16(iv) along $a\colon \widetilde{\mathbf{P}}\mathscr{E} \to \mathbf{P}\mathscr{E}'$ gives a canonical map

$$a_{*}(\tilde{q})\colon \mathscr{O}_{\mathbf{P}\mathscr{E}'} \to \mathscr{G}^{\vee} \otimes \mathscr{O}_{\pi'}(1) \otimes \pi'^{*}\mathscr{L}.$$

Since $\tilde{q}$ is the family of linear forms along the projective bundle $\mathbf{P}\mathscr{G} \to \mathbf{P}\mathscr{E}'$ corresponding to linear sections of Q containing $\mathbf{P}\mathscr{F}$, the vanishing locus of $a_{*}(\tilde{q})$ is supported on Q'. The following explicitly verifies that they agree scheme-theoretically:

4.17. Lemma. — *Q' is the vanishing locus of $a_*(\tilde{q})$ in $\mathbf{P}\mathscr{E}'$.*

Proof. Having defined Q' and $a_*(\tilde{q})$ globally, the question is local on S. Passing to a Zariski open, we may thus assume that $\mathscr{L}$, $\mathscr{E}$, and $\mathscr{F}$ are free $\mathscr{O}_S$-modules of ranks 1, $n+1$, and r, and that there are projective coordinates

$$\mathbf{P}\mathscr{E} \cong \{(x_0 : \cdots : x_{r-1} : y_0 : \cdots : y_{n-r}) \in \mathbf{P}^n_S\}$$

such that $\mathbf{P}\mathscr{F} \cong \mathrm{V}(y_0, \ldots, y_{n-r})$ and $Q \cong \mathrm{V}(\sum_{i=0}^{r-1} x_i y_i + q')$ where q' is a quadratic form in the remaining $n - 2r + 1$ variables (by the "splitting off a hyperbolic space lemma;" see [Bae78, Proof of Theorem 3.6] which applies also when q is degenerate but the isotropic subbundle is regular). In this way, $\mathbf{P}\mathscr{E}'$ may be identified with the $\mathbf{P}^{n-r}_S$ on the y_i-coordinates. Consider now the $\mathbf{P}^r$-bundle The coordinates provide a splitting

$$\mathscr{G} \cong \pi'^*\mathscr{F} \oplus \mathscr{O}_{\pi'}(-1)$$

and local fibre coordinates $(t_0 : \cdots : t_r)$ for the associated $\mathbf{P}^r$-bundle $a \colon \mathbf{P}\mathscr{G} \to \mathbf{P}\mathscr{E}'$ such that $\tilde{q}$ from 4.16(iv) is given above a point $y = (y_0 : \cdots : y_{n-r})$ of $\mathbf{P}\mathscr{E}'$ by

$$\tilde{q}_y = y_0 t_0 + \cdots + y_{r-1} t_{r-1} + q'(y_r, \ldots, y_{n-r}) t_r.$$

Thus $a_*(\tilde{q})$ is the section of $\mathscr{G}^\vee \otimes \mathscr{O}_{\pi'}(1) \cong \mathscr{O}_{\pi'}(1)^{\oplus r} \oplus \mathscr{O}_{\pi'}(2)$ with components $(y_0, \ldots, y_{r-1}, q')$. Since the orthogonal of $\mathscr{F}$ in $\mathscr{E}$ corresponds to the linear subspace $\mathrm{V}(y_0, \ldots, y_{r-1})$, the result follows upon comparing with the definition of Q' from 4.14. ■

This has a crucial consequence in the special case when the regular subbundle $\mathscr{F}$ has rank $r = 1$. The discussion of 4.14 means that the morphism $\widetilde{Q} \to \mathbf{P}\bar{\mathscr{E}}$ is an isomorphism away from Q', and the inverse image of Q' has codimension 1. In fact, $\widetilde{Q}$ is often the blowup of $\mathbf{P}\bar{\mathscr{E}}$ along Q'. We give a proof below, though compare with [KS18, Remark 2.6]. It may also be interesting to note that something of this nature holds when $\mathscr{F}$ is not necessarily regular: see [CPZ25, Lemma 4.3].

4.18. Lemma. — *Assume that $\mathscr{F}$ has rank 1. If Q' is an effective Cartier divisor in $\mathbf{P}(\mathscr{F}^\perp/\mathscr{F})$, then $a \colon \widetilde{Q} \to \mathbf{P}\mathscr{E}'$ is isomorphic to the blowup of $\mathbf{P}\mathscr{E}'$ along Q' and its exceptional divisor Λ satisfies*

$$\mathscr{O}_{\widetilde{Q}}(\Lambda) \cong \mathscr{O}_a(1) \otimes \mathscr{O}_{\widetilde{Q}}(-2E) \otimes \tilde{\pi}^*(\mathscr{F}^\vee \otimes \mathscr{L}).$$

Proof. Since Q' is an effective Cartier divisor in $\mathbf{P}(\mathscr{F}^\perp/\mathscr{F})$, which itself is a hyperplane in $\mathbf{P}\mathscr{E}'$, its defining equation $a_*(\tilde{q})$ from 4.17 is a regular section of $\mathscr{G}^\vee \otimes \mathscr{O}_{\pi'}(1) \otimes \pi'^*\mathscr{L}$. The ideal sheaf $\mathscr{I}$ of Q' in $\mathbf{P}\mathscr{E}'$ therefore admits a Koszul resolution

$$0 \longrightarrow \mathscr{O}_{\mathbf{P}\mathscr{E}'} \xrightarrow{a_*(\tilde{q})} \mathscr{G}^\vee \otimes \mathscr{O}_{\pi'}(1) \otimes \pi'^*\mathscr{L} \longrightarrow \Delta \otimes \mathscr{I} \longrightarrow 0$$

where $\Delta := \det(\mathscr{G}^\vee \otimes \mathscr{O}_{\pi'}(1) \otimes \pi'^*\mathscr{L})$. The sequence induces a surjection from the symmetric algebra on the centre term to the Rees algebra on the twisted ideal sheaf on the right; since line bundle twists only change the relative hyperplane class when taking Proj-constructions, this embeds the blowup of $\mathbf{P}\mathscr{E}'$ along Q' into the projective bundle $\mathbf{P}\mathscr{G}$. Furthermore, the sequence shows that the blowup is cut out by the ideal generated in degree 1 by $a_*(q)$, and this is precisely $\widetilde{Q}$ by 4.16(iv).

To determine the line bundle on $\widetilde{Q}$ associated with the exceptional divisor Λ, recall that this is the relative $\mathscr{O}(-1)$ for the Proj-construction: see, for example, [Stacks, 02OS]. The Koszul resolution above relates twists of relative $\mathscr{O}(1)$'s, yielding

$$\mathscr{O}_a(1) \otimes a^*(\mathscr{O}_{\pi'}(1) \otimes \pi'^*\mathscr{L})|_{\widetilde{Q}} \cong a^*\Delta|_{\widetilde{Q}} \otimes \mathscr{O}_{\widetilde{Q}}(-\Lambda).$$

The diagram of 4.15 implies that $\Delta \cong \mathscr{O}_{\pi'}(3) \otimes \pi'^*(\mathscr{F}^\vee \otimes \mathscr{L}^{\otimes 2})$. Using additionally the fact that $a^*\mathscr{O}_{\pi'}(1) \cong \mathscr{O}_a(1) \otimes \mathscr{O}_{\widetilde{\mathbf{P}\mathscr{E}}}(E)$ from 4.16(ii) and rearranging then gives the result. ■

The following gives an algebraic reformulation of the hypothesis in 4.18. See, for example, [Stacks, 0547 and 056L] for the definition of weakly associated points.

4.19. Lemma. — *Assume that $\mathscr{F}$ has rank 1. Then Q' is an effective Cartier divisor in $\mathbf{P}(\mathscr{F}^{\perp}/\mathscr{F})$ if and only if S_{n-1} does not contain any weakly associated points of S.*

Proof. This is local on S, so pass to a Zariski open and adopt notation as in the proof of 4.17 with $r = 1$. Then Q is cut out in $\mathbf{P}^n_S$ by a quadratic form $q = xy + q'$ and Q' is cut out by q' in a complementary $\mathbf{P}^{n-2}_S$. That Q' is an effective Cartier divisor means that q' is not a zero divisor in $\mathscr{O}_S[y_1, \ldots, y_{n-1}]$. Either by a minimal multidegree argument or else by [McC57], the polynomial q' is a zero divisor if and only if its coefficients are simultaneously killed by a nonzero scalar; this means that their vanishing locus S_{n-1} contains a weakly associated point by [Stacks, 05C3]. ■

5. Clifford algebras and spinor sheaves

This section develops a theory of spinor sheaves, following [Add11, Xie23]. The new results here are 5.13 and 5.14, which show how spinor bundles are related through special hyperplane sections; and 5.17, which relates spinor sheaves on quadric cones with spinors from its base. Throughout, unless otherwise stated, $\rho \colon Q \to S$ is a family of quadrics associated with a quadratic form $q \colon \mathscr{E} \to \mathscr{L}$, and we additionally assume that Q is an effective Cartier divisor in $\mathbf{P}\mathscr{E}$: see also 4.19.

5.1. Clifford algebras. — Following [BK94], the sheaf of *generalized Clifford algebras* associated with the quadratic form $q \colon \mathscr{E} \to \mathscr{L}$ is the sheaf of $\mathscr{O}_S$-algebras with presentation

$$\mathcal{C}\ell(\mathscr{E}, q) := \Big(\bigoplus\nolimits_{i=0}^{\infty} \mathscr{E}^{\otimes i}\Big) \otimes \Big(\bigoplus\nolimits_{k \in \mathbf{Z}} \mathscr{L}^{\otimes k}\Big) / \langle (v \otimes v) \otimes 1 - 1 \otimes 1 \otimes q(v) : v \in \mathscr{E} \rangle.$$

Placing $\mathscr{E}^{\otimes i} \otimes \mathscr{L}^{\otimes k}$ in degree $i + 2k$, the ideal of relations is generated by homogeneous elements of degree 2, whereupon $\mathcal{C}\ell(\mathscr{E}, q)$ becomes a sheaf of graded $\mathscr{O}_S$-algebras, with graded decomposition

$$\mathcal{C}\ell(\mathscr{E}, q) = \bigoplus\nolimits_{d \in \mathbf{Z}} \mathcal{C}\ell_d(\mathscr{E}, q).$$

The sheaf $\mathcal{C}\ell_0(\mathscr{E}, q)$ is the sheaf of *even Clifford algebras*; $\mathcal{C}\ell_1(\mathscr{E}, q)$ is the *odd Clifford bimodule*; and there are isomorphisms

$$\mathcal{C}\ell_d(\mathscr{E}, q) \cong \begin{cases} \mathcal{C}\ell_0(\mathscr{E}, q) \otimes \mathscr{L}^{\otimes k} & \text{if } d = 2k, \text{ and} \\ \mathcal{C}\ell_1(\mathscr{E}, q) \otimes \mathscr{L}^{\otimes k} & \text{if } d = 2k+1. \end{cases}$$

Writing the rank $n+1$ of $\mathscr{E}$ as 2ℓ or $2\ell + 1$, the tensor algebra induces $\mathscr{O}_S$-module filtrations

$$\begin{aligned} \mathscr{O}_S = \mathrm{Fil}_0 \subset \mathrm{Fil}_2 \subset \cdots \subset \mathrm{Fil}_{2\ell} \quad &= \mathcal{C}\ell_0(\mathscr{E}, q) \text{ and} \\ \mathscr{E} = \mathrm{Fil}_1 \subset \mathrm{Fil}_3 \subset \cdots \subset \mathrm{Fil}_{2\ell+1} &= \mathcal{C}\ell_1(\mathscr{E}, q), \end{aligned}$$

with associated graded pieces $\mathrm{Fil}_i / \mathrm{Fil}_{i-2} \cong \wedge^i \mathscr{E} \otimes (\mathscr{L}^{\vee})^{\otimes \lfloor i/2 \rfloor}$. In particular, for each integer d, $\mathcal{C}\ell_d(\mathscr{E}, q)$ is a locally free $\mathscr{O}_S$-module of rank 2^n.

5.2. Clifford ideals. — Functoriality of the Clifford algebra construction, verified in [BK94, Lemma 3.3] for instance, provides, for each subbundle $\mathscr{E}' \subseteq \mathscr{E}$, an inclusion of graded $\mathscr{O}_S$-algebras

$$\mathcal{C}\ell(\mathscr{E}', q') \subseteq \mathcal{C}\ell(\mathscr{E}, q),$$

where $q' \colon \mathscr{E}' \to \mathscr{L}$ is the restriction of q. In particular, if $\mathscr{W} \subset \mathscr{E}$ is an isotropic subbundle of rank r, this provides an inclusion of graded $\mathscr{O}_S$-algebras

$$\Big(\bigoplus\nolimits_{i=0}^{r} \wedge^i \mathscr{W}\Big) \otimes \Big(\bigoplus\nolimits_{k \in \mathbf{Z}} \mathscr{L}^{\otimes k}\Big) = \mathcal{C}\ell(\mathscr{W}, 0) \subset \mathcal{C}\ell(\mathscr{E}, q).$$

The *Clifford ideal* $\mathscr{I}^{\mathscr{W}} = \mathscr{I}$ associated with $\mathscr{W}$ is the graded left ideal in $C\ell(\mathscr{E},q)$ generated by the degree r summand $\det\mathscr{W}$. Its d-th graded piece $\mathscr{I}_d$ is a locally free $\mathscr{O}_S$-module of rank 2^{n-r}, as can be seen by considering the tensor algebra filtration. Since the annihilator of $\det\mathscr{W}$ in $C\ell(\mathscr{E},q)$ is the left ideal generated by $\mathscr{W}$, writing $[k]$ for shift-by-k in grading, there is a presentation

$$C\ell(\mathscr{E},q)[-r-1]\otimes\mathscr{W}\otimes\det\mathscr{W}\to C\ell(\mathscr{E},q)[-r]\otimes\det\mathscr{W}\to\mathscr{I}\to 0$$

of graded $C\ell(\mathscr{E},q)$-modules: see also [Xie23, Lemma 2.5].

5.3. Spinor sheaves. — Let $\pi\colon \mathbf{P}\mathscr{E}\to S$ be the projective bundle associated with $\mathscr{E}$ and write $\iota\colon Q\to\mathbf{P}\mathscr{E}$ for the inclusion morphism. For each $d\in\mathbf{Z}$, the *d-th spinor sheaf* $\mathscr{S}_d^{\mathscr{W}} = \mathscr{S}_d$ associated with $\mathscr{W}$ is the $\mathscr{O}_Q$-module characterized as by the short exact sequence

$$0\to\mathscr{O}_\pi(-1)\otimes\pi^*\mathscr{I}_{d-1}\xrightarrow{\phi_d}\pi^*\mathscr{I}_d\to\iota_*\mathscr{S}_d\to 0,$$

where ϕ_d is the map induced by Clifford multiplication upon viewing $\mathscr{O}_\pi(-1)$ as the tautological subbundle of $\pi^*\mathscr{E}\subset C\ell_1(\mathscr{E},q)$. The point here is that ϕ_d is part of a matrix factorization of q, that is, $\phi_d\circ\phi_{d-1} = q$, where we abuse notation and write ϕ_{d-1} for its twist by $\mathscr{O}_\pi(-1)$. This implies that the cokernel of ϕ_d is supported on Q, and gives injectivity of ϕ_d under our assumption that Q is an effective Cartier divisor in $\mathbf{P}\mathscr{E}$. Moreover, [Add11, Proposition 2.1] shows that if $\operatorname{corank}(\mathscr{W}\subset\mathscr{E}) = n+1-r\geq 2$, then the restriction of $\mathscr{S}_d$ to

- $Q\setminus(\mathbf{P}\mathscr{W}\cap\operatorname{Sing}\rho)$ is locally free of rank 2^{n-r-1}; and
- $\mathbf{P}\mathscr{W}\cap\operatorname{Sing}\rho$ coincides with $\rho^*\mathscr{I}_d$, and so is locally free of rank 2^{n-r}.

The spinor sheaves also enjoy the following properties relative to S:

5.4. Lemma. — *$\mathscr{S}_d$ is S-perfect; it is furthermore S-flat if $\rho\colon Q\to S$ is flat.*

Proof. By definition, $\mathscr{S}_d$ is S-perfect if and only if its pushforward to $\mathbf{P}\mathscr{E}$ is perfect, which follows from the defining exact sequence. If $\rho\colon Q\to S$ is flat, then $Q_s\subset\mathbf{P}\mathscr{E}_s$ is an effective Cartier divisor for each $s\in S$, so $\phi_d|_{\mathbf{P}\mathscr{E}_s}$ is injective. Then [Stacks, 046Y] shows that $\iota_*\mathscr{S}_d$ is S-flat, at which point [Stacks, 0FLM] implies that $\mathscr{S}_d$ itself is S-flat. ■

5.5. — Restricting the defining presentation of $\iota_*\mathscr{S}_d$ to Q and observing that $\phi_i\circ\phi_{i-1}$ is a matrix factorization of q provides two exact complexes of $\mathscr{O}_Q$-modules

$$\cdots\longrightarrow\mathscr{O}_\rho(-2)\otimes\rho^*\mathscr{I}_{d-2}\xrightarrow{\phi_{d-1}}\mathscr{O}_\rho(-1)\otimes\rho^*\mathscr{I}_{d-1}\xrightarrow{\phi_d}\rho^*\mathscr{I}_d\longrightarrow\mathscr{S}_d\longrightarrow 0,\text{ and}$$

$$0\longrightarrow\mathscr{S}_d\longrightarrow\mathscr{O}_\rho(1)\otimes\rho^*\mathscr{I}_{d+1}\xrightarrow{\phi_{d+2}}\mathscr{O}_\rho(2)\otimes\rho^*\mathscr{I}_{d+2}\xrightarrow{\phi_{d+3}}\cdots,$$

see [Add11, §4] and [Xie23, p.168]. In particular, this shows that

$$\mathscr{S}_d\cong\operatorname{im}(\phi_{d+1}\colon\rho^*\mathscr{I}_d\to\mathscr{O}_\rho(1)\otimes\rho^*\mathscr{I}_{d+1}).$$

By relating the dual of these complexes with the corresponding right Clifford ideals associated with $\mathscr{W}$, as is done in [Xie23, Lemma 2.7 and Remark 2.9], this shows that the derived dual of a spinor sheaf is simply a twist of another spinor sheaf; this is summarized by the following duality relations, as in [Add11, Proposition 4.1] and [Xie23, Corollary 2.11]:

$$\mathscr{S}_d^\vee\cong\begin{cases}\mathscr{S}_{r-d-1}\otimes\mathscr{O}_\rho(-1)\otimes\rho^*(\det\mathscr{W}^\vee\otimes\det\mathscr{E}^\vee\otimes\mathscr{L}^{\otimes\ell}) & \text{if }\operatorname{rank}\mathscr{E}=2\ell,\text{ and}\\ \mathscr{S}_{r-d}\quad\otimes\mathscr{O}_\rho(-1)\otimes\rho^*(\det\mathscr{W}^\vee\otimes\det\mathscr{E}^\vee\otimes\mathscr{L}^{\otimes\ell}) & \text{if }\operatorname{rank}\mathscr{E}=2\ell+1.\end{cases}$$

Finally, there is the degree shift relation $\mathscr{S}_d\otimes\rho^*\mathscr{L}\cong\mathscr{S}_{d+2}$.

The defining presentation of $\mathscr{S}_d$ makes it easy to compute its derived pushforward along $\rho\colon Q\to S$. For instance, $R\rho_*\mathscr{S}_d\cong\mathscr{I}_d$. For later use, some vanishing pushforwards are as follows:

5.6. Lemma. — *For any $d \in \mathbf{Z}$ and each integer $0 \le i \le n-2$,*

$$R\rho_*(\mathscr{S}_d \otimes \mathscr{O}_\rho(-i-1)) = R\rho_* R\mathscr{H}om_{\mathscr{O}_Q}(\mathscr{O}_\rho(i), \mathscr{S}_d^\vee) = 0.$$

Proof. Vanishing for $\mathscr{S}_d \otimes \mathscr{O}_\rho(-i-1)$ follows from the defining presentation of $\mathscr{S}_d$ and the fact that $\pi\colon \mathbf{P}\mathscr{E} \to S$ is a $\mathbf{P}^n$-bundle. Together with the duality relations from 5.5, this implies the vanishing of the $R\rho_* R\mathscr{H}om_{\mathscr{O}_Q}(\mathscr{O}_\rho(i), \mathscr{S}_d^\vee)$. ∎

To illustrate the theory and for later use, the following two paragraphs explicitly describe spinor sheaves when $\mathscr{E}$ has small even rank. In short: when $\operatorname{rank}\mathscr{E} = 2$, spinors are structure sheaves of points; whereas when $\operatorname{rank}\mathscr{E} = 4$, spinors are essentially ideal sheaves of lines.

5.7. Spinors in relative dimension 0. — Let $\rho\colon Q \to S$ be a family of quadrics of relative dimension 0, and consider a closed subscheme $i\colon \mathbf{P}\mathscr{W} \to Q$ given by an isotropic line subbundle of $\mathscr{E}$; this is the effective Cartier divisor in $\mathbf{P}\mathscr{E}$ defined by a section $s_{\mathbf{P}\mathscr{W}}\colon \mathscr{O}_\pi \to \mathscr{O}_\pi(1) \otimes \pi^*(\mathscr{E}/\mathscr{W})$. The equation of Q in $\mathbf{P}\mathscr{E}$ factors as

$$s_Q\colon \mathscr{O}_\pi(-2) \xrightarrow{s_{\mathbf{P}\mathscr{W}}} \mathscr{O}_\pi(-1) \otimes \pi^*(\mathscr{E}/\mathscr{W}) \xrightarrow{s_Z} \pi^*\mathscr{L}$$

where s_Z is the equation of the *residual subscheme* $j\colon Z \to Q$ to $\mathbf{P}\mathscr{W}$ in Q: this is an effective Cartier divisor of relative degree 1 over S away from the corank 2 locus S_2, and is otherwise the entire fibre. With this notation, the d-th spinor sheaf associated with $\mathscr{W}$ is:

$$\mathscr{S}_d \cong \begin{cases} i_*\mathscr{O}_{\mathbf{P}\mathscr{W}} \otimes \rho^*(\det\mathscr{E} \otimes \mathscr{L}^{\otimes k-1}) & \text{if } d = 2k, \text{ and} \\ j_*\mathscr{O}_Z \quad \otimes \rho^*(\mathscr{W} \otimes \mathscr{L}^{\otimes k}) & \text{if } d = 2k+1. \end{cases}$$

Proof. Since $\mathscr{W}$ is an isotropic line subbundle, the associated Clifford ideal $\mathscr{I}$ satisfies $\mathscr{I}_1 = \mathscr{W}$ and $\mathscr{I}_2 \cong (\mathscr{E}/\mathscr{W}) \otimes \mathscr{W} \cong \det\mathscr{E}$. Therefore the degree 2 spinor sheaf $\mathscr{S}_2$ is the cokernel of the map

$$\phi_2\colon \mathscr{O}_\pi(-1) \otimes \pi^*\mathscr{W} \to \pi^*\det\mathscr{E}$$

induced by Clifford multiplication. It is straightforward to see, via a local computation for instance, that this is the map $s_{\mathbf{P}\mathscr{W}}$, so $\mathscr{S}_2 \cong i_*\mathscr{O}_{\mathbf{P}\mathscr{W}} \otimes \rho^*\det\mathscr{E}$. Similarly, $\mathscr{S}_1$ is the cokernel of

$$\mathscr{O}_\pi(-1) \otimes \pi^*\big((\mathscr{E}/\mathscr{W}) \otimes \mathscr{W} \otimes \mathscr{L}^\vee\big) \to \pi^*\mathscr{W},$$

which may be identified as s_Z. Therefore $\mathscr{S}_1 \cong j_*\mathscr{O}_Z \otimes \rho^*\mathscr{W}$. The remaining degrees then follow from the degree shift relation from 5.5. ∎

5.8. Spinors in relative dimension 2. — Let $\rho\colon Q \to S$ be a quadric surface bundle. If $\mathscr{W}$ is a rank 2 isotropic subbundle of $\mathscr{E}$, then its dual spinor sheaf of degree d is

$$\mathscr{S}_d^\vee \cong \begin{cases} \mathscr{I}_{\mathbf{P}\mathscr{W}/Q} \otimes \rho^*(\det\mathscr{W}^\vee \otimes \mathscr{L}^{\otimes -k+1}) & \text{if } d = 2k, \text{ and} \\ \mathscr{H}om_{\mathscr{O}_Q}(\mathscr{I}_{\mathbf{P}\mathscr{W}/Q}, \mathscr{O}_Q) \otimes \mathscr{O}_\rho(-1) \otimes \rho^*(\det\mathscr{E}^\vee \otimes \mathscr{L}^{\otimes -k+1}) & \text{if } d = 2k+1. \end{cases}$$

Proof. It suffices to treat the case $d = 2k$ is even, as the odd case then follows from the duality relations of 5.5. Since the pushforward of $\mathscr{S}_{2k}$ along $\rho\colon Q \to S$ is the $2k$-th Clifford ideal $\mathscr{I}_{2k}$, taking the canonical line subbundle $\mathscr{L}^{\otimes k-1}$ inside $C\ell_{2k-2}(\mathscr{E}, q) \cong C\ell_0(\mathscr{E}, q) \otimes \mathscr{L}^{\otimes k-1}$ in the presentation of $\mathscr{I}_{2k}$ from 5.2, evaluating sections along ρ, and dualizing provides a canonical map

$$\sigma\colon \mathscr{S}_{2k}^\vee \to \rho^*(\det\mathscr{W}^\vee \otimes \mathscr{L}^{\otimes -k+1}).$$

It suffices to show that σ is injective and that its cokernel is locally isomorphic to the structure sheaf of $\mathbf{P}\mathscr{W}$, reducing the problem to a local one on S.

Shrinking S then reduces the problem to the case where all of $\mathscr{E}$, $\mathscr{L}$, and $\mathscr{W}$ are trivial $\mathscr{O}_S$-modules; in particular, since $\mathscr{L}$ is trivial, only the parity of the degree matters in Clifford algebra considerations. Choose global projective coordinates $(x_0 : x_1 : x_2 : x_3)$ on $\mathbf{P}\mathscr{E} \cong \mathbf{P}^3_S$ so that $\mathbf{P}\mathscr{W} = \mathrm{V}(x_0, x_1)$ and

$$Q = \left\{(x_0 : x_1 : x_2 : x_3) \in \mathbf{P}^3_S : x_0 L_0 + x_1 L_1 = 0\right\} \text{ where } L_0, L_1 \in \Gamma(\mathbf{P}^3_S, \mathscr{O}_{\mathbf{P}^3_S}(1)),$$

and such that L_1 does not contain the variable x_0. Writing e_i for the basis vector of $\mathscr{E}$ dual to the coordinate x_i, the odd and even Clifford ideals associated with $\mathscr{W}$ have bases

$$\mathscr{I}_- \cong (\mathscr{O}_S \cdot e_0 e_2 e_3) \oplus (\mathscr{O}_S \cdot e_1 e_2 e_3) \text{ and } \mathscr{I}_+ \cong (\mathscr{O}_S \cdot e_2 e_3) \oplus (\mathscr{O}_S \cdot e_0 e_1 e_2 e_3)$$

so that $\det \mathscr{W}$ includes as the first summand of the even Clifford ideal. A direct computation using the presentation from 5.3 identifies the even spinor sheaf $\mathscr{S}_+$ as the cokernel of the map

$$\begin{pmatrix} L_0 & L_1 \\ -x_1 & x_0 \end{pmatrix} : \mathscr{O}_Q(-1) \otimes \rho^* \mathscr{I}_- \longrightarrow \rho^* \mathscr{I}_+.$$

The map dual to σ is the inclusion $\rho^* \det \mathscr{W} \to \rho^* \mathscr{I}_+$ of the first summand followed by the projection $\rho^* \mathscr{I}_+ \to \mathscr{S}_+$ onto the quotient. From this, it follows that $\sigma^\vee$ vanishes precisely at points where the bottom row of the above matrix vanishes, identifying its cokernel as $j_* \mathscr{O}_{\mathbf{P}\mathscr{W}}$. The kernel of $\sigma^\vee$ may be identified as the submodule of $\mathscr{O}_Q$ spanned by local sections of the form $aL_0 + bL_1$ where $a, b \in \mathscr{O}_Q$ satisfy $-ax_1 + bx_0 = 0$. The relation implies that the kernel is both x_0- and x_1-torsion, and therefore must be zero. Thus there is a right exact sequence

$$\rho^* \det \mathscr{W} \xrightarrow{\sigma^\vee} \mathscr{S}_+ \to j_* \mathscr{O}_{\mathbf{P}\mathscr{W}} \to 0.$$

The resolutions in 5.5 imply that spinors do not have higher $\mathscr{E}xt$-sheaves, so the dual exact sequence takes the form

$$0 \to \mathscr{S}_+^\vee \xrightarrow{\sigma} \rho^* \det \mathscr{W}^\vee \to \mathscr{E}xt^1_{\mathscr{O}_Q}(j_* \mathscr{O}_{\mathbf{P}\mathscr{W}}, \mathscr{O}_Q) \to 0.$$

Grothendieck duality together with transitivity of relative dualizing sheaves along the inclusions $\mathbf{P}\mathscr{W} \to Q \to \mathbf{P}\mathscr{E}$ implies that the last term is $j_* \mathscr{O}_{\mathbf{P}\mathscr{W}}$, from which the conclusion follows. ■

5.9. Dependence on the subbundle. — The *special Clifford group scheme* $\mathbf{S\Gamma}(\mathscr{E}, q)$ of the quadratic form $q : \mathscr{E} \to \mathscr{L}$ is the group scheme over S whose T-points are

$$\mathbf{S\Gamma}(\mathscr{E}, q)(T) := \{u \in C\ell_0(\mathscr{E}_T, q_T)^\times : u \cdot \mathscr{E}_T \cdot u^{-1} \subseteq \mathscr{E}_T\},$$

consisting of units in the base change $C\ell_0(\mathscr{E}_T, q_T)$ of the 0-th Clifford algebra to T such that conjugation on $C\ell_1(\mathscr{E}_T, q_T)$ preserves the subbundle $\mathscr{E}_T$. This preserves the quadratic form and fits into an exact sequence of group schemes over S

$$1 \to \mathbf{G}_m \to \mathbf{S\Gamma}(\mathscr{E}, q) \to \mathbf{SO}(\mathscr{E}, q)$$

which is furthermore exact on the right when $q : \mathscr{E} \to \mathscr{L}$ is regular: see [Knu91, IV.8.2.3].

Suppose that the conjugation action of a section u of $\mathbf{S\Gamma}(\mathscr{E}, q)$ takes an isotropic subbundle $\mathscr{W} \subseteq \mathscr{E}$ to $\mathscr{V}$. Letting the special Clifford group act on the right,

$$\mathscr{I}^{\mathscr{W}} \cdot u^{-1} = C\ell(\mathscr{E}, q) \cdot (u \cdot \det \mathscr{W} \cdot u^{-1}) = C\ell(\mathscr{E}, q) \cdot \det \mathscr{V} = \mathscr{I}^{\mathscr{V}}.$$

Thus the action of u provides an isomorphism $\mathscr{I}^{\mathscr{W}} \cong \mathscr{I}^{\mathscr{V}}$ of $C\ell(\mathscr{E}, q)$-modules, whence an isomorphism $\mathscr{S}^{\mathscr{W}} \cong \mathscr{S}^{\mathscr{V}}$ of the corresponding spinor sheaves.

As an application of this principle which will be useful in §7, the following shows that, over an algebraically closed field, once the quadric is at least 4-dimensional and its singular locus is at most 1-dimensional, then a given spinor sheaf is isomorphic to the spinor sheaf associated with a linear space containing any given smooth point:

5.10. Lemma. — *Let $Q \subset \mathbf{P}V$ be a quadric of dimension $n-1 \geq 4$ and corank ≤ 2 over an algebraically closed field $\mathbf{k}$. Write n as $2\ell+1$ or $2\ell+2$. Given any smooth point $x \in Q$ and any ℓ-plane $\mathbf{P}W \subset Q$, there exists an ℓ-plane $\mathbf{P}W' \subset Q$ containing x whose associated spinor sheaf is isomorphic to that of $\mathbf{P}W$.*

Proof. If $x \in \mathbf{P}W$, then there is nothing to show. So suppose that $x \notin \mathbf{P}W$. Then it suffices to show that there is $x' \in \mathbf{P}W$ which is a smooth point of Q in the orbit of x under the special Clifford group: If $g \cdot x = x'$ for $g \in \mathbf{S\Gamma}(V,q)$, then $\mathbf{P}W' := g^{-1} \cdot \mathbf{P}W \subset Q$ is an ℓ-plane containing x whose spinor sheaf is isomorphic to that of $\mathbf{P}W$ by 5.9. Construct x' in three steps:

First, there exists a point $x' \in \mathbf{P}W$ which is a smooth point of Q and such that the line in $\mathbf{P}V$ spanned by x and x' is disjoint from the singular locus of Q: Writing $v \in V$ for a basis vector of the linear space underlying x, this means there exists $v' \in W$ not contained in $\mathbf{k}\cdot v + \mathrm{rad}\, q$; in other words, it suffices to show that

$$(\mathbf{k}\cdot v + \mathrm{rad}\, q) \cap W \subsetneq W.$$

Since $\dim_{\mathbf{k}}(\mathbf{k}\cdot v + \mathrm{rad}\, q) \leq 3$ and $\dim_{\mathbf{k}} W = \ell + 1 \geq 3$, the only way equality can occur is if $\mathbf{k}\cdot v + \mathrm{rad}\, q = W$. This is impossible since $v \notin W$. Now fix any such $x' = \mathbf{P}v' \in \mathbf{P}W$.

Second, there is a 4-dimensional subspace $U \subset V$ such that $q|_U$ is non-degenerate and $v, v' \in U$. Set $U_1 := \mathbf{k}\cdot v \oplus \mathbf{k}\cdot v' \subset V$. There are then two cases depending on the pairing between v and v': If $b_q(v,v') = 0$, then U_1 is isotropic and intersects $\mathrm{rad}\, q$ trivially, so by the theory of hyperbolic planes, there is a subspace $U_2 \subset V$ mapped isomorphically to $U_1^\vee$ by b_q. If $b_q(v,v') \neq 0$, then U_1 is itself a hyperbolic plane, so we have a splitting $V = U_1 \oplus U_1^\perp$. The orthogonal $U_1^\perp$ has dimension $n-1 \geq 4$ and carries a quadratic form of corank ≤ 2, so it has an isotropic vector not contained in its radical. This isotropic vector can be completed to a second hyperbolic plane U_2. In either case, $U := U_1 \oplus U_2 \subset V$ is a hyperbolic subspace of dimension 4 containing v and v', as desired.

Finally, to conclude, since $q|_U$ is non-degenerate, there is an orthogonal decomposition $V = U \oplus U^\perp$ and a corresponding inclusion of special Clifford groups

$$\mathbf{S\Gamma}(U, q|_U) \times_{\mathbf{k}} \mathbf{S\Gamma}(U^\perp, q|_{U^\perp}) \subset \mathbf{S\Gamma}(V,q).$$

Since U is nonsingular, the morphism $\mathbf{S\Gamma}(U,q|_U) \to \mathbf{SO}(U,q|_U)$ is surjective, as in 5.9. Since $\dim_{\mathbf{k}} U \geq 2$, the special orthogonal group acts transitively on its isotropic lines; this implies that there exists an element $g' \in \mathbf{S\Gamma}(U,q|_U)$ taking $\mathbf{k}\cdot v$ to $\mathbf{k}\cdot v'$. Then the element $g := (g',1) \in \mathbf{S\Gamma}(V,q)$ has the sought-after properties. ∎

5.11. Hyperplane sections. — The next few paragraphs describe the relationship between spinor sheaves on a quadric and a hyperplane section when the hyperplane contains the defining subspace. Let $\mathcal{E}' \subset \mathcal{E}$ be a corank 1 subbundle and let $q' : \mathcal{E}' \to \mathcal{L}$ be the restriction of q thereon. The Clifford algebra of $\mathcal{E}'$ is then a graded subalgebra of the Clifford algebra of $\mathcal{E}$, and the quotient can be identified as follows: Writing $\mathcal{E}'' := \mathcal{E}/\mathcal{E}'$ for the quotient line bundle and $[-1]$ for shift of grading, there is a short exact sequence of graded $C\ell(\mathcal{E}',q')$-modules:

$$0 \to C\ell(\mathcal{E}',q') \to C\ell(\mathcal{E},q) \to \mathcal{E}'' \otimes_{\mathcal{O}_S} C\ell(\mathcal{E}',q')[-1] \to 0.$$

Proof. Clifford multiplication gives a map $\mathcal{E} \otimes_{\mathcal{O}_S} C\ell(\mathcal{E}',q')[-1] \to C\ell(\mathcal{E},q)/C\ell(\mathcal{E}',q')$ which vanishes on $\mathcal{E}' \otimes_{\mathcal{O}_S} C\ell(\mathcal{E}',q')[-1]$ and therefore factors through a map

$$\mathcal{E}'' \otimes_{\mathcal{O}_S} C\ell(\mathcal{E}',q')[-1] \to C\ell(\mathcal{E},q)/C\ell(\mathcal{E}',q').$$

The degree d part of each side is a locally free $\mathcal{O}_S$-module of rank 2^{n-1} and easy to see that a local basis on the left is sent to one on the right, so the map is an isomorphism. ∎

More generally, $\mathscr{W} \subset \mathscr{E}'$ be an isotropic subbundle, and write $\mathscr{I}$ and $\mathscr{I}'$ for the associated Clifford ideals in $C\ell(\mathscr{E},q)$ and $C\ell(\mathscr{E}',q')$, respectively. Since both ideals are generated by $\det\mathscr{W}$, $\mathscr{I}'$ may be identified as a sub-$C\ell(\mathscr{E}',q')$-module of $\mathscr{I}$. The argument above generalizes to yield:

5.12. Lemma. — *In the setting of* **5.11**, *inclusion induces an exact sequence graded of $C\ell(\mathscr{E}',q')$-modules*

$$0 \to \mathscr{I}' \to \mathscr{I} \to \mathscr{E}'' \otimes_{\mathscr{O}_S} \mathscr{I}'[-1] \to 0. \qquad \blacksquare$$

The sequence of Clifford ideals induces a corresponding sequence of spinor sheaves. To set notation, write $\pi' \colon \mathbf{P}\mathscr{E}' \to S$ and $\rho' \colon Q' \to S$ for the projective and quadric bundles associated with $q' \colon \mathscr{E}' \to \mathscr{L}$, and let $\mathscr{S}$ and $\mathscr{S}'$ be the spinor sheaves corresponding to $\mathscr{W}$ on Q and Q', respectively. Then:

5.13. Lemma. — *In the setting of* **5.11**, *there is an exact sequence*

$$0 \to \mathscr{S}'_d \to \mathscr{S}_d|_{Q'} \to \mathscr{E}'' \otimes \mathscr{S}'_{d-1} \to 0.$$

Proof. This can be deduced from **5.12** via the following general construction: For any quasi-coherent graded $C\ell(\mathscr{E}',q')$-module $\mathscr{M}$, Clifford multiplication as in **5.3** provides a short exact sequence

$$0 \to \mathscr{O}_{\pi'}(-1) \otimes \pi'^*\mathscr{M}_{-1} \to \pi'^*\mathscr{M}_0 \to \iota_*\mathscr{F} \to 0$$

where $\mathscr{F}$ is a quasi-coherent sheaf on Q' and $\iota \colon Q' \to \mathbf{P}\mathscr{E}'$ is the inclusion, and the assignment $\mathscr{M} \mapsto \mathscr{F}$ determines an exact functor from the category of quasi-coherent graded $C\ell(\mathscr{E}',q')$-modules to the category of quasi-coherent modules on Q'. $\blacksquare$

Going the other direction, the spinor sheaf $\mathscr{S}$ on Q is related to the Clifford ideal $\rho^*\mathscr{I}'$ via modification by $\mathscr{S}'$ along the closed subscheme $j' \colon Q' \to Q$. This relation comes from considering the quotient of the Clifford ideal $\rho^*\mathscr{I}_d$ by the span of the Clifford ideal $\rho^*\mathscr{I}'_d$ for the hyperplane section, and the twisted spinor sheaf $\mathscr{O}_\rho(-1) \otimes \mathscr{S}_{d-1}$ identified as the image of the Clifford multiplication map ϕ_d from the first complex in **5.5**:

5.14. Proposition. — *In the setting of* **5.11**, *assume additionally that $j' \colon Q' \to Q$ is the inclusion of an effective Cartier divisor. Then $\rho^*\mathscr{I}'_d \cap \mathscr{O}_\rho(-1) \otimes \mathscr{S}_{d-1} = 0$ as submodules of $\rho^*\mathscr{I}_d$ and*

$$\rho^*\mathscr{I}_d/(\rho^*\mathscr{I}'_d \oplus \mathscr{O}_\rho(-1) \otimes \mathscr{S}_{d-1}) \cong \rho^*\mathscr{E}'' \otimes j'_*\mathscr{S}_{d-1}.$$

In particular, there is a short exact sequence

$$0 \to \mathscr{O}_\rho(-1) \otimes \mathscr{S}_{d-1} \to \rho^*(\mathscr{E}'' \otimes \mathscr{I}'_{d-1}) \to \rho^*\mathscr{E}'' \otimes j'_*\mathscr{S}'_{d-1} \to 0.$$

Proof. We first show that $\mathscr{O}_\rho(-1) \otimes \mathscr{S}_{d-1}$, viewed as the image of $\phi_d \colon \mathscr{O}_\rho(-1) \otimes \rho^*\mathscr{I}_{d-1} \to \rho^*\mathscr{I}_d$ as above, intersects $\rho^*\mathscr{I}'_d$ trivially away on $Q \setminus Q'$. There, the inclusion $\mathscr{E}' \subset \mathscr{E}$ is split by the composite $\mathscr{O}_\rho(-1) \hookrightarrow \rho^*\mathscr{E} \to \rho^*\mathscr{E}''$, and this induces a corresponding splitting of the sequence from **5.12**:

$$\rho^*\mathscr{I}|_{Q\setminus Q'} \cong \rho^*\mathscr{I}'|_{Q\setminus Q'} \oplus \mathscr{O}_\rho(-1) \otimes \rho^*\mathscr{I}'[-1]|_{Q\setminus Q'}.$$

With respect to this splitting, $\phi_d|_{Q\setminus Q'}$ may be identified as the map

$$(\varphi_1,\varphi_2) \colon \mathscr{O}_\rho(-1) \otimes \rho^*\mathscr{I}'_{d-1}|_{Q\setminus Q'} \oplus \mathscr{O}_\rho(-2) \otimes \rho^*\mathscr{I}'_{d-2}|_{Q\setminus Q'} \longrightarrow \rho^*\mathscr{I}_d|_{Q\setminus Q'}$$

where φ_i is induced by i-fold Clifford multiplication. Now $\varphi_2 = 0$ since it is multiplication by the equation of Q as in **5.3**, and φ_1 is the inclusion of a direct complement of $\rho^*\mathscr{I}'_d|_{Q\setminus Q'}$ since any local section contains a local section in $\mathscr{E} \setminus \mathscr{E}'$. This shows that $\rho^*\mathscr{I}_d \cap \mathscr{O}_\rho(-1) \otimes \mathscr{S}_{d-1}$ is zero upon restriction to $Q \setminus Q'$. Since this is a submodule of a locally free $\mathscr{O}_Q$-module, and the complement $Q \setminus Q'$ of an effective Cartier divisor is scheme-theoretically dense in Q, in fact $\rho^*\mathscr{I}_d \cap \mathscr{O}_\rho(-1) \otimes \mathscr{S}_{d-1} = 0$.

The quotient of $\rho^*\mathcal{I}_d$ by the span of these two submodules may be identified as the cokernel of

$$\psi\colon \mathcal{O}_\rho(-1)\otimes\mathcal{S}_{d-1}\to\rho^*\mathcal{I}_d\to\rho^*\mathcal{I}_d/\rho^*\mathcal{I}'_d\cong\rho^*(\mathcal{E}''\otimes\mathcal{I}'_{d-1})$$

where the rightmost identification is from 5.12. We claim it is scheme-theoretically supported on Q'. This is a statement local on S, so assume that $\mathcal{L}\cong\mathcal{O}_S$ and that

$$\mathcal{E}\cong\mathcal{O}_S\cdot e_0\oplus\mathcal{E}'\cong\big(\bigoplus\nolimits_{i=0}^{n-r-1}\mathcal{O}_S\cdot e_i\big)\oplus\mathcal{W}\cong\bigoplus\nolimits_{i=0}^{n}\mathcal{O}_S\cdot e_i.$$

With these trivializations, $\rho^*(\mathcal{E}''\otimes\mathcal{I}'_{d-1})\cong e_0\cdot\rho^*C\ell_{d-r-1}(\mathcal{E}',q')\cdot\det\mathcal{W}$ and, writing x_i for the coordinates of $\mathbf{P}\mathcal{E}\cong\mathbf{P}^n$ dual to the basis e_i, the image of ψ may be identified with the submodule

$$\big(x_0e_0\cdot\rho^*C\ell_{d-r-1}(\mathcal{E}',q')\cdot\det\mathcal{W}\big)+\big(e_0\cdot(x_1e_1+\cdots+x_ne_n)\cdot\rho^*C\ell_{d-r-2}(\mathcal{E}',q')\cdot\det\mathcal{W}\big).$$

The first summand shows that $\operatorname{coker}\psi$ is supported on the vanishing locus $Q'=Q\cap\mathbf{P}\mathcal{E}'$ of x_0.

Writing $\operatorname{coker}\psi\cong j'_*\mathcal{F}$, it remains to show that $\mathcal{F}\cong\rho'^*\mathcal{E}''\otimes\mathcal{S}'_{d-1}$. We have $\mathcal{F}=\operatorname{coker}j^*\psi=\operatorname{coker}\alpha$ where α is as the diagonal map in

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \rho'^*\mathcal{I}'_d & \longrightarrow & \rho'^*\mathcal{I}_d & \longrightarrow & \rho'^*(\mathcal{E}''\otimes\mathcal{I}'_{d-1}) & \longrightarrow & 0\\
 & & \uparrow{\scriptstyle\phi'_d} & & \uparrow{\scriptstyle\phi_d|_{Q'}} & \nearrow{\scriptstyle\alpha} & \uparrow{\scriptstyle\phi'_{d-1}} & & \\
0 & \longrightarrow & \mathcal{O}_{\rho'}(-1)\otimes\rho'^*\mathcal{S}'_{d-1} & \longrightarrow & \mathcal{O}_{\rho'}(-1)\otimes\rho'^*\mathcal{S}_{d-1} & \longrightarrow & \mathcal{O}_{\rho'}(-1)\otimes\rho'^*(\mathcal{E}''\otimes\mathcal{S}'_{d-2}) & \longrightarrow & 0
\end{array}$$

where the rows are exact and the squares are commutative by 5.12. We see that the cokernel of α is equal to the cokernel of ϕ'_{d-1} which is equal to $\rho^*\mathcal{E}''\otimes j'\mathcal{S}'_{d-1}$. ■

5.15. Cones. — The final few statements in this section describe the behaviour of spinor sheaves along cones over quadrics. Let $\mathcal{K}\subset\mathcal{E}$ be a rank c subbundle contained in the radical of q, set $\bar{\mathcal{E}}:=\mathcal{E}/\mathcal{K}$, and let $\bar{q}\colon\bar{\mathcal{E}}\to\mathcal{L}$ be the induced quadratic form. The quotient map $\mathcal{E}\to\bar{\mathcal{E}}$ induces a short exact sequence of graded $C\ell(\mathcal{E},q)$-modules

$$0\to\mathcal{J}\to C\ell(\mathcal{E},q)\to C\ell(\bar{\mathcal{E}},\bar{q})\to 0$$

where the kernel is the sheaf of ideals generated by $\mathcal{K}$. Since $\mathcal{K}$ lies in the radical of q, it is central in $C\ell(\mathcal{E},q)$, and so $\mathcal{J}$ is a two-sided ideal. The quotient map endows each $C\ell(\bar{\mathcal{E}},\bar{q})$-module with the structure of a $C\ell(\mathcal{E},q)$-module.

Let $\mathcal{W}\subset\mathcal{E}$ be an isotropic subbundle of rank r containing $\mathcal{K}$, and let $\overline{\mathcal{W}}\subset\bar{\mathcal{E}}$ be the corresponding isotropic bundle in the quotient. The Clifford ideals $\mathcal{I}$ and $\bar{\mathcal{I}}$ associated with $\mathcal{W}$ and $\overline{\mathcal{W}}$, respectively, are related as follows:

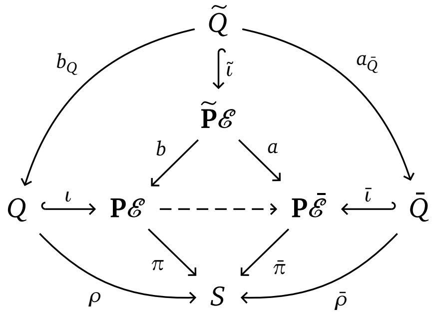
5.16. Lemma. — *$\bar{\mathcal{I}}_d\cong\mathcal{I}_{d+c}\otimes_{\mathcal{O}_S}\det\mathcal{K}^\vee$ as $C\ell_0(\mathcal{E},q)$-modules for each $d\in\mathbf{Z}$.*

Proof. The annihilator in $C\ell(\mathcal{E},q)$ of the generator $\det\overline{\mathcal{W}}$ of $\bar{\mathcal{I}}$ is the preimage of the sheaf of left ideals in $C\ell(\bar{\mathcal{E}},\bar{q})$ generated by $\overline{\mathcal{W}}$. Since $\overline{\mathcal{W}}=\mathcal{W}/\mathcal{K}$, and $\mathcal{K}$ is central in $C\ell(\mathcal{E},q)$, this is the sheaf of left ideals in $C\ell(\mathcal{E},q)$ generated by $\mathcal{W}$. So Clifford multiplication gives rise to a presentation

$$C\ell_{d-r+c-1}(\mathcal{E},q)\otimes\mathcal{W}\otimes\det\overline{\mathcal{W}}\to C\ell_{d-r+c}(\mathcal{E},q)\otimes\det\overline{\mathcal{W}}\to\bar{\mathcal{I}}_d\to 0.$$

Since $\det\overline{\mathcal{W}}\cong\det\mathcal{W}\otimes\det\mathcal{K}^\vee$, this is also the presentation for $\mathcal{I}_{d+c}\otimes\det\mathcal{K}^\vee$. ■

As usual, linear projection centred along $\mathbf{P}\mathscr{K}$ results in a commutative diagram



where $b\colon \widetilde{\mathbf{P}}\mathscr{E}$ and $b_Q\colon \widetilde{Q} \to Q$ are the blowups of $\mathbf{P}\mathscr{E}$ and Q along $\mathbf{P}\mathscr{K}$, respectively, and $a\colon \widetilde{\mathbf{P}}\mathscr{E} \to \mathbf{P}\bar{\mathscr{E}}$ is the morphism resolving linear projection. The blowup $\widetilde{Q}$ may be identified as the restriction of linear projection $a\colon \widetilde{\mathbf{P}}\mathscr{E} \to \mathbf{P}\bar{\mathscr{E}}$ over the base of the cone $\bar{Q}$; in other words, the top right square in the diagram is Cartesian. The spinor sheaves $\mathscr{S}$ and $\bar{\mathscr{S}}$ associated with $\mathscr{W}$ and $\overline{\mathscr{W}}$, respectively, are related as follows:

5.17. Proposition. — *$Rb_{Q,*}La^*_{\bar{Q}}\bar{\mathscr{S}}_d \cong \mathscr{S}_{d+c} \otimes \rho^* \det \mathscr{K}^\vee$ in $\mathrm{D}_{\mathrm{qc}}(Q)$ for each $d \in \mathbf{Z}$.*

Proof. View the defining presentation of $\bar{\mathscr{S}}_d$ on $\mathbf{P}\bar{\mathscr{E}}$ as an exact triangle in $\mathrm{D}_{\mathrm{qc}}(\mathbf{P}\bar{\mathscr{E}})$ of the form

$$\mathscr{O}_{\bar{\pi}}(-1) \otimes \bar{\pi}^* \bar{\mathscr{I}}_{d-1} \longrightarrow \bar{\pi}^* \bar{\mathscr{I}}_d \longrightarrow R\bar{\iota}_* \bar{\mathscr{S}}_d \xrightarrow{+1}$$

Apply Rb_*La^* to obtain a triangle in $\mathrm{D}_{\mathrm{qc}}(\mathbf{P}\mathscr{E})$, the first two terms of which are easy to identify: Commutativity of the diagram, the projection formula, and **5.16** together give the first line of

$$\begin{aligned} Rb_*La^*(\mathscr{O}_{\bar{\pi}}(-i) \otimes \bar{\pi}^* \bar{\mathscr{I}}_{d-i}) &\cong Rb_*a^*\mathscr{O}_{\bar{\pi}}(-i) \otimes \pi^*(\mathscr{I}_{d+c-i} \otimes \det \mathscr{K}^\vee) \\ &\cong \mathscr{O}_{\pi}(-i) \otimes \pi^*(\mathscr{I}_{d+c-i} \otimes \det \mathscr{K}^\vee) \text{ for } i = 0, 1. \end{aligned}$$

For the second line, note that $a^*\mathscr{O}_{\bar{\pi}}(-1) \cong \mathscr{O}_{\widetilde{\mathbf{P}}\mathscr{E}}(E) \otimes b^*\mathscr{O}_{\pi}(-1)$ where $E \subset \widetilde{\mathbf{P}}\mathscr{E}$ is the exceptional divisor, and so $Rb_*a^*\mathscr{O}_{\bar{\pi}}(-i) \cong \mathscr{O}_{\pi}(-i)$ for $i = 0, 1$. As for the third term in the new triangle, observe that $a\colon \widetilde{\mathbf{P}}\mathscr{E} \to \mathbf{P}\bar{\mathscr{E}}$ is a projective bundle, so it is tor independent with $\bar{\iota}\colon \bar{Q} \to \mathbf{P}\bar{\mathscr{E}}$, and thus $La^* \circ R\bar{\iota}_* = R\tilde{\iota} \circ La^*_{\bar{Q}}$ as functors $\mathrm{D}_{\mathrm{qc}}(\bar{Q}) \to \mathrm{D}_{\mathrm{qc}}(\widetilde{\mathbf{P}}\mathscr{E})$ by [Stacks, 0E23]. Commutativity of the top left square above then shows that $Rb_* \circ R\tilde{\iota} = R\iota_* \circ Rb_{Q,*}$. Therefore the new triangle is

$$\mathscr{O}_{\pi}(-1) \otimes \pi^*(\mathscr{I}_{d+c-1} \otimes \det \mathscr{K}^\vee) \longrightarrow \pi^*(\mathscr{I}_{d+c} \otimes \det \mathscr{K}^\vee) \longrightarrow R\iota_*Rb_{Q,*}La^*_{\bar{Q}}\bar{\mathscr{S}}_d \xrightarrow{+1}$$

Comparing with the defining sequence for $\mathscr{S}_{d+c}$ now shows that

$$R\iota_*Rb_{Q,*}L^*_{\bar{Q}}\bar{\mathscr{S}}_d \cong R\iota_*(\mathscr{S}_{d+c} \otimes \rho^* \det \mathscr{K}^\vee) \in \mathrm{D}_{\mathrm{qc}}(\mathbf{P}\mathscr{E}).$$

Since the closed immersion $\iota\colon Q \to \mathbf{P}\mathscr{E}$ is, in particular, affine, this at least shows that $Rb_{Q,*}L^*_{\bar{Q}}\bar{\mathscr{S}}_d$ is a complex concentrated in degree 0. Using that ι_* is fully faithful on quasi-coherent modules, see [Stacks, 01QY], the sheaf in degree 0 is identified with $\mathscr{S}_{d+c} \otimes \rho^* \deg \mathscr{K}^\vee$, whence the result. ■

6. Kuznetsov components under hyperbolic reduction

Returning to the setting of **4.14**, suppose the quadric bundle $\rho\colon Q \to S$ admits a regular section corresponding to a rank 1 subbundle $\mathscr{F} \subset \mathscr{E}$, and let $\rho'\colon Q' \to S$ be the associated hyperbolic reduction. Assume throughout that S_{n-1} does not contain any weakly associated points of S, so that Q' is an effective Cartier divisor in $\mathbf{P}(\mathscr{F}^\perp/\mathscr{F})$ by **4.19**. The aim in the first half of this section is to relate the Kuznetsov components of Q and Q'. Specifically, a mutations argument gives in **6.1** an S-linear equivalence $\Phi\colon \mathrm{Ku}(Q') \to \mathrm{Ku}(Q)$. We are then able to explicitly identify the kernel underlying Φ in **6.2**, with which we show in **6.3** that Φ sends the dual spinor sheaves of Q' to those of Q.

To begin, continue with the notation from the diagram in 4.15. By its construction together with 4.18, the scheme $\widetilde{\rho}\colon \widetilde{Q} \to S$ is a blowup in two ways, and fits into a commutative diagram

$$\begin{array}{ccccccc}
E & \xhookrightarrow{\tilde{\imath}} & \widetilde{Q} & & \xhookleftarrow{\tilde{\jmath}} & & \Lambda \\
{\scriptstyle b_E}\downarrow & & \swarrow{\scriptstyle b} & \searrow{\scriptstyle a} & & & \downarrow{\scriptstyle a_\Lambda} \\
\mathbf{P}\mathscr{F} & \hookrightarrow & Q & & \mathbf{P}\mathscr{E}' & \xhookleftarrow{j'} & Q'
\end{array}$$

of schemes over S. Orlov's blowup formula from [Orl92, Theorem 4.3], see also [Kuz14, Theorem 1.6], therefore gives two semiorthogonal decompositions of $\mathrm{D}_{\mathrm{qc}}(\widetilde{Q})$: On the one hand, viewing $\widetilde{Q}$ as a blowup of Q along $\mathbf{P}\mathscr{F}$ yields a S-linear semiorthogonal decomposition of $\mathrm{D}_{\mathrm{qc}}(\widetilde{Q})$ of the form

$$\text{(B)}\quad \big\langle Lb^*\mathrm{Ku}(Q),\, L\widetilde{\rho}^*\mathrm{D}_{\mathrm{qc}}(S)\otimes^L \mathscr{O}_{\widetilde{Q}},\, \ldots,\, L\widetilde{\rho}^*\mathrm{D}_{\mathrm{qc}}(S)\otimes^L \mathscr{O}_{\widetilde{Q}}((n-2)H),$$
$$L\widetilde{\rho}^*\mathrm{D}_{\mathrm{qc}}(S)\otimes^L R\tilde{\imath}_*\mathscr{O}_E,\, \ldots,\, L\widetilde{\rho}^*\mathrm{D}_{\mathrm{qc}}(S)\otimes^L R\tilde{\imath}_*\mathscr{O}_E(-(n-3)E)\big\rangle$$

where $\mathscr{O}_{\widetilde{Q}}(H) := b^*\mathscr{O}_\rho(1)$. On the other hand, viewing $\widetilde{Q}$ as a blowup of $\mathbf{P}\mathscr{E}'$ along Q' and writing $\mathscr{O}_{\widetilde{Q}}(h) := a^*\mathscr{O}_{\rho'}(1)$ yields the S-linear semiorthogonal decomposition

$$\text{(A)}\quad \big\langle R\tilde{\jmath}_*(La_\Lambda^*\mathrm{Ku}(Q')\otimes^L \mathscr{O}_\Lambda(\Lambda)),\, L\widetilde{\rho}^*\mathrm{D}_{\mathrm{qc}}(S)\otimes^L R\tilde{\jmath}_*\mathscr{O}_\Lambda(\Lambda),\, \ldots,\, L\widetilde{\rho}^*\mathrm{D}_{\mathrm{qc}}(S)\otimes^L R\tilde{\jmath}_*\mathscr{O}_\Lambda(\Lambda+(n-4)h),$$
$$L\widetilde{\rho}^*\mathrm{D}_{\mathrm{qc}}(S)\otimes^L \mathscr{O}_{\widetilde{Q}},\, \ldots,\, L\widetilde{\rho}^*\mathrm{D}_{\mathrm{qc}}(S)\otimes^L \mathscr{O}_{\widetilde{Q}}((n-1)h)\big\rangle.$$

The following matches the two Kuznetsov components via a series of mutations:

6.1. Proposition. — *As S-linear subcategories of $\mathrm{D}_{\mathrm{qc}}(\widetilde{Q})$,*

$$Lb^*\mathrm{Ku}(Q) = \mathbf{L}_{L\widetilde{\rho}^*\mathrm{D}_{\mathrm{qc}}(S)\otimes^L\mathscr{O}_{\widetilde{Q}}(-E)}\big(R\tilde{\jmath}_*(La_\Lambda^*\mathrm{Ku}(Q')\otimes^L \mathscr{O}_\Lambda(\Lambda))\big).$$

This determines an S-linear equivalence of categories $\Phi\colon \mathrm{Ku}(Q') \to \mathrm{Ku}(Q)$.

Compare the proof below to the proof of [Xie23, Theorem 4.2].

Proof. Starting with the semiorthogonal decomposition (B), we perform a series of mutations, which are justified by 3.4, to arrive at a decomposition which may be compared with (A). The overall strategy is to carefully move the components supported on E completely toward the left, so that they end up adjacent to the Kuznetsov component of Q. In what follows, all functors are derived, so we suppress the L and R for left and right derived functors. Moreover, all constructions are linear over S, so to ease the notation, we suppress the tensor over $L\widetilde{\rho}^*\mathrm{D}_{\mathrm{qc}}(S)$ and simply denote by an exceptional object the corresponding S-linear subcategory it generates.

Step 1. Starting from $\tilde{\imath}_*\mathscr{O}_E$ and moving right, for each $k = 0, \ldots, n-3$, move $\tilde{\imath}_*\mathscr{O}_E(-kE)$ to the left via $n-2-k$ right mutations. Each mutation is of the form

$$\langle \mathscr{O}_{\widetilde{Q}}(mH-kE), \tilde{\imath}_*\mathscr{O}_E(-kE)\rangle = \langle \tilde{\imath}_*\mathscr{O}_E(-kE), \mathbf{R}_{\tilde{\imath}_*\mathscr{O}_E(-kE)}(\mathscr{O}_{\widetilde{Q}}(mH-(k+1)E))\rangle.$$

To determine the right mutation, note $\mathscr{O}_{\widetilde{Q}}(H)$ restricts trivially to E because $\widetilde{Q} \to Q$ is a blowup along a section, so that

$$\rho_*b_*\mathscr{H}om_{\mathscr{O}_{\widetilde{Q}}}(\mathscr{O}_{\widetilde{Q}}(mH-kE), \tilde{\imath}_*\mathscr{O}_E(-kE)) \cong \rho_*b_*\tilde{\imath}_*\mathscr{O}_E \cong \mathscr{O}_S,$$

generated by the equation of E in $\widetilde{Q}$. Therefore the right mutation of $\mathscr{O}_{\widetilde{Q}}(mH-kE)$, which is the cone of the coevaluation map above up to a twist, is

$$\mathrm{cone}(\mathscr{O}_{\widetilde{Q}} \to \tilde{\imath}_*\mathscr{O}_E)\otimes \mathscr{O}_{\widetilde{Q}}(mH-kE) \cong \mathscr{O}_{\widetilde{Q}}(mH-(k+1)E).$$

This series of right mutations produces a semiorthogonal decomposition

$$\langle b^*\mathrm{Ku}(Q), \mathscr{A}_0, \ldots, \mathscr{A}_{n-3}, \mathscr{O}_{\widetilde{Q}}((n-2)h)\rangle$$

where $\mathscr{A}_k = \langle \mathcal{O}_{\widetilde{Q}}(kH-kE), \tilde{\imath}_*\mathcal{O}_E(-kE)\rangle$ for each $k = 0,\dots,n-3$.

Step 2. Within each of the blocks $\mathscr{A}_k$, left mutate $\tilde{\imath}_*\mathcal{O}_E(-kE)$ through $\mathcal{O}_{\widetilde{Q}}(kH-kE)$. The computation from step 1 implies that this produces decompositions

$$\mathscr{A}_k = \langle \mathcal{O}_{\widetilde{Q}}(kH-(k+1)E), \mathcal{O}_{\widetilde{Q}}(kH-kE)\rangle = \langle \mathcal{O}_{\widetilde{Q}}(\Lambda+(k-1)h), \mathcal{O}_{\widetilde{Q}}(kh)\rangle,$$

for $k = 0,\dots,n-3$, where the second equality arises from the relations

$$\mathcal{O}_{\widetilde{Q}}(H-E) \cong \mathcal{O}_{\widetilde{Q}}(h) \text{ and } \mathcal{O}_{\widetilde{Q}}(H-2E) \cong \mathcal{O}_{\widetilde{Q}}(\Lambda) \otimes \tilde{\rho}^*(\mathscr{F}\otimes\mathscr{L}^\vee)$$

from 4.16(iii) and 4.18, respectively, and the fact that twisting by $\tilde{\rho}^*(\mathscr{F}\otimes\mathscr{L}^\vee)$ does not affect the S-linear subcategory an object generates. Regroup the resulting decomposition of $\mathrm{D}_{\mathrm{qc}}(\widetilde{Q})$ as follows:

$$\langle b^*\mathrm{Ku}(Q), \mathcal{O}_{\widetilde{Q}}(-E), \mathscr{B}_0, \dots, \mathscr{B}_{n-4}, \mathcal{O}_{\widetilde{Q}}((n-3)h), \mathcal{O}_{\widetilde{Q}}((n-2)h)\rangle$$

where $\mathscr{B}_k := \langle \mathcal{O}_{\widetilde{Q}}(kh), \mathcal{O}_{\widetilde{Q}}(\Lambda+kh)\rangle$ for $k = 0,\dots,n-4$.

Step 3. Within each of the blocks $\mathscr{B}_k$, left mutate $\mathcal{O}_{\widetilde{Q}}(\Lambda+kh)$ through $\mathcal{O}_{\widetilde{Q}}(kh)$. Since Λ is the exceptional divisor of the blowup a,

$$\tilde{\rho}_*\mathscr{H}\!om_{\mathcal{O}_{\widetilde{Q}}}(\mathcal{O}_{\widetilde{Q}}(kh), \mathcal{O}_{\widetilde{Q}}(\Lambda+kh)) \cong \tilde{\rho}_*\mathcal{O}_{\widetilde{Q}}(\Lambda) \cong \rho'_*a_*\mathcal{O}_{\widetilde{Q}}(\Lambda) \cong \mathcal{O}_S,$$

the generator being an equation of Λ. Therefore the left mutation of $\mathcal{O}_{\widetilde{Q}}(\Lambda+kh)$, being the cone of the evaluation map, is

$$\mathrm{cone}(\mathcal{O}_{\widetilde{Q}} \to \mathcal{O}_{\widetilde{Q}}(\Lambda)) \otimes \mathcal{O}_{\widetilde{Q}}(kh) \cong \tilde{\jmath}_*\mathcal{O}_\Lambda(\Lambda+kh).$$

This gives $\mathscr{B}_k = \langle \tilde{\jmath}_*\mathcal{O}_\Lambda(\Lambda+kh), \mathcal{O}_{\widetilde{Q}}(kh)\rangle$ for $k = 0,\dots,n-4$.

Step 4. Since $\mathcal{O}_\Lambda(\Lambda)$ restricts to $\mathcal{O}_{\mathbf{P}^1}(-1)$ along the fibres of $\Lambda \to Q'$,

$$\begin{aligned}\tilde{\rho}_*\mathscr{H}\!om_{\widetilde{Q}}(\mathcal{O}_{\widetilde{Q}}(kh), \tilde{\jmath}_*\mathcal{O}_\Lambda(\Lambda+\ell h)) &\cong \rho'_*a_*\tilde{\jmath}_*\mathcal{O}_\Lambda(\Lambda+(\ell-k)h) \\ &\cong \rho'_*j'_*a_{\Lambda,*}\mathcal{O}_\Lambda(\Lambda+(\ell-k)h) = 0.\end{aligned}$$

Therefore the $\mathcal{O}_{\widetilde{Q}}(kh)$ are totally orthogonal to the $\tilde{\jmath}_*\mathcal{O}_\Lambda(\Lambda+\ell h)$ for all integers k and ℓ, and so the decomposition can be reordered to

$$\langle b^*\mathrm{Ku}(Q), \mathcal{O}_{\widetilde{Q}}(-E), \tilde{\jmath}_*\mathcal{O}_\Lambda(\Lambda), \dots, \tilde{\jmath}_*\mathcal{O}_\Lambda(\Lambda+(n-4)h), \mathcal{O}_{\widetilde{Q}}, \dots, \mathcal{O}_{\widetilde{Q}}((n-2)h)\rangle.$$

Step 5. Right mutate $b^*\mathrm{Ku}(Q)$ through $\mathcal{O}_{\widetilde{Q}}(-E)$ and apply Serre duality for the morphism $\tilde{\rho}\colon \widetilde{Q} \to S$ to move $\mathcal{O}_{\widetilde{Q}}(-E)$ to the right side. Since

$$\mathcal{O}_{\widetilde{Q}}(-E) \otimes \omega_{\tilde{\rho}}^\vee \cong \mathcal{O}_{\widetilde{Q}}((n-1)h) \otimes \tilde{\rho}^*\mathscr{M}$$

for some invertible $\mathcal{O}_S$-module $\mathscr{M}$, this yields the S-linear semiorthgonal decomposition

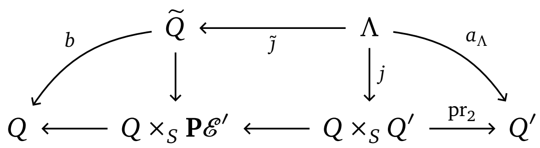
$$\langle \mathbf{R}_{\mathcal{O}_{\widetilde{Q}}(-E)}(b^*\mathrm{Ku}(Q)), \tilde{\jmath}_*\mathcal{O}_\Lambda(\Lambda), \dots, \tilde{\jmath}_*\mathcal{O}_\Lambda(\Lambda+(n-4)h), \mathcal{O}_{\widetilde{Q}}, \dots, \mathcal{O}_{\widetilde{Q}}((n-1)h)\rangle.$$

Comparing with the semiorthogonal decomposition (A) shows that

$$\mathbf{R}_{\mathcal{O}_{\widetilde{Q}}(-E)}(b^*\mathrm{Ku}(Q)) = \tilde{\jmath}_*(a_\Lambda^*\mathrm{Ku}(Q') \otimes \mathcal{O}_\Lambda(\Lambda))$$

as S-linear subcategories of $\mathrm{D}_{\mathrm{qc}}(\widetilde{Q})$. Applying left mutation through $\mathcal{O}_{\widetilde{Q}}(-E)$ on both sides then completes the proof. ■

Since $\Phi\colon \mathrm{Ku}(Q') \to \mathrm{Ku}(Q)$ is a composition of pullbacks, pushforwards, tensor products, and mutations, it is of Fourier–Mukai type. To determine a kernel, consider the commutative diagram



where $j\colon \Lambda \to Q \times_S Q'$ is the closed immersion as the point-line incidence correspondence, resulting from the geometric interpretation of $\rho'\colon Q' \to S$ from 4.11 as the relative scheme of lines in $\rho\colon Q \to S$ incident with the section $\mathbf{P}\mathscr{F}$:

$$\Lambda = \big\{(x,[\ell]) \in Q \times_S Q' : x \in \ell,\ \ell \subset Q \text{ a line through the section } \mathbf{P}\mathscr{F}\big\}.$$

Let $\mathscr{J}$ be the ideal sheaf of Λ in $Q \times_S Q'$. The result is as follows:

6.2. Proposition. — $\Phi\colon \mathrm{Ku}(Q') \to \mathrm{Ku}(Q)$ *is the Fourier–Mukai functor with kernel*

$$\mathrm{pr}_2^*\big(\mathscr{O}_{\rho'}(1) \otimes \rho'^*(\mathscr{F}^\vee \otimes \mathscr{L})\big) \otimes \mathscr{J} \in \mathrm{D}_{\mathrm{qc}}(Q \times_S Q').$$

Proof. Functors in this proof are all derived. By construction, Φ sends an object $F \in \mathrm{Ku}(Q')$ to $M \in \mathrm{Ku}(Q)$ characterized by

$$\begin{aligned} Lb^*M &\cong \mathrm{cone}\big(\tilde\rho^*\mathscr{H} \otimes \mathscr{O}_{\tilde Q}(-E) \to \tilde\jmath_*(a_\Lambda^* F \otimes \mathscr{N}_{\Lambda/\tilde Q})\big) \\ &\cong \mathrm{cone}\big(\tilde\rho^*\mathscr{H} \to \tilde\jmath_* a_\Lambda^*(F \otimes \mathscr{O}_{\rho'}(1)) \otimes \tilde\rho^*(\mathscr{F}^\vee \otimes \mathscr{L})\big) \otimes \mathscr{O}_{\tilde Q}(-E), \end{aligned}$$

where $\mathscr{N}_{\Lambda/\tilde{\mathscr{Q}}}$ is identified as $\tilde\jmath^*(\mathscr{O}_{\tilde Q}(h-E) \otimes \tilde\rho^*(\mathscr{F}^\vee \otimes \mathscr{L}))$ via 4.16(iii) and 4.18, and $\mathscr{H}$ is the object

$$\begin{aligned} \mathscr{H} &:= \tilde\rho_*\mathscr{H}\!om_{\mathscr{O}_{\tilde Q}}(\mathscr{O}_{\tilde Q}(-E), \tilde\jmath_*(a^*F \otimes \mathscr{N}_{\Lambda/\tilde Q})) \\ &\cong \tilde\rho_*\tilde\jmath_* a^*(F \otimes \mathscr{O}_{\rho'}(1)) \otimes (\mathscr{F}^\vee \otimes \mathscr{L}) \cong \rho'_*(F \otimes \mathscr{O}_{\rho'}(1)) \otimes (\mathscr{F}^\vee \otimes \mathscr{L}) \in \mathrm{D}_{\mathrm{qc}}(S). \end{aligned}$$

Since $b\colon \tilde Q \to Q$ is a blowup along a regular section, $b_*\mathscr{O}_{\tilde Q}(E) = \mathscr{O}_Q$, so the projection formula gives an isomorphism $M \cong b_*(b^*M \otimes \mathscr{O}_{\tilde Q}(E))$. Therefore there is a canonical isomorphism

$$\Phi(F) \cong b_*\,\mathrm{cone}\big(\tilde\rho^*\mathscr{H} \to \tilde\jmath_* a^*(F \otimes \mathscr{O}_{\rho'}(1)) \otimes \tilde\rho^*(\mathscr{F}^\vee \otimes \mathscr{L})\big).$$

Writing $\rho^*\mathscr{H} = \mathrm{pr}_{1,*}\mathrm{pr}_2^*(F \otimes \mathscr{O}_{\rho'}(1)) \otimes \rho^*(\mathscr{F}^\vee \otimes \mathscr{L})$ and $\mathrm{pr}_1\colon Q \times_S Q' \to Q$ for the first projection, the relations $a = \mathrm{pr}_2 \circ j$ and $b \circ \tilde\jmath = \mathrm{pr}_1 \circ j$ from the diagram above simplifies this to

$$\begin{aligned} \Phi(F) &\cong \mathrm{pr}_{1,*}\,\mathrm{cone}\big(\mathrm{pr}_2^*(F \otimes \mathscr{O}_{\rho'}(1)) \to j_*j^*\mathrm{pr}_2^*(F \otimes \mathscr{O}_{\rho'}(1))\big) \otimes \rho^*(\mathscr{F}^\vee \otimes \mathscr{L}) \\ &\cong \mathrm{pr}_{1,*}\,\mathrm{cone}\big(\mathrm{pr}_2^*(F \otimes \mathscr{O}_{\rho'}(1)) \to \mathrm{pr}_2^*(F \otimes \mathscr{O}_{\rho'}(1)) \otimes j_*\mathscr{O}_\Lambda\big) \otimes \rho^*(\mathscr{F}^\vee \otimes \mathscr{L}) \end{aligned}$$

where the map is restriction to Λ. Thus the cone is $\mathrm{pr}_2^*(F \otimes \mathscr{O}_{\rho'}(1))$ tensored with the ideal sheaf $\mathscr{J}$ of Λ in $Q \times_S Q'$, and so

$$\Phi(F) \cong \mathrm{pr}_{1,*}(\mathrm{pr}_2^*(F) \otimes \mathrm{pr}_2^*\big(\mathscr{O}_{\rho'}(1) \otimes \rho'^*(\mathscr{F}^\vee \otimes \mathscr{L})\big) \otimes \mathscr{J}). \qquad \blacksquare$$

A particularly useful feature of the equivalence $\Phi\colon \mathrm{Ku}(Q') \to \mathrm{Ku}(Q)$ is that it is compatible with the construction of spinor sheaves. More precisely, let $\mathscr{W}' \subset \mathscr{F}^\perp/\mathscr{F}$ be an isotropic subbundle, and let $\mathscr{W}$ be the corresponding subbundle of in $\mathscr{E}$. For each integer d, these define spinor sheaves $\mathscr{S}'_d$ and $\mathscr{S}_d$ on Q' and Q, respectively. By 5.6, these may be viewed as objects in $\mathrm{Ku}(Q')$ and $\mathrm{Ku}(Q)$, respectively. The statement is that Φ sends the dual of one to the dual of the other:

6.3. Proposition. — $\Phi(\mathscr{S}'^\vee_d) \cong \mathscr{S}^\vee_d$ *for all* $d \in \mathbf{Z}$.

Proof. All functors are again derived in this proof. Continuing with the notation from the diagram preceding 6.2, tensor the ideal sheaf sequence for $j\colon \Lambda \to Q \times_S Q'$ with $\mathrm{pr}_2^*\mathscr{S}'_d$ and pushforward along $\mathrm{pr}_1\colon Q \times_S Q' \to Q$ to obtain the triangle

$$\mathrm{pr}_{1,*}(\mathrm{pr}_2^*\mathscr{S}'_d \otimes \mathscr{J}) \longrightarrow \mathrm{pr}_{1,*}\mathrm{pr}_2^*\mathscr{S}'_d \longrightarrow (\mathrm{pr}_1 \circ j)_*(\mathrm{pr}_2 \circ j)^*\mathscr{S}'_d \xrightarrow{+1}$$

in $\mathrm{D}_{\mathrm{qc}}(Q)$. The defining presentation for the spinor sheaf from 5.3 identifies the middle term as

$$\mathrm{pr}_{1,*}\mathrm{pr}_2^*\mathscr{S}'_d \cong \rho^*\rho'_*\mathscr{S}'_d \cong \rho^*\mathscr{I}'_d$$

where $\mathscr{I}_d$ is the d-th Clifford ideal on S associated with $\mathscr{W}'$. To identify the term on the right, let Q'' be the intersection of Q with its tangent space along $\mathbf{P}\mathscr{F}$; in other words, let $q''\colon \mathscr{F}^\perp \to \mathscr{L}$ be the restriction of q to the orthogonal of $\mathscr{F}$, and let Q'' be the associated quadric bundle in $\mathbf{P}\mathscr{F}^\perp$. Then Q'' is a cone over Q' with vertex $\mathbf{P}\mathscr{F}$, and the morphism $\mathrm{pr}_1 \circ j\colon \Lambda \to Q$ factors through a morphism $b_\Lambda\colon \Lambda \to Q''$ identifying it with the blowup along the vertex. Thus there is a commutative diagram

$$\begin{array}{ccccc} \Lambda & \xrightarrow{j} & Q \times_S Q' & \xrightarrow{\mathrm{pr}_2} & Q' \\ {\scriptstyle b_\Lambda}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} & & \\ Q'' & \xrightarrow{i''} & Q. & & \end{array}$$

Writing $a_\Lambda = \mathrm{pr}_2 \circ j\colon \Lambda \to Q'$, the term on the right is identified via 5.17 as the pushforward to Q of a spinor sheaf on Q'' associated with $\mathscr{W}$:

$$(\mathrm{pr}_1 \circ j)_*(\mathrm{pr}_2 \circ j)^*\mathscr{S}'_d \cong i''_* b_{\Lambda,*} a^*_\Lambda \mathscr{S}'_d \cong i''_* \mathscr{S}''_{d+1} \otimes \rho^*\mathscr{F}^\vee.$$

Identifying furthermore the Clifford ideal $\mathscr{I}'_d$ for $\rho'\colon Q' \to S$ with the twisted Clifford ideal $\mathscr{I}''_{d+1} \otimes \mathscr{F}^\vee$ for $\rho''\colon Q'' \to S$ via 5.16 transforms the triangle to

$$\mathrm{pr}_{1,*}(\mathrm{pr}_2^*\mathscr{S}'_d \otimes \mathscr{J}) \longrightarrow \rho^*\mathscr{I}''_{d+1} \otimes \rho^*\mathscr{F}^\vee \longrightarrow i''_*\mathscr{S}''_{d+1} \otimes \rho^*\mathscr{F}^\vee \xrightarrow{+1}$$

where the second map is induced by the evaluation map. Comparing with 5.14 and using the fact that $\mathscr{E}/\mathscr{F}^\perp \cong \mathscr{F}^\vee \otimes \mathscr{L}$ shows that

$$(\star) \qquad \mathrm{pr}_{1,*}(\mathrm{pr}_2^*\mathscr{S}'_d \otimes \mathscr{J}) \cong \mathscr{S}_{d+1} \otimes \mathscr{O}_\rho(-1) \otimes \rho^*\mathscr{L}^\vee.$$

To conclude, it remains to compare the two spinor sheaves to their duals via the relations in 5.3. Writing the rank of $\mathscr{E}$ as $2k$ or $2k+1$, and that of $\mathscr{W}$ as r, the relations give

$$\mathscr{S}'_d \otimes \mathscr{O}_{\rho'}(-1) \cong \mathscr{S}'^\vee_{r-d-2} \otimes \rho'^*(\det \mathscr{W}' \otimes \det(\mathscr{F}^\perp/\mathscr{F}) \otimes \mathscr{L}^{\vee,\otimes k-1}), \text{ and}$$
$$\mathscr{S}_{d+1} \otimes \mathscr{O}_\rho(-1) \cong \mathscr{S}^\vee_{r-d-2} \otimes \rho^*(\det \mathscr{W} \otimes \det \mathscr{E} \otimes \mathscr{L}^{\vee,\otimes k}).$$

Since $\det \mathscr{W}' \cong \det \mathscr{W} \otimes \mathscr{F}^\vee$ and $\det(\mathscr{F}^\perp/\mathscr{F}) \cong \det \mathscr{E} \otimes \mathscr{L}^\vee$, substituting these identifications into ($\star$), cancelling out common invertible factors, and multiplying through by $\rho^*\mathscr{L}$ shows

$$\mathrm{pr}_{1,*}\big(\mathrm{pr}_2^*\mathscr{S}'^\vee_d \otimes \mathrm{pr}_2^*\big(\mathscr{O}_{\rho'}(1) \otimes \rho'^*(\mathscr{F}^\vee \otimes \mathscr{L})\big) \otimes \mathscr{J}\big) \cong \mathscr{S}^\vee_d.$$

The functor on the left is Φ by 6.2, yielding the result. ■

One simple application of 6.3 is to show that spinor sheaves are relatively simple in certain situations. Note that this is not always the case: see [Add11, Proposition 6.2].

6.4. Corollary. — *Let $\rho\colon Q \to S$ be a quadric 2ℓ-fold bundle, $\mathbf{P}\mathscr{W} \to S$ an isotropic ℓ-plane intersecting $\mathrm{Sing}\,\rho$ in at most 1 point in each fibre, and $\mathscr{S}^\vee_d$ the associated spinor sheaf. Then the canonical map*

$$\mathscr{O}_S \to \rho_*\mathscr{H}om_{\mathscr{O}_Q}(\mathscr{S}^\vee_d, \mathscr{S}^\vee_d)$$

is an isomorphism for any $d \in \mathbf{Z}$. In other words, $\mathscr{S}^\vee_d$ is simple over S.

Proof. The assertion is local on S. Passing to an étale cover, we may assume that $\mathscr{W}$ admits a line subbundle $\mathscr{N}$. Let $\rho'\colon Q' \to S$ be the associated hyperbolic reduction, set $\mathscr{W}' := \mathscr{W}/\mathscr{N}$, and write $\mathscr{S}'^\vee_d$ for the corresponding d-th dual spinor sheaf. There is an S-linear equivalence $\Phi\colon \mathrm{Ku}(Q') \to \mathrm{Ku}(Q)$ satisfying $\mathscr{S}^\vee_d \cong \Phi(\mathscr{S}'^\vee_d)$ by 6.1 and 6.3, so this induces an isomorphism

$$R\rho_*R\mathscr{H}om_{\mathscr{O}_Q}(\mathscr{S}^\vee_d, \mathscr{S}^\vee_d) \cong R\rho'_*R\mathscr{H}om_{\mathscr{O}_{Q'}}(\mathscr{S}'^\vee_d, \mathscr{S}'^\vee_d).$$

By induction, then, it suffices to treat the case when $\rho\colon Q \to S$ has relative dimension 0 and $\mathbf{P}\mathscr{W} \to S$ provides a section $w\colon S \to Q$ of ρ. In this case, 5.7 shows that the spinor sheaves $\mathscr{S}^\vee_d$ are, up to invertible factors, of the form $i_*\mathcal{O}_T$ for a closed subscheme $i\colon T \to Q$ cut out in the $\mathbf{P}^1$-bundle $\pi\colon \mathbf{P}\mathscr{E} \to S$ by a line bundle of relative degree 1. Noting that ρ is flat and i is affine, taking degree 0 cohomology sheaves then gives the result:

$$\rho_*\mathscr{H}om_{\mathcal{O}_Q}(\mathscr{S}_d^\vee,\mathscr{S}_d^\vee) \cong \rho_*\mathscr{H}om_{\mathcal{O}_Q}(i_*\mathcal{O}_T, i_*\mathcal{O}_T) \cong \rho_* i_*\mathscr{H}om_{\mathcal{O}_T}(\mathcal{O}_T,\mathcal{O}_T) \cong \rho_* i_*\mathcal{O}_T \cong \mathcal{O}_S. \qquad \blacksquare$$

6.5. Maximal hyperbolic reduction. — Let $\rho : Q \to S$ be a quadric bundle of even relative dimension 2ℓ. Suppose that there exists a regular isotropic subbundle $\mathscr{F} \subset \mathscr{E}$ of rank ℓ; in particular, this implies that $S_3 = \varnothing$. Assuming furthermore that S_2 contains no weakly associated points of S, the corresponding hyperbolic reduction $\mu\colon M \to S$ is a family of quadrics of relative dimension 0, having 1-dimensional fibres over points of S_2. As such, we refer to $\mu\colon M \to S$ as a *maximal hyperbolic reduction* of the quadric bundle $\rho : Q \to S$.

Identify M via 4.11 as the subscheme of the relative Fano scheme $\mathbf{F}_\ell(Q/S)$ parameterizing ℓ-planes along $\rho : Q \to S$ that contain $\mathbf{P}\mathscr{F}$, and let $\widetilde{\mathscr{F}}$ be the tautological isotropic subbundle of rank $\ell+1$ in $\mu^*\mathscr{E}$. In this way, the quadric bundle

$$Q \times_S M \subset \mathbf{P}\mathscr{E} \times_S M \cong \mathbf{P}(\mu^*\mathscr{E})$$

over M via $\mathrm{pr}_2\colon Q \times_S M \to M$ carries a tautological family $\mathbf{P}\widetilde{\mathscr{F}}$ of isotropic ℓ-planes. Writing $\mathscr{S}_d$ for the associated degree d spinor sheaf on $Q \times_S M$, we have the following:

6.6. Proposition. — *In the setting above, the Fourier–Mukai functor* $\mathrm{D}_{\mathrm{qc}}(M) \to \mathrm{D}_{\mathrm{qc}}(Q)$ *given by*

$$\Psi_d(F) := R\mathrm{pr}_{1,*}(L\mathrm{pr}_2^* F \otimes^L \mathscr{S}_d^\vee)$$

factors through $\mathrm{Ku}(Q)$ *and defines an S-linear equivalence* $\Psi_d : \mathrm{D}_{\mathrm{qc}}(M) \to \mathrm{Ku}(Q)$ *for each* $d \in \mathbf{Z}$.

Proof. To ease notation, all functors in this proof are derived. That the functor factors through $\mathrm{Ku}(Q)$ follows from 5.6: Indeed, using $\rho \circ \mathrm{pr}_1 = \mu \circ \mathrm{pr}_2$, the fact that the relative tautological bundle of $\mathrm{pr}_2\colon Q \times_S M \to M$ is $\mathcal{O}_{\mathrm{pr}_2}(1) := \mathrm{pr}_1^*\mathcal{O}_\rho(1)$, and the projection formula,

$$\begin{aligned}\rho_*\mathscr{H}om_{\mathcal{O}_Q}(\mathcal{O}_\rho(i), \Psi_d(F)) &\cong \rho_*\mathrm{pr}_{1,*}(\mathrm{pr}_2^*F \otimes \mathscr{H}om_{\mathcal{O}_{Q\times_S M}}(\mathcal{O}_{\mathrm{pr}_2}(i), \mathscr{S}_d^\vee)) \\ &\cong \mu_*(F \otimes \mathrm{pr}_{2,*}\mathscr{H}om_{\mathcal{O}_{Q\times_S M}}(\mathcal{O}_{\mathrm{pr}_2}(i), \mathscr{S}_d^\vee)) \\ &= 0 \ \text{ for } 0 \le i \le 2r-1.\end{aligned}$$

Henceforth, view Ψ_d as a functor $\mathrm{D}_{\mathrm{qc}}(M) \to \mathrm{Ku}(Q)$. Show that this is an equivalence via induction on the integer $\ell \geq 1$. When $\ell = 1$, the even degree $d = 2k$ dual spinors $\mathscr{S}_{2k}^\vee$ are by 5.8, up to twists by invertible objects pulled back from S, the ideal sheaf $\mathscr{J}$ of the tautological line $j\colon \Lambda \hookrightarrow Q \times_S M$. In this case, 6.1 and 6.2 together imply that Ψ_{2k} is an equivalence. For odd $d = 2k+1$, 5.8 shows that, up to twists by invertible objects pulled up from S, Ψ_{2k+1} is of the form

$$F \mapsto \mathrm{pr}_{1,*}\big(\mathrm{pr}_2^*F \otimes \mathscr{H}om_{\mathcal{O}_{Q\times_S M}}(\mathscr{J}, \mathcal{O}_{Q\times_S M}) \otimes \mathrm{pr}_1^*\mathcal{O}_\rho(-1)\big).$$

Since the ideal sheaf $\mathscr{J}$ is pseudo-coherent, pr_2^*F may be brought into the $\mathscr{H}om$-complex; since $\mu\colon M \to S$ is an effective Cartier divisor of relative degree 2 in a $\mathbf{P}^1$-bundle over S, it has a dualizing sheaf which is the pullback of a line bundle on S. Therefore, the functor above is, up to invertible objects from S, isomorphic to

$$F \mapsto \mathscr{H}om_{\mathcal{O}_Q}\big(\mathrm{pr}_{1,*}(\mathrm{pr}_2^*\mathscr{H}om_{\mathcal{O}_M}(F, \mathcal{O}_M) \otimes \mathscr{J}), \mathcal{O}_Q(-1)\big).$$

This shows that Ψ_{2k+1} is obtained from the Fourier–Mukai equivalence of 6.1 and 6.2 upon applying autoequivalences to the source and target, and so Ψ_{2k+1} itself is an equivalence. This establishes the base case of the induction.

Assume $\ell \geq 2$. By 1.23, which applies thanks to 6.7, it suffices to show that Ψ_d is an equivalence after replacing S by an étale cover. Doing so, assume further that $\mathscr{F}$ admits a line subbundle $\mathscr{N}$, and let $\rho'\colon Q' \to S$ be the associated hyperbolic reduction. Then $\mu\colon M \to S$ is the maximal hyperbolic reduction of the quadric $(2\ell-2)$-fold bundle $\rho'\colon Q' \to S$ along the regular isotropic $(\ell-1)$-bundle $\mathscr{F}' := \mathscr{F}/\mathscr{N}$. The constructions of 6.5 then apply to yield, for each integer d, dual spinor sheaves

$$\mathscr{S}_d'^{\vee} \text{ on the quadric bundle } \mathrm{pr}_2'\colon Q' \times_S M \to M$$

constructed from the tautological isotropic rank ℓ subbundle $\widetilde{\mathscr{F}}'$ of $\mu^*(\mathscr{N}^{\perp}/\mathscr{N})$. The inductive hypothesis implies that the corresponding Fourier–Mukai functor $\Psi_d'\colon \mathrm{D_{qc}}(M) \to \mathrm{Ku}(Q')$ is an equivalence. Let $\Phi\colon \mathrm{Ku}(Q') \to \mathrm{Ku}(Q)$ be the equivalence from 6.1. We now show that Ψ_d and $\Phi \circ \Psi_d'$ are isomorphic as functors $\mathrm{D_{qc}}(M) \to \mathrm{Ku}(Q)$, from which the result follows.

On the one hand, $\Phi \circ \Psi_d'$ is a Fourier–Mukai functor whose kernel, in view of 6.2, is

$$\mathscr{K} := \mathrm{pr}_{1,3,*}\big(\mathrm{pr}_{1,2}^*\mathscr{J} \otimes \mathrm{pr}_2^*(\mathscr{O}_{\rho'}(1) \otimes \rho'^*(\mathscr{N}^{\vee} \otimes \mathscr{L})) \otimes \mathrm{pr}_{2,3}^*\mathscr{S}_d'^{\vee}\big) \in \mathrm{D_{qc}}(Q \times_S M),$$

where $\mathscr{J}$ is the ideal sheaf of the incidence correspondence $\Lambda \hookrightarrow Q \times_S Q'$ as before. On the other hand, base change along $\mu\colon M \to S$, and view the quadric bundle $\mathrm{pr}_2'\colon Q' \times_S M \to M$ as the hyperbolic reduction of $\mathrm{pr}_2\colon Q \times_S M \to M$ along the regular isotropic line subbundle $\mu^*\mathscr{N} \subset \mu^*\mathscr{E}$. Applying 6.1 to this situation thus gives an equivalence

$$\Phi_M\colon \mathrm{Ku}(Q' \times_S M) \to \mathrm{Ku}(Q \times_S M).$$

The tautological incidence correspondence between $Q \times_S M$ and $Q' \times M$ is none other than

$$\Lambda \times_S M = \mathrm{pr}_{1,2}^{-1}(\Lambda) \hookrightarrow Q \times_S Q' \times_S M \cong (Q \times_S M) \times_M (Q' \times_S M),$$

so its ideal sheaf is $\mathrm{pr}_{1,2}^*\mathscr{J}$. Comparing with 6.2 then shows that

$$\mathscr{K} \cong \Phi_M(\mathscr{S}_d'^{\vee}) \in \mathrm{D_{qc}}(Q \times_S M).$$

Since $\mathscr{S}_d'^{\vee}$ is the d-th dual spinor bundle associated with $\widetilde{\mathscr{F}}' = \widetilde{\mathscr{F}}/\mathscr{N}$ on $Q' \times_S M$, and Φ_M respects dual spinor sheaves by 6.3, it follows that $\mathscr{K} \cong \mathscr{S}_d^{\vee}$. Thus $\Phi \circ \Psi_d'$ and Ψ_d are both Fourier–Mukai transforms $\mathrm{D_{qc}}(M) \to \mathrm{Ku}(Q)$ with isomorphic kernels, and so the functors are isomorphic. ■

In order to apply 1.23, we need to verify that the tautological dual spinor sheaves $\mathscr{S}_d^{\vee}$ are relatively perfect relative to both factors of $Q \times_S M$. That it is M-perfect follows from 5.4, whereas Q-perfectness can be established by induction using the technique of 6.3:

6.7. Lemma. — *In the setting of 6.5, the dual spinor sheaf $\mathscr{S}_d^{\vee}$ is perfect relative to Q.*

Proof. Induct on the integer $\ell \geq 1$. The base case $\ell = 1$ concerns a quadric surface bundle $\rho: Q \to S$ and by 5.8 and 6.2, ignoring twists by line bundles, the ideal sheaf $\mathscr{J}$ of the tautological line $j\colon \Lambda \to Q \times_S M$. By the ideal sheaf sequence, it suffices to verify that $\mathscr{O}_{Q \times_S M}$ and $j_*\mathscr{O}_{\Lambda}$ are Q-perfect. For the former, observe that first projection factors as

$$\mathrm{pr}_1 : Q \times_S M \hookrightarrow Q \times_S \mathbf{P}(\mathscr{E}/\mathscr{F}) \to Q$$

where the first arrow is a closed immersion, which is regular by the standing assumption on weakly associated points together with 4.19, and the second arrow is a projective bundle. Thus pr_1 is a composition of perfect morphisms, and thus is itself perfect, meaning that $\mathscr{O}_{Q \times_S M}$ is Q-perfect. For $j_*\mathscr{O}_{\Lambda}$, as explained above 6.2, the map $\Lambda \to Q$ factors as $b \circ \tilde{\jmath}$ where $\tilde{\jmath}$ is the inclusion of an effective Cartier divisor and b is a blowup in a local complete intersection closed subscheme. Therefore $\Lambda \to Q$ is an local complete intersection morphism in the sense of [Stacks, 068E], and so $j_*\mathscr{O}_{\Lambda}$ is perfect relative to Q by [Stacks, 069H]. This establishes the base case.

Assume $\ell > 1$ and that the tautological dual spinors between a quadric $2(\ell-1)$-bundle and its maximal hyperbolic reduction is perfect relative to the quadric. Let $\rho\colon Q \to S$ be a quadric 2ℓ-fold bundle with maximal hyperbolic reduction $\mu\colon M \to S$ with respect to a regular isotropic subbundle $\mathscr{F} \subset \mathscr{E}$ of rank ℓ. The problem is local on S, so pass to an étale cover to assume that $\mathscr{F}$ admits a line subbundle $\mathscr{N}$ so that the associated hyperbolic reduction $\rho'\colon Q' \to S$ shares $\mu\colon M \to S$ as its maximal hyperbolic reduction along $\mathscr{F}/\mathscr{N}$. The proof of 6.3 shows, using 5.14 and 5.17, that the spinors of $Q \times_S M$ and $Q' \times_S M$ fit into a short exact sequence

$$0 \to \mathscr{S}_d \otimes \mathscr{O}_{\mathrm{pr}_2}(-1) \otimes (\rho \circ \mathrm{pr}_1)^*\mathscr{L}^\vee \to \mathrm{pr}_2^*\mathscr{I}'_{d-1} \to (i'' \times \mathrm{id}_M)_*(b \times \mathrm{id}_M)_*(a \times \mathrm{id}_M)^*\mathscr{S}'_{d-1} \to 0$$

where $\mathscr{I}'_{d-1}$ is the Clifford ideal associated with the tautological $(\ell-1)$-plane in $Q' \times_S M$, $i''\colon Q'' \to Q$ is the inclusion of the tangent hyperplane section along $\mathbf{P}\mathscr{N}$, $a\colon \widetilde{Q} \to Q'$ is a $\mathbf{P}^1$-bundle, and $b\colon \widetilde{Q} \to Q''$ is the blowup along the vertex $\mathbf{P}\mathscr{N}$.

The morphism $\mathrm{pr}_1\colon Q \times_S M \to Q$ is Gorenstein, so 2.6(i) implies that it suffices to show that $\mathscr{S}_d$ is Q-perfect, and this would follow from seeing that the second and third terms of this sequence are Q-perfect. This is true for $\mathrm{pr}_2^*\mathscr{I}'_{d-1}$ since it is locally free and $\mathrm{pr}_1\colon Q \times_S M \to Q$ is perfect by the argument in the base case. As for the term on the right, induction and 2.6(i) together mean that $\mathscr{S}'_{d-1}$ is Q'-perfect. Relative perfectness is clearly preserved upon pullback along the $\mathbf{P}^1$-bundle $a \times \mathrm{id}_M$, and also pushforward along $b \times \mathrm{id}_M$ by 1.20. Finally, $i''\colon Q'' \to Q$ is the inclusion of a effective Cartier divisor since it is the tangent hyperplane section along a smooth point, so the pushforward of a Q''-perfect object yields a Q-perfect object. Together, this completes the proof. ■

7. Stack of spinor sheaves

The goal of this section is to prove a global version of 6.6, in which the quadric bundle may not possess a regular isotropic subbundle of maximal dimension. This is achieved in 7.25, wherein a global incarnation of the maximal hyperbolic reduction is constructed as an open subspace of a moduli space of spinor sheaves. Much of this section is therefore devoted to a study of the moduli stack of spinor sheaves on quadric bundles. Throughout this section, unless otherwise specified, $\rho\colon Q \to S$ denotes a quadric 2ℓ-fold bundle with $\ell \geq 1$ and $S_3 = \varnothing$.

7.1. Definition and local structure. — Applying the constructions of 5.3 to the tautological rank $\ell+1$ subbundle on $\mathbf{F}_\ell(Q/S)$ provides, for each $d \in \mathbf{Z}$, a morphism

$$\zeta_d\colon \mathbf{F}_\ell(Q/S) \to \mathscr{C}oh_{Q/S}$$

from the relative Fano scheme of ℓ-planes of $\rho\colon Q \to S$ to the stack of coherent sheaves on Q over S which sends an ℓ-plane to its associated degree d dual spinor sheaf. We aim to prove:

7.2. Proposition. — *The morphism $\zeta_d\colon \mathbf{F}_\ell(Q/S) \to \mathscr{C}oh_{Q/S}$ is smooth.*

Since smooth morphisms are open, the image of ζ_d is an open substack $\overline{\mathscr{M}}_d(Q/S) \subset \mathscr{C}oh_{Q/S}$: this is the *stack of spinor sheaves* of $\rho\colon Q \to S$ of degree d. Of primary interest is the open substack

$$\mathscr{M}_d(Q/S) \subseteq \overline{\mathscr{M}}_d(Q/S)$$

consisting of spinors locally free away from at most one point in each fibre of $\rho\colon Q \to S$; by 5.3, this is the image of the restriction $\zeta_d^\circ\colon \mathbf{F}_\ell(Q/S)^\circ \to \mathscr{C}oh_{Q/S}$ of the morphism ζ_d to the open subscheme of $\mathbf{F}_\ell(Q/S)$ parameterizing ℓ-planes which fibrewise intersect the singular locus of $\rho\colon Q \to S$ in at most one point. Notably, thanks to 6.4, $\mathscr{M}_d(Q/S)$ is contained in the open substack $s\mathscr{C}oh_{Q/S} \subset \mathscr{C}oh_{Q/S}$ of simple sheaves, and so it admits a coarse moduli space $\mathrm{M}_d(Q/S)$—an algebraic space over S—over which $\mathscr{M}_d(Q/S)$ is a $\mathbf{G}_m$-gerbe.

The local picture of $\mathscr{M}_d(Q/S)$ can be described as follows: Suppose that $\rho: Q \to S$ admits a family of $(\ell-1)$-planes $L \subset Q$ contained in its smooth locus, and let $\mu: M \to S$ be the associated maximal hyperbolic reduction as in 6.5. Identify M via 4.11 as the closed subscheme of $\mathbf{F}_\ell(Q/S)$ parametrizing ℓ-planes in Q containing L. In fact, $M \subset \mathbf{F}_\ell(Q/S)^\circ$ since an ℓ-plane containing $L \subset Q \setminus \operatorname{Sing}\rho$ can fibrewise intersect the singular locus of $\rho: Q \to S$ in at most one point. Then:

7.3. Proposition. — *Let $\mu: M \to S$ be a maximal hyperbolic reduction of $\rho: Q \to S$ as above. Then*

$$\zeta_d|_M : M \subset \mathbf{F}_\ell(Q/S)^\circ \to \mathscr{M}_d(Q/S)$$

is smooth of relative dimension 1*, and the composite $M \to \mathscr{M}_d(Q/S) \to \mathrm{M}_d(Q/S)$ is an open immersion.*

In general, $\rho: Q \to S$ may not admit a maximal hyperbolic reduction as there may not be a global family of $(\ell-1)$-planes contained in the smooth locus. Nonetheless, étale locally on S, the space $\mathrm{M}_d(Q/S)$ is covered by opens of the form M as above:

7.4. Lemma. — *Let $f: T \to \mathscr{M}_d(Q/S)$ be a morphism from an S-scheme T. Then there exists an étale covering $U \to T$ and a family of $(\ell-1)$-planes $L \subset Q_U$ contained in the smooth locus of $\rho_U : Q_U \to U$ such that the composite $U \to T \to \mathscr{M}_d(Q/S)$ factors through the maximal hyperbolic reduction $\mu_U : M_U \to U$ of $\rho_U : Q_U \to U$ with respect to L:*

$$\begin{array}{ccc} U & \longrightarrow & M_U \\ \downarrow & & \downarrow{\scriptstyle \zeta_d|_{M_U}} \\ T & \xrightarrow{f} & \mathscr{M}_d(Q/S). \end{array}$$

Proof. After replacing T by an étale cover, we may assume f factors through ζ_d°: Indeed, this is because, by 7.2, the fibre product $T \times_{\mathscr{M}_d(Q/S)} \mathbf{F}_\ell(Q/S)^\circ$ is a smooth algebraic space over T mapping surjectively onto T, and so it admits a section after an étale covering. This means that f may be taken to be the classifying map for the spinor sheaf associated with a family $P \subset Q_T$ of ℓ-planes which intersects the singular locus of $Q_T \to T$ fibrewise in at most one point. Since most hyperplanes avoid a given point, there is an open subscheme $\mathbf{G}$ of the Grassmannian $\mathbf{G}(\ell-1, P) \to T$ of hyperplanes in $P \to T$ which is nonempty over every point of T and which parameterizes those $(\ell-1)$-planes $L \subset P$ contained in the smooth locus of $Q_T \to T$. Since $\mathbf{G}$ is smooth over T, replacing T by yet another étale covering provides a section, whence a family $L \subset Q_T$ of $(\ell-1)$-planes contained in the smooth locus. Writing $M_T \to T$ for the associated maximal hyperbolic reduction, the containment $L \subset P$ means via 4.11 that the classifying morphism $T \to \mathbf{F}_\ell(Q/S)^\circ$ for P factors through $M_T \subseteq \mathbf{F}_\ell(Q/S)^\circ$, from which the result follows. ■

Propositions 7.2 and 7.3 will be proven over the course of the following few paragraphs. We first treat the special case where $\rho: Q \to S$ is a quadric surface bundle in 7.9 and 7.11, respectively. Then we reduce to the case of relative dimension 2 using the following, the statement of which only makes sense after 7.2 is established:

7.5. Lemma. — *Let $\rho: Q \to S$ be a quadric 2ℓ-bundle with $\ell > 1$ and let $\rho': Q' \to S$ be the family of quadrics of relative dimension $2\ell-2$ obtained via hyperbolic reduction of $\rho: Q \to S$ along a regular section. Then there is an isomorphism $\alpha: \overline{\mathscr{M}}_d(Q'/S) \cong \overline{\mathscr{M}}_d(Q/S)$ fitting into a commutative square*

$$\begin{array}{ccc} \mathbf{F}_{\ell-1}(Q'/S) & \xrightarrow[\zeta_d']{} & \overline{\mathscr{M}}_d(Q'/S) \\ {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \mathbf{F}_\ell(Q/S) & \xrightarrow{\zeta_d} & \overline{\mathscr{M}}_d(Q/S), \end{array}$$

where β is the closed immersion of 4.11 and $\zeta'_d : \mathbf{F}_{\ell-1}(Q'/S) \to \mathscr{M}_d(Q'/S)$ is the tautological degree d dual spinor morphism of 7.1 associated with $\rho' : Q' \to S$.

Proof. Embedding the stack of coherent sheaves as an open substack of the stack of complexes as in 2.3, view ζ_d and ζ'_d as taking values in their respective stacks of complexes. Furthermore, by 5.6, we may actually view ζ_d and ζ'_d as taking values in the open substack parametrizing complexes in the respective residual categories. The morphism $\alpha : \overline{\mathscr{M}}_d(Q'/S) \to \overline{\mathscr{M}}_d(Q/S)$ is that induced by the open immersion from 3.12 coming from the equivalence Φ of 6.1; that α factors through $\overline{\mathscr{M}}_d(Q/S)$ and that the square commutes is because Φ preserves dual spinor sheaves by 6.3. Since α is an open immersion, it remains to show α induces a bijection on geometric points. We must now show that a given spinor sheaf on Q is isomorphic to one defined by a linear space containing the regular section defining Q', and this follows from 5.10. ∎

7.6. Stack of spinors for quadric surfaces. — Until 7.12, assume that $\ell = 1$, meaning that $\rho : Q \to S$ is a quadric surface bundle. The standing hypothesis that $S_3 = \varnothing$ is equivalent to the statement that every geometric fiber $Q_{\bar{s}}$ is a reduced quadric surface. Begin with a more concrete description of the morphism $\zeta_d : \mathbf{F}_1(Q/S) \to \mathscr{C}oh_{Q/S}$:

According to 5.8, the d-th spinor sheaf corresponding to a flat family of lines is identified with the corresponding ideal sheaf, at least up to duals and tensoring with a line bundle. Since the derived dual of a spinor coincides with its linear dual as in 5.5, and since quadric bundles are Gorenstein, up to automorphisms of the stack $\mathscr{C}omplexes_{Q/S}$ as in 2.5, this means that the morphism ζ_d may be identified with the morphism

$$\zeta_{\mathrm{ideal}} : \mathbf{F}_1(Q/S) \to \mathscr{C}oh_{Q/S} \subset \mathscr{C}omplexes_{Q/S}$$

which on T-points takes a family of lines $L \subset Q_T$ to its ideal sheaf $\mathscr{I}_{L/Q_T}$. Since both statements 7.2 and 7.3 are insensitive to modification by automorphisms of the target, it suffices to prove the analogous statements for ζ_{ideal}. These proofs are based on the following two computations:

7.7. Lemma. — *Let $L \subset Q$ be an S-flat family of lines with ideal sheaf $\mathscr{I}$. Then the object*

$$R\rho_* R\mathscr{H}om_{\mathscr{O}_Q}(\mathscr{I}, \mathscr{O}_Q) \in \mathrm{D}_{\mathrm{qc}}(S)$$

is a rank two vector bundle in degree zero. In particular, the sheaf $\rho_*\mathscr{H}om_{\mathscr{O}_Q}(\mathscr{I}, \mathscr{O}_Q)$ is a vector bundle of rank two whose formation commutes with arbitrary base change.

Proof. The statement is local on S, so we may assume that $Q \subset \mathbf{P}^3_S$ is defined by a section of $\mathscr{O}_{\mathbf{P}^3_S}(2)$ and $L \cong \mathbf{P}^1_S$. Grothendieck duality gives

$$R\rho_* R\mathscr{H}om_{\mathscr{O}_Q}(\mathscr{I}, \mathscr{O}_Q) = R\rho_* R\mathscr{H}om_{\mathscr{O}_Q}(\mathscr{I}(-2), \mathscr{O}_Q(-2)) = R\mathscr{H}om_{\mathscr{O}_S}(R\rho_*\mathscr{I}(-2), \mathscr{O}_S)[2].$$

Twisting and pushing the ideal sheaf sequence for L provides a triangle

$$R\rho_*\mathscr{I}(-2) \longrightarrow R\rho_*\mathscr{O}_Q(-2) \longrightarrow R\rho_*\mathscr{O}_L(-2) \xrightarrow{+1}$$

in $\mathrm{D}_{\mathrm{qc}}(S)$, whose long exact sequence in cohomology sheaves reduces to a short exact sequence

$$0 \to R^1\rho_*\mathscr{O}_L(-2) \to R^2\rho_*\mathscr{I}_L(-2) \to R^2\rho_*\mathscr{O}_Q(-2) \to 0.$$

Since the external terms are line bundles by standard computations, this shows that $R\rho_*\mathscr{I}(-2)$ is a rank 2 vector bundle in degree 2, yielding the first statement. The underived second statement then follows from the derived one by 1.21. ∎

7.8. Lemma. — *Let $L \subset Q$ be an S-flat family of lines and let $\sigma\colon S \to Q$ be a regular section of $\rho\colon Q \to S$. Let $\mathscr{I}$ and $\mathscr{J}$ be the ideal sheaves of L and $\sigma(S)$ in Q, respectively. Then the object*

$$R\rho_* R\mathscr{H}om_{\mathscr{O}_Q}(\mathscr{I}, \mathscr{J}) \in \mathrm{D}_{\mathrm{qc}}(S)$$

is a line bundle in degree zero. In particular, the sheaf $\rho_*\mathscr{H}om_{\mathscr{O}_Q}(\mathscr{I}, \mathscr{J})$ is a line bundle whose formation commutes with arbitrary base change.

Proof. The ideal sheaf sequence for $\sigma(S) \subset Q$ gives rise to a triangle

$$R\rho_* R\mathscr{H}om_{\mathscr{O}_Q}(\mathscr{I}, \mathscr{J}) \longrightarrow R\rho_* R\mathscr{H}om_{\mathscr{O}_Q}(\mathscr{I}, \mathscr{O}_Q) \longrightarrow R\rho_* R\mathscr{H}om_{\mathscr{O}_Q}(\mathscr{I}, \sigma_*\mathscr{O}_S) \xrightarrow{+1}$$

The middle term was identified in 7.7 as a rank 2 vector bundle in degree 0, and the term on the right is a line bundle in degree 0 because σ is a smooth section (so $\mathscr{I}$ is a line bundle in a neighborhood of $\sigma(S)$) and

$$R\rho_* R\mathscr{H}om_{\mathscr{O}_Q}(\mathscr{I}, \sigma_*\mathscr{O}_S) = R\mathscr{H}om_{\mathscr{O}_S}(\sigma^*\mathscr{I}, \mathscr{O}_S).$$

To complete the proof, it suffices to show that the second arrow induces a surjection

$$\rho_*\mathscr{H}om_{\mathscr{O}_Q}(\mathscr{I}, \mathscr{O}_Q) \to \mathscr{H}om_{\mathscr{O}_S}(\sigma^*\mathscr{I}, \mathscr{O}_S)$$

of locally free $\mathscr{O}_S$-modules on degree 0 cohomology sheaves. Over a geometric point $\bar{s} \to S$, this is the evaluation at $\sigma(\bar{s})$ map. There is always a line M whose ideal sheaf is isomorphic to that of $L_{\bar{s}}$ and such that $\sigma(\bar{s}) \notin M$, and this defines a morphism $\mathscr{I}_{\bar{s}} \hookrightarrow \mathscr{O}_{Q_{\bar{s}}}$ which is not zero after evaluation at $\sigma(\bar{s})$. If $Q_{\bar{s}}$ has corank 2 and $L_{\bar{s}}$ is equal to the singular locus, then this is clear because σ is a regular section. Otherwise, $L_{\bar{s}}$ intersects the singular locus in at most one point and the fact that one can find such an M follows from the description of lines with the same ideal sheaf as $L_{\bar{s}}$, see [Add11, Section 3]. ∎

We now prove smoothness of $\zeta_{\mathrm{ideal}}\colon \mathbf{F}_1(Q/S) \to \mathscr{C}oh_{Q/S}$ via the infinitesimal lifting criterion:

7.9. Proof of 7.2 when $\ell = 1$. — We may assume S is Noetherian. Consider a solid commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(R/J) & \longrightarrow & \mathbf{F}_1(Q/S) \\ \downarrow & \nearrow & \downarrow{\scriptstyle \zeta_{\mathrm{ideal}}} \\ \mathrm{Spec}(R) & \longrightarrow & \mathscr{C}oh_{Q/S} \end{array}$$

in which $(R, \mathfrak{m}_R)$ is an Artinian local ring over S with residue field κ and $J \subset R$ is an ideal such that $\mathfrak{m}_R \cdot J = 0$. The task is to construct a dashed arrow making the diagram commute.

Denote by $i\colon Q_{R/J} \to Q_R$ the inclusion. The data of the diagram gives an R/J-flat family of lines $L' \subset Q_{R/J}$ with ideal sheaf $\mathscr{I}'$, an R-flat coherent sheaf $\mathscr{F}$ on Q_R, and an isomorphism $\alpha : i^*\mathscr{F} \cong \mathscr{I}'$. Flatness of $Q \to S$ means that the ideal sheaf of the closed subscheme $Q_{R/J} \subset Q_R$ is isomorphic to $J \otimes_\kappa \mathscr{O}_{Q_\kappa}$, so there is a distinguished triangle

$$J \otimes_\kappa \mathscr{O}_{Q_\kappa} \longrightarrow \mathscr{O}_{Q_R} \longrightarrow i_*\mathscr{O}_{Q_{R/J}} \longrightarrow J \otimes_\kappa \mathscr{O}_{Q_\kappa}[1]$$

in the derived category of Q_R. The isomorphism α gives rise to a morphism $\beta\colon \mathscr{F} \to i_*\mathscr{O}_{Q_{R/J}}$. The triangle together with the vanishing of $\mathrm{Ext}^1_{Q_\kappa}(\mathscr{I}, \mathscr{O}_{Q_\kappa})$ from 7.7 over $\mathrm{Spec}(\kappa)$ shows that β lifts to a morphism $\gamma\colon \mathscr{F} \to \mathscr{O}_{Q_R}$. Applying [Stacks, 046Y], we see that γ is injective and has flat cokernel, which is necessarily the structure sheaf of a family of lines in Q_R. This family gives the dashed arrow of the diagram. ∎

Having established smoothness of ζ_{ideal}, define as in 7.1 open substacks

$$\mathscr{M} \subset \overline{\mathscr{M}} \subset \mathscr{C}oh_{Q/S}$$

as the images of $\mathbf{F}_1(Q/S)^\circ$ and $\mathbf{F}_1(Q/S)$. Thus T-points of $\overline{\mathscr{M}}$ consist of sheaves $\mathscr{F} \in \mathscr{C}oh_{Q/S}(T)$ such that $\mathscr{F}_{\bar{t}}$ is isomorphic to the ideal sheaf of a line on $Q_{\bar{t}}$ for every geometric point $\bar{t} \to T$. Similarly, T-points of $\mathscr{M}$ are those $\mathscr{F}$ such that each $\mathscr{F}_{\bar{t}}$ is isomorphic to the ideal of a line not contained in the singular locus of $Q_{\bar{t}}$.

Writing $\mathscr{U}$ for the universal sheaf on $Q \times_S \mathscr{M}$, the following provides a useful alternative description of $\mathbf{F}_1(Q/S)^\circ$ in terms of maps from an ideal sheaf into the structure sheaf:

7.10. Lemma. — $\zeta^\circ_{\mathrm{ideal}} \colon \mathbf{F}_1(Q/S)^\circ \to \mathscr{M}$ *factors through an open immersion* $j \colon \mathbf{F}_1(Q/S)^\circ \to \mathbf{A}(\mathscr{H})$ *where* $\mathscr{H}$ *is the rank* 2 *vector bundle on* $\mathscr{M}$ *given by*

$$\mathscr{H} := \mathrm{pr}_{2,*}\mathscr{H}om_{\mathscr{O}_{Q \times_S \mathscr{M}}}(\mathscr{U}, \mathscr{O}_{Q \times_S \mathscr{M}}).$$

Proof. That $\mathscr{H}$ is a rank 2 vector bundle whose formation commutes with arbitrary base change may be verified after pullback along the smooth covering $\zeta^\circ_{\mathrm{ideal}} \colon \mathbf{F}_1(Q/S)^\circ \to \mathscr{M}$, whereupon this is the statement of 7.7 for the quadric bundle $Q \times_S \mathbf{F}_1(Q/S)^\circ \to \mathbf{F}_1(Q/S)^\circ$. Given an S-scheme T, an object of the fiber category $\mathbf{A}(\mathscr{H})(T)$ is a pair $(\mathscr{F}, \varphi)$ where $\mathscr{F} \in \mathscr{M}_d(Q/S)(T)$ and $\varphi \in \Gamma(T, \mathscr{H}om_{\mathscr{O}_{Q_T}}(\mathscr{F}, \mathscr{O}_{Q_T})) = \mathrm{Hom}_{Q_T}(\mathscr{F}, \mathscr{O}_{Q_T})$, and a morphism $(\mathscr{F}, \varphi) \to (\mathscr{G}, \psi)$ is an isomorphism $\alpha \colon \mathscr{F} \to \mathscr{G}$ such that the diagram

$$\begin{array}{ccc} \mathscr{F} & \xrightarrow{\varphi} & \mathscr{O}_{Q_T} \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle =} \\ \mathscr{G} & \xrightarrow{\psi} & \mathscr{O}_{Q_T} \end{array}$$

commutes. Consider the open substack $\mathscr{V} \subset \mathbf{A}(\mathscr{H})$ parametrizing pairs $(\mathscr{F}, \varphi)$ with φ injective; that this is an open substack can be seen using [Stacks, 046Y]. The stack $\mathscr{V}$ is in fact an algebraic space: Since $\mathscr{M}_d(Q/S)$ is a $\mathbf{G}_m$-gerbe by 6.4, étale locally on T, any two morphisms $(\mathscr{F}, \varphi) \to (\mathscr{G}, \psi)$ differ by some $u \in \mathscr{O}(T)^\times$. But the only way both α and $u\alpha$ can make the square above commute when φ and ψ are injective is if $u = 1$. Thus $\mathscr{V}$ is the algebraic space whose T-points are the set of subobjects $\mathscr{F} \subset \mathscr{O}_{Q_T}$ with $\mathscr{F} \in \mathscr{M}_d(Q/S)$, and this is precisely $\mathbf{F}_1(Q/S)^\circ$. ■

Suppose now that $\sigma \colon S \to Q$ is a regular section of $\rho \colon Q \to S$. Let $\mu \colon M \to S$ be the corresponding maximal hyperbolic reduction and, as usual, identify M via 4.11 as the closed subscheme of $\mathbf{F}_1(Q/S)^\circ$ parameterizing lines through σ. This puts us in the setting of:

7.11. Proof of 7.3 when $\ell = 1$. — To show that $\zeta_{\mathrm{ideal}}|_M \colon M \to \mathscr{M}$ is smooth of relative dimension 1, we claim that under the identification from 7.10 of $\mathbf{F}_1(Q/S)^\circ$ as an open subscheme of the affine 2-space bundle $\mathbf{A}(\mathscr{H}) \to \mathscr{M}$, M is an open subscheme of the subbundle on the line subbundle

$$\mathscr{N} := \mathrm{pr}_{2,*}\mathscr{H}om_{\mathscr{O}_{Q \times_S \mathscr{M}}}(\mathscr{U}, \mathscr{J}_{\mathscr{M}}) \subseteq \mathrm{pr}_{2,*}\mathscr{H}om_{\mathscr{O}_{Q \times_S \mathscr{M}}}(\mathscr{U}, \mathscr{O}_{Q \times_S \mathscr{M}}) = \mathscr{H}$$

where $\mathscr{J}_{\mathscr{M}}$ is the pullback of the ideal sheaf $\mathscr{J} \subset \mathscr{O}_Q$ of $\sigma(S)$ to $Q \times_S \mathscr{M}$; that $\mathscr{N}$ is a line bundle on $\mathscr{M}$ whose formation commutes with arbitrary base change follows from 7.8. Since the structure map $\mathbf{A}(\mathscr{N}) \to \mathscr{M}$ is smooth of relative dimension 1, this would give the result. With the notation of 7.10, observe that a T-point $(\mathscr{F}, \varphi)$ of $\mathbf{A}(\mathscr{H}) \supseteq \mathbf{F}_1(Q/S)^\circ$ is in the locally closed subscheme M if and only if φ is injective and the composition

$$\mathscr{F} \xrightarrow{\varphi} \mathscr{O}_{Q_T} \to \sigma_{T,*}\mathscr{O}_T$$

is zero. Equivalently, this means that φ is injective and factors through the ideal sheaf $\mathscr{J}_T \subset \mathscr{O}_{Q_T}$ of $\sigma_T(T) \subset Q_T$. The latter condition defines the subbundle $\mathbf{A}(\mathscr{N}) \subset \mathbf{A}(\mathscr{H})$ and the former condition, as before, is an open condition on φ, proving the claim.

For the remaining statement of 7.3, observe that $\mathbf{G}_m$ acts on the line bundle $\mathscr{N}$ with weight -1, and this means that $\mathscr{M}$ is, in fact, a trivial $\mathbf{G}_m$-gerbe: see 1.13. Furthermore, its $\mathbf{G}_m$-rigidification may be identified as the complement of the zero section of $\mathbf{A}(\mathscr{N})$. The same is true for the $\mathbf{G}_m$-gerbe given by the open substack $\zeta_{\mathrm{ideal}}(M) \subseteq \mathscr{M}$. The previous paragraph now shows that the complement of the zero section of $\mathbf{A}(\mathscr{N}) \times_{\mathscr{M}} \zeta_{\mathrm{ideal}}(M)$ is the maximal hyperbolic reduction M—the space of φ is 1-dimensional, so φ is injective if and only it is nonzero!—from which the result follows. ■

7.12. Stack of spinors in general. — Having established the surface cases, we now prove 7.2 and 7.3 for quadric 2ℓ-fold bundles $\rho : Q \to S$ with $\ell > 1$. Smoothness of ζ_d is established via induction on dimension together with, once again, the infinitesimal lifting criterion:

Proof of 7.2. View ζ_d as a morphism $\mathbf{F}_\ell(Q/S) \to \mathscr{C}omplexes_{\mathrm{Ku}(Q)/S}$, as is possible by 5.6. Proceed by induction on $\ell \geq 1$. The base case $\ell = 1$ handled in 7.9. So let $\ell > 1$ and assume that 7.2 holds for all quadric bundles of relative dimension $2(\ell - 1)$.

We may assume S is Noetherian. Consider a solid commutative diagram

$$(\star\star) \qquad \begin{array}{ccc} \mathrm{Spec}(R/J) & \xrightarrow{\;t\;} & \mathbf{F}_\ell(Q/S) \\ \downarrow & \overset{\nu}{\dashrightarrow} & \downarrow{\scriptstyle \zeta_d} \\ \mathrm{Spec}(R) & \xrightarrow{\;u\;} & \mathscr{C}omplexes_{\mathrm{Ku}(Q)/S} \end{array}$$

in which $(R, \mathfrak{m}_R)$ is an Artinian local ring over S with algebraically closed residue field κ and $J \subset R$ is an ideal such that $\mathfrak{m}_R \cdot J = 0$. The task is to construct a dashed arrow ν making the diagram commute.

The morphism t corresponds to a flat family $L \subset Q_{R/J}$ of ℓ-planes. The standing assumption that $S_3 = \varnothing$ means that the singular locus of Q_κ has dimension ≤ 1. Since L_κ has dimension $\ell > 1$, there exists a point $x \in L_\kappa(\kappa)$ not contained in the singular locus of Q_κ. Then since L is smooth over R/J, there exists a section $\bar{\sigma}$ of $L \to \mathrm{Spec}(R/J)$ whose restriction to $\mathrm{Spec}(\kappa)$ is x. Moreover, since $\rho : Q \to S$ is smooth at x, there is a section σ of $\rho_R : Q_R \to \mathrm{Spec}(R)$ extending $\bar{\sigma}$. Then σ is a regular section of Q_R over R, the associated hyperbolic reduction Q' is a quadric $2(\ell-1)$-fold bundle over R, and L induces a flat family $L' \subset Q'_{R/J}$ of $(\ell-1)$-planes. This provides the morphism t' in the commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(R/J) & \xrightarrow{\;t'\;} & \mathbf{F}_{\ell-1}(Q'/R) & \xrightarrow{\;\zeta'_d\;} & \mathscr{C}omplexes_{\mathrm{Ku}(Q')/R} \\ & \searrow^{t} & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \alpha} \\ & & \mathbf{F}_\ell(Q_R/R) & \xrightarrow{\;\zeta_d\;} & \mathscr{C}omplexes_{\mathrm{Ku}(Q_R)/R} \end{array}$$

where ζ_d and ζ'_d are the morphisms from 7.1; β is the closed immersion coming from the identification from 4.11 of $(\ell-1)$-planes on Q' with ℓ-planes on Q_R going through σ; α is the open immersion 3.12 coming from the equivalence of 6.1; and that the square commutes follows from 6.3.

The two commutative diagrams together show that $u : \mathrm{Spec}(R) \to \mathscr{C}omplexes_{\mathrm{Ku}(Q_R)/R}$ restricted to the closed subscheme $\mathrm{Spec}(R/I)$ may be written as

$$u|_{\mathrm{Spec}(R/I)} = \zeta_d \circ t = \alpha \circ \zeta'_d \circ t'.$$

Since α is an open immersion and $\mathrm{Spec}(R)$ and $\mathrm{Spec}(R/I)$ have the same support, u uniquely factors through α, providing a morphism $u' : \mathrm{Spec}(R) \to \mathscr{C}omplexes_{\mathrm{Ku}(Q')/R}$ whose restriction to $\mathrm{Spec}(R/J)$ is the composition $\zeta'_d \circ t'$; in other words, t' and u' together fit into a solid diagram as in $(\star\star)$ for Q'. By induction, ζ'_d is smooth, and so there exists a morphism $\nu' : \mathrm{Spec}(R) \to \mathbf{F}_{\ell-1}(Q'/R)$ filling in the corresponding dashed arrow. It is now straightforward to see that $\nu := \beta \circ \nu' : \mathrm{Spec}(R) \to \mathbf{F}_\ell(Q/S)$ fits as a dashed arrow making the square $(\star\star)$ commute. ■

Suppose now that $\rho : Q \to S$ admits a family $L \subset Q$ of $(\ell-1)$-planes contained in its smooth locus, and let $\mu : M \to S$ be the associated maximal hyperbolic reduction.

Proof of 7.3. The statements may be verified étale locally on S, so passing to a cover, we may additionally assume that L contains a flat family $L' \subset L$ of $(\ell-2)$-planes. Hyperbolic reduction along L' yields a quadric surface bundle $\rho' \colon Q' \to S$. The ℓ-plane L induces a regular section of ρ' and, by comparing functors of points using 4.11, the assoicated maximal hyperbolic reduction may be canonically identified with $\mu \colon M \to S$. Now, iteratively applying 7.5 shows that $\zeta_d|_M = \alpha \circ \zeta'_d|_M$ where α is the isomorphism $\mathscr{M}_d(Q'/S) \cong \mathscr{M}_d(Q/S)$ induced by the equivalence 6.1. In this way, the result is reduced to the surface case, which was established in 7.11. ∎

7.13. Coarse moduli space. — As already mentioned following 7.2, $\mathscr{M}_d(Q/S)$ consists of simple sheaves by 6.4 and is therefore a $\mathbf{G}_m$-gerbe over an algebraic space $\mathrm{M}_d(Q/S)$ which we will refer to as the *coarse moduli space of spinors* on $\rho \colon Q \to S$. By 7.3 and 7.4, étale locally on the source and target, $\mathrm{M}_d(Q/S) \to S$ is identified with a maximal hyperbolic reduction of $\rho \colon Q \to S$. In fact, away from S_2, the morphism $\mathrm{M}_d(Q/S) \to S$ is isomorphic to a maximal hyperbolic reduction étale locally on the target. Over geometric points of S_2 on the other hand, a maximal hyperbolic reduction provides one of the two connected components:

7.14. Lemma. — *The fibre of* $\mathrm{M}_d(Q/S) \to S$ *over a geometric point* $\bar{s} \to S$ *is isomorphic to*

$$\mathrm{M}_d(Q/S)_{\bar{s}} \cong \begin{cases} \bar{s} \sqcup \bar{s} & \text{if } \bar{s} \to S \setminus S_1, \\ \bar{s}[\epsilon] & \text{if } \bar{s} \to S_1 \setminus S_2, \text{ and} \\ \mathbf{P}^1_{\bar{s}} \sqcup \mathbf{P}^1_{\bar{s}} & \text{if } \bar{s} \to S_2. \end{cases}$$

If $\mu \colon M \to S$ *is a maximal hyperbolic reduction of* $\rho \colon Q \to S$*, then the morphism* $\zeta_d|_M \colon M \to \mathrm{M}_d(Q/S)$ *is an isomorphism over* $S \setminus S_2$.

Proof. The statement is local on S, so assume that $\rho \colon Q \to S$ admits a maximal hyperbolic reduction $\mu \colon M \to S$. By 7.3, the morphism $\zeta_d|_M$ is an open immersion. So for the statements away from S_2, it remains to see it is surjective on geometric points. On the one hand, the geometric fibres of $\mu \colon M \to S$ have cardinality 2 over points of $S \setminus S_1$, and 1 over points of $S_1 \setminus S_2$. On the other hand, the fibres of $\mathrm{M}_d(Q/S) \to S$ away from S_2 are 0-dimensional by 7.3, and its geometric fibers have cardinality at most 2 over points of $S \setminus S_1$ and at most 1 over $S_1 \setminus S_2$ since they are surjected on respectively by the Fano scheme of a smooth quadric 2ℓ-fold and a corank 1 quadric 2ℓ-fold. Together, this implies that $\zeta_d|_M$ must be surjective on geometric points away from S_2, as desired.

To analyze geometric fibres over S_2, apply 7.5 to reduce this to the case of a quadric surface bundle, at which point the problem is to show that if Q is a reducible, reduced quadric surface over an algebraically closed field $\mathbf{k}$, then $\mathrm{M}_d(Q/\mathbf{k})$ is a disjoint union of two projective lines. Since $\mathrm{M}_d(Q/\mathbf{k})$ admits a surjection from

$$\mathbf{F}_1(Q/\mathbf{k})^\circ \cong \mathbf{P}^2_{\mathbf{k}} \setminus \{\mathrm{pt}\} \sqcup \mathbf{P}^2_{\mathbf{k}} \setminus \{\mathrm{pt}\}$$

by construction, it too has at most two connected components. Recall from 7.6 that this surjection may be identified with the morphism $\zeta^\circ_{\mathrm{ideal}}$ which takes a line to its ideal sheaf. This implies that the images of the two connected components of $\mathbf{F}_1(Q/\mathbf{k})^\circ$ are disjoint: The ideal sheaf of a line on one irreducible component of Q cannot be isomorphic to the ideal sheaf of a line on the other. Since the images are open by 7.2, $\mathrm{M}_d(Q/\mathbf{k})$ must have exactly two connected components and they consist of ideal sheaves of lines on the respective irreducible components of Q.

Finally, let M be the hyperbolic reduction of Q with respect to a smooth point on an irreducible component $W \subset Q$. It now suffices to show that the open immersion

$$\zeta_{\mathrm{ideal}}|_M \colon M \cong \mathbf{P}^1_{\mathbf{k}} \to \mathrm{M}_d(Q/\mathbf{k})$$

of 7.3 surjects onto the connected component parametrizing ideal sheaves of lines on W. This follows from the fact that the ideal sheaves of two lines on the reducible quadric Q which are not equal to the singular locus of Q are isomorphic if and only if they lie on the same irreducible component and have the same intersection point with the singular locus. See [Add11, Section 3]. ■

These arguments show that, if $\rho\colon Q \to S$ admits a maximal hyperbolic reduction $\mu\colon M \to S$, then the base change of the morphism $M \to \mathrm{M}_d(Q/S)$ of 7.3 to a geometric point of S_2 is the inclusion of one of two connected components of $\mathbf{P}^1 \sqcup \mathbf{P}^1$; moreover, this means that $\mathrm{M}_d(Q/S)$ is in general not separated over S if $S_2 \neq \varnothing$ as points over $S \setminus S_2$ can specialize to points of both connected components of a geometric fiber over a point of S_2. We aim to construct an open subspace of $\mathrm{M}_d(Q/S)$ which is étale locally on the base isomorphic to a maximal hyperbolic reduction. Toward this, observe that the restriction of $\mathrm{M}_d(Q/S) \to S$ to S_2 factors through the canonical double cover $\tilde{S}_2 \to S_2$ of the corank 2 locus provided by 4.12, separating the two projective lines in the fibre. More precisely:

7.15. Proposition. — *Assume that $\rho\colon Q \to S$ satisfies $S = S_2$ scheme-theoretically and $S_3 = \varnothing$. Then the structure morphism $\mathrm{M}_d(Q/S) \to S$ factors through the canonical étale double cover $\tilde{S} \to S$ constructed in 4.12. Furthermore, $\mathrm{M}_d(Q/S) \to \tilde{S}$ has geometrically connected fibres.*

One way to prove this might go as follows: Generalizing [Add11, Lemma 6.1 and Proposition 6.2], the spinor sheaf $\mathscr{S}_d$ associated with an isotropic family of ℓ-planes in $Q \to S$, fibrewise intersecting the singular locus at a single point, has a unique simple Clifford submodule isomorphic to the spinor associated with the corresponding family of $(\ell+1)$-planes. Let $\bar{Q} \to S$ be the smooth quadric bundle of relative dimension $2\ell - 2$ obtained by projecting away from the radical. An analogue of 5.17 shows that this determines a spinor bundle on $\bar{Q}$, thereby producing a morphism $\mathscr{M}_d(Q/S) \to \mathscr{M}_d(\bar{Q}/S)$. Passing to coarse moduli spaces and comparing with 4.12 would then give the result.

Rather than pursuing this route, we prove 7.15 directly by observing that $\tilde{S}_2$ parameterizes geometric connected components of the fibres $\mathbf{F}_\ell(Q/S)^\circ \to S$ over S_2, and that the morphism $\mathbf{F}_\ell(Q/S)^\circ \to \mathrm{M}_d(Q/S)$ induced by ζ_d reflects connected components as it has geometrically connected fibres. We begin with a series of general results, cumulating in a rigidity lemma 7.19, from which we may directly deduce 7.15.

7.16. Lemma. — *Let $f: X \to S$ be a continuous map of topological spaces. Assume that f is open, S is connected, and the fibers $f^{-1}(s) \subseteq X$ are connected. Then X is connected.*

Proof. Since $S \neq \varnothing$ and each fiber of f is non-empty, $X \neq \varnothing$. Suppose $U \subseteq X$ is a non-empty open and closed subset. Then $V := f(U) \subseteq S$ is open and non-empty, and for $s \in S$ we have

$$s \in V \iff f^{-1}(s) \cap U \neq \varnothing \iff f^{-1}(s) \subseteq U \iff s \notin f(X \setminus U)$$

since $f^{-1}(s)$ is connected. Thus $V = S \setminus f(X \setminus U)$ is closed and open, so $V = S$. Applying the equivalences again shows that $f^{-1}(s) \subseteq U$ for all $s \in S$, hence $U = X$. ■

7.17. Lemma. — *Let $f, g: X \to Y$ be morphisms of algebraic spaces over an algebraic space S. Assume that X is connected, Y is separated and étale over S, and that there is $x \in |X|$ such that $f(x) = g(x) \in |Y|$. Then $f = g$.*

Proof. The hypotheses on Y mean that the diagonal $\Delta_Y: Y \to Y \times_S Y$ is both an open and a closed immersion. Thus $U := (f,g)^{-1}(\Delta_Y) \subseteq X$ is an open and closed subspace containing the point x, and so $U = X$ by connectedness. This implies $f = g$. ■

7.18. Lemma. — *Let $p: X \to S$ and $q: Y \to S$ be morphisms of algebraic spaces. Assume p is flat, finitely presented, and has geometrically connected fibers. Assume q is separated and étale. Then any morphism $f: X \to Y$ over S is of the form $\eta \circ p$ for some unique section η of q.*

Proof. Uniqueness of η follows from the fact that $X \to S$ is flat, surjective, and finitely presented since Y is a sheaf for the fppf topology. For existence, it suffices to show that the two morphisms $f \circ \mathrm{pr}_i : X \times_S X \to Y$ are equal. For this, one reduces to the case when S is the spectrum of a local ring, hence connected. Then $X \times_S X$ is connected by 7.16: $X \times_S X \to S$ is open since it is flat and finitely presented, and has connected fibers since the product of geometrically connected spaces of finite type over a field is geometrically connected. Now, for any $x \in |X|$, there is a point $(x,x) \in |X \times_S X|$ and $f \circ \mathrm{pr}_1$ and $f \circ \mathrm{pr}_2$ agree on this point, so by 7.17, the two morphisms are equal. ■

7.19. Lemma. — *Let S be an algebraic space. Let $f: P \to X$ be a morphism of algebraic spaces over S which is flat, finitely presented, and has geometrically connected fibers. Let $Y \to S$ be a separated, étale morphism of algebraic spaces. Then any S-morphism $h: P \to Y$ factors through uniquely through X.*

Proof. We must show that there is a unique S-morphism $g: X \to Y$ such that $h = g \circ f$. Consider the X-morphism $(f,h): P \to X \times_S Y$. Then the data of g is equivalent to a section $s: X \to X \times_S Y$ of pr_1 satisfying $(f,h) = s \circ f$. Replacing S by X, Y by $X \times_S Y$, and X by P then reduces this to 7.18. ■

Proof of 7.15. Consider the Fano incidence correspondence

$$\mathbf{F}_{\ell,\ell+1}(Q/S)^\circ := \left\{([P_\ell],[P_{\ell+1}]) \in \mathbf{F}_\ell(Q/S)^\circ \times_S \mathbf{F}_{\ell+1}(Q/S) : P_\ell \subset P_{\ell+1}\right\}$$

parameterizing flags of ℓ- and $(\ell+1)$-planes in the corank 2 quadric 2ℓ-fold bundle $\rho: Q \to S$. First projection maps this isomorphically onto $\mathbf{F}_\ell(Q/S)^\circ$: Indeed, as in the argument of 4.12, the $(\ell+1)$-plane $P_{\ell+1}$ contains $\mathrm{Sing}\,\rho$, so it may be uniquely determined as the span of $\mathrm{Sing}\,\rho$ and the ℓ-plane P_ℓ—note that P_ℓ intersects the singular locus in exactly one point in each fibre, and that this works in families follows from 4.13. Identifying $\mathbf{F}_\ell(Q/S)^\circ$ with the incidence correspondence provides a morphism $\mathbf{F}_\ell(Q/S)^\circ \to \mathbf{F}_{\ell+1}(Q/S)$, and hence a commutative diagram of solid arrows:

$$\begin{array}{ccc}
\mathbf{F}_\ell(Q/S)^\circ & \xrightarrow{\quad} & \mathrm{M}_d(Q/S) \\
\downarrow & \overset{\zeta_d^\circ}{\swarrow} & \downarrow \\
\tilde{S} & \xrightarrow{\quad\quad} & S\,.
\end{array}$$

We seek to show that there is a morphism $\mathrm{M}_d(Q/S) \to \tilde{S}$ filling in the dashed arrow and making the diagram commute. This would follow from 7.19 upon verifying that ζ_d° has geometrically connected fibres, and this reduces to the situation where S is the spectrum of an algebraically closed field $\mathbf{k}$. The arguments of 7.14 show that points of $\mathrm{M}_d(Q/\mathbf{k})$ correspond to a choice of connected component of $\mathbf{F}_\ell(Q)^\circ$ and a point $x \in \mathrm{Sing}\,Q$. Furthermore, the fibre of ζ_d° over consists of ℓ-planes in the chosen connected component which intersects $\mathrm{Sing}\,Q$ precisely at x. Thus it remains to show that

$$\mathbf{F}_\ell(Q;x)^\circ := \{[P] \in \mathbf{F}_\ell(Q) : P \cap \mathrm{Sing}\,Q = \{x\}\}$$

has just two connected components. But projection from the singular locus identifies this with $\mathbf{F}_{\ell-1}(\bar{Q})$, where $\bar{Q}$ is the smooth quadric $(2\ell-2)$-fold at the base of the cone over $\mathrm{Sing}\,Q$, and the latter has two connected components. ■

7.20. Choosing a family of spinors. — The discussion of 7.14 shows that in order to construct a space locally isomorphic to a maximal hyperbolic reduction of $\rho: Q \to S$ out of $\mathrm{M}_d(Q/S)$, we must be able to consistently choose one of the two connected components in the fibres over the corank 2 locus S_2. This is possible precisely when the double cover $\tilde{S}_2 \to S_2$ splits.

Consider this case, and suppose $\sigma\colon S_2 \to \tilde{S}_2$ is a section of the double cover $\tilde{S}_2 \to S_2$ from 4.12. Pulling back the morphism $\mathrm{M}_d(Q/S)|_{S_2} \to \tilde{S}_2$ from 7.15 along σ produces a closed subspace of $\mathrm{M}_d(Q/S)$ whose open complement

$$M_\sigma := \mathrm{M}_d(Q/S) \setminus (S_2 \times_{\sigma, \tilde{S}_2} \mathrm{M}_d(Q/S)|_{S_2})$$

we shall see is separated over S. Note that here $\tilde{S}_2 = \sigma(S_2) \coprod T$ where T is the image of the complementary section, so the fiber product above is simply the preimage of the closed and open set $\sigma(S_2) \subset \tilde{S}_2$.

7.21. Maximal hyperbolic reduction and sections. — One situation in which $\tilde{S}_2 \to S_2$ splits is when $\rho\colon Q \to S$ carries a regular isotropic $(\ell-1)$-bundle $\mathbf{P}\mathscr{F}$. Let $\mu\colon M \to S$ be the associated maximal hyperbolic reduction, as in 6.5. Consider the closed subscheme

$$Z := \{[P] \in \mathbf{F}_{\ell+1}(Q/S)|_{S_2} : \mathbf{P}\mathscr{F}|_{S_2} \subset P\}$$

parameterizing isotropic $(\ell+1)$-planes over S_2 containing the restriction of $\mathbf{P}\mathscr{F}$. Then 4.11 identifies Z as the scheme of lines of $M \to S$ restricted to S_2: this implies that the map $Z \to S_2$ is an isomorphism. Post-composing the inclusion of Z with the map $\mathbf{F}_{\ell+1}(Q/S)|_{S_2} \to \tilde{S}_2$ from 4.12 arising from the Stein factorization provides a section $S_2 \xrightarrow{\sim} Z \to \tilde{S}_2$. Call the complementary section $\sigma\colon S_2 \to \tilde{S}_2$.

This situation thus provides two distinct ways of obtaining open subspaces of $\mathrm{M}_d(Q/S)$: On the one hand, the maximal hyperbolic reduction $\mu\colon M \to S$ canonically embeds as an open via 7.3 and 7.14. On the other hand, the section $\sigma\colon S_2 \to \tilde{S}_2$ provides an open subspace M_σ via 7.20. The two methods produce the same open subspace:

7.22. Lemma. — *The open immersion $M \to \mathrm{M}_d(Q/S)$ provides an identification $M \cong M_\sigma$.*

Proof. The two opens agree away from the corank 2 locus by 7.14, so we may assume $\rho\colon Q \to S$ is everywhere of corank 2. In this situation, $\tilde{S} \cong \sigma(S) \sqcup Z$ decomposes as a disjoint union of the two sections over S, and the open subspace corresponding to the section σ is taken in 7.20 to be

$$M_\sigma := \mathrm{M}_d(Q/S) \times_{\tilde{S}} Z.$$

As explained in the proof of 7.15, the structure map $\mathbf{F}_\ell(Q/S)^\circ \to S$ factors through $\tilde{S}$, meaning it has two connected components over S. The embedding of 4.11 identifies M as the closed subscheme

$$M \cong \{[P] \in \mathbf{F}_\ell(Q/S)^\circ : \mathbf{P}\mathscr{F} \subset P\}$$

and so, comparing with the definition of $Z \subset \mathbf{F}_{\ell+1}(Q/S)$ from 7.21, the composite map $M \to \tilde{S}$ factors through Z. This implies the result. ■

Write $\mathscr{M}_\sigma$ for the restriction of the $\mathbf{G}_m$-gerbe $\mathscr{M}_d(Q/S) \to \mathrm{M}_d(Q/S)$ to the open $M_\sigma \subset \mathrm{M}_d(Q/S)$.

7.23. Corollary. — *There is an equivalence $M \times B\mathbf{G}_m \cong \mathscr{M}_\sigma$ such that, under the associated equivalence*

$$(M \times B\mathbf{G}_m) \times_S Q \cong \mathscr{M}_\sigma \times_S Q,$$

the universal degree d dual spinor sheaf $\tilde{\mathscr{S}}_d^\vee$ on the right hand side corresponds to the degree d dual spinor sheaf $\mathscr{S}_d^\vee$ associated to the tautological family of ℓ-planes in $M \times_S Q$ placed in $\mathbf{G}_m$-weight one.

Proof. By its definition, the open immersion $M \to M_d(Q/S)$ with image M_σ factors through the morphism $M \to \mathscr{M}_d(Q/S)$ for which the pullback of $\tilde{\mathscr{S}}_d^\vee$ along $M \times_S Q \to \mathscr{M}_d(Q/S) \times_S Q$ is $\mathscr{S}_d^\vee$. By general properties of gerbes, $M \to \mathscr{M}_d(Q/S)$ factors as $M \to M \times B\mathbf{G}_m \to \mathscr{M}_d(Q/S)$ where the first morphism is the tautological section and the second induces the identity on autormorphism groups, see [Stacks, 06QG]. Thus there is a commutative diagram

$$\begin{array}{ccc} M \times B\mathbf{G}_m & \xrightarrow{\cong} & \mathscr{M}_\sigma \\ \uparrow & \nearrow & \downarrow \\ M & \xrightarrow{\cong} & M_\sigma \end{array}$$

The sheaf $\tilde{\mathscr{S}}_d^\vee$ on $\mathscr{M}_\sigma$ has $\mathbf{G}_m$-weight one, and so it corresponds to an object of $\mathbf{G}_m$-weight one under $M \times B\mathbf{G}_m \cong \mathscr{M}_\sigma \times_S Q$ since this equivalence induces the identity on bands. Since the pullback of the corresponding object under the tautological section $M \to M \times B\mathbf{G}_m$ is $\mathscr{S}_d^\vee$, the result follows. ■

7.24. Main result. — We now arrive at the main result of this section, which geometrically identifies the Kuznetsov component of a quadric bundle of even relative dimension in a rather general setting. The description is explicit in the sense that the algebraic space M_σ below is, by 7.22, étale locally on S isomorphic to a maximal hyperbolic reduction of $Q \to S$. In particular, $M_\sigma \to S$ is proper, finite locally free over $S \setminus S_1$, and a $\mathbf{P}^1$-bundle over S_2, with geometric fibres isomorphic to two points over $S \setminus S_1$, $\mathrm{Spec}(\mathbf{k}[\epsilon])$ over $S_1 \setminus S_2$, and $\mathbf{P}^1$ over S_2.

7.25. Theorem. — *Let $\rho\colon Q \to S$ be a quadric bundle of relative dimension 2ℓ. Assume that:*

(i) *$S_3 = \varnothing$;*
(ii) *S_2 contains no weakly associated points of S; and*
(iii) *the étale double cover $\tilde{S}_2 \to S_2$ splits.*

Then each section $\sigma\colon S_2 \to \tilde{S}_2$ provides an open subspace $M_\sigma \subset \mathrm{M}_d(Q/S)$, a Brauer class $\beta \in \mathrm{Br}(M_\sigma)$, and an S-linear equivalence

$$\Phi_\sigma\colon \mathrm{D}_{\mathrm{qc}}(M_\sigma, \beta) \to \mathrm{Ku}(Q).$$

Proof. Fix a section $\sigma\colon S_2 \to \tilde{S}_2$ and let M_σ be the open subspace of $\mathrm{M}_d(Q/S)$ constructed in 7.20. Restricting the coarse moduli map $\mathscr{M}_d(Q/S) \to \mathrm{M}_d(Q/S)$ yields a $\mathbf{G}_m$-gerbe $\mathscr{M}_\sigma \to M_\sigma$. Let $\tilde{\mathscr{S}}_d^\vee$ be the universal d-th dual spinor sheaf on $Q \times_S \mathscr{M}_\sigma$, and consider the associated Fourier–Mukai functor $\Phi\colon \mathrm{D}_{\mathrm{qc}}(\mathscr{M}_\sigma) \to \mathrm{D}_{\mathrm{qc}}(Q)$. As recalled in 1.11, the derived category of $\mathscr{M}_\sigma$ decomposes into homogeneous components

$$\mathrm{D}_{\mathrm{qc}}(\mathscr{M}_\sigma) \cong \prod\nolimits_{k \in \mathbf{Z}} \mathrm{D}_{\mathrm{qc},k}(\mathscr{M}_\sigma) \cong \prod\nolimits_{k \in \mathbf{Z}} \mathrm{D}_{\mathrm{qc}}(M_\sigma, \beta^k)$$

where $\beta \in \mathrm{Br}(M_\sigma)$ is the Brauer class associated with the $\mathbf{G}_m$-gerbe $\mathscr{M}_\sigma \to M_\sigma$.

We claim that the restriction $\Phi_{-1}\colon \mathrm{D}_{\mathrm{qc}}(M_\sigma, \beta^{-1}) \to \mathrm{D}_{\mathrm{qc}}(Q)$ of Φ to the weight -1 component induces an equivalence onto the subcategory $\mathrm{Ku}(Q)$. By 1.23, it suffices to check this upon passing to an étale cover of S. Doing so, we may assume that $\mathscr{M}_\sigma$ is identified with $M \times B\mathbf{G}_m$ with M a maximal hyperbolic reduction of $\rho\colon Q \to S$ as in 7.21 and 7.22. By 7.23, under this equivalence, the kernel $\tilde{S}_d^\vee$ corresponds to the d-th dual spinor sheaf $\mathscr{S}_d^\vee$ on $Q \times_S M_\sigma$ given $\mathbf{G}_m$-weight one. It therefore follows from 6.6 that this object induces the sought-after equivalence

$$\mathrm{D}_{\mathrm{qc},-1}(\mathscr{M}_\sigma) \to \mathrm{Ku}(Q).$$

■

7.26. On the hypotheses. — Theorem 7.25 says that, along with the usual assumptions on the quadric bundle $\rho\colon Q \to S$, once it is possible to consistently choose one of the two families of spinor sheaves over the corank 2 locus—precisely, if the canonical étale cover $\tilde{S}_2 \to S_2$ splits—then there is a Fourier–Mukai equivalence $\Phi_\sigma\colon \mathrm{D}_{\mathrm{qc}}(M_\sigma, \beta) \to \mathrm{Ku}(Q)$. This is, of course, always possible étale locally on S, wherein M_σ may be identified with a suitable maximal hyperbolic reduction of $\rho\colon Q \to S$ by 7.14 and 7.22; moreover, the equivalence Φ_σ is locally isomorphic to the Fourier–Mukai equivalence defined by the kernel featured in **6.6**. The hypothesis **7.25**(iii) turns out to be essentially necessary for a result of this form: in the setting of the Theorem, if for some étale cover of S, the local maximal hyperbolic reductions and kernels glue and descend to an algebraic space $M \to S$ and equivalence $\mathrm{D}_{\mathrm{qc}}(M,\beta) \to \mathrm{Ku}(Q)$ for some $\beta \in \mathrm{Br}(M)$, then the double cover $\tilde{S}_2 \to S$ is necessarily split:

7.27. Proposition. — *Let $\rho\colon Q \to S$ be a quadric 2ℓ-fold bundle with $S_3 = \varnothing$ and S_2 containing no weakly associated points. Suppose there exists*

(i) *an algebraic space $Y \to S$, a $\mathbf{G}_m$-gerbe $\mathscr{Y} \to Y$, and an object $K \in \mathrm{D}_{\mathrm{qc},1}(\mathscr{Y} \times_S Q)$;*
(ii) *an étale cover $\{U_i \to S\}_i$, maximal hyperbolic reductions $\mu_i\colon M_i \to U_i$ of $\rho_{U_i}\colon Q_{U_i} \to U_i$, and objects $K_i \in \mathrm{D}_{\mathrm{qc}}(M_i \times_{U_i} Q_{U_i})$ as in* **6.6***; and*
(iii) *equivalences $\mathscr{Y}_{U_i} \cong M_i \times B\mathbf{G}_m$ over U_i under which K_{U_i} corresponds to K_i with $\mathbf{G}_m$-weight one.*

Then the étale double cover $\tilde{S}_2 \to S_2$ admits a section.

Proof. The object $K \in \mathrm{D}_{\mathrm{qc},1}(\mathscr{Y} \times_S Q)$ induces an S-morphism $\mathscr{Y} \to \mathscr{M}_d(Q/S)$ since this is true locally by (ii) and (iii). This induces an S-morphism $Y \to \mathrm{M}_d(Q/S)$ of $\mathbf{G}_m$-rigidifications. Over S_2, post-composing this with the morphism from **7.15** provides a morphism $f : Y \times_S S_2 \to \tilde{S}_2$. Since each $M_i \to U_i$ is a $\mathbf{P}^1$-bundle, $Y \times_S S_2 \to S_2$ is a Brauer–Severi space of relative dimension 1 and, in particular, has geometrically connected fibres. Thus the rigidity lemma **7.18** applies with f to show that $\tilde{S}_2 \to S$ must admit a section. ■

7.28. Example. — The étale double cover $\tilde{S}_2 \to S_2$ often will not split when $\dim S_2 \geq 1$, suggesting that the Kuznetsov component of quadric bundle over fourfolds, for instance, may not be twisted geometric. When $\dim S_2 = 0$, the cover also might not split for arithmetic reasons: let $S := \mathbf{A}^1_{\mathbf{R}}$ be the affine line with coordinate t, and consider the quadric surface bundle over S given by

$$Q := \{(x_0 : x_1 : x_2 : x_3) \in \mathbf{P}^3_S : tx_0x_1 + x_2^2 + x_3^2 = 0\}.$$

Then $S_2 \subset S$ is the closed subscheme $t = 0$, and the canonical double covering $\tilde{S}_2 \to S_2$ from **4.12** is the Fano scheme of planes in the real quadric surface $x_2^2 + x_3^2 = 0$. It is straightforward to see that this is isomorphic to the non-split étale cover $\operatorname{Spec}\mathbf{C} \to \operatorname{Spec}\mathbf{R}$. ■

8. Complements and applications

In this section, we sketch applications of our results to the derived categories of intersections of quadrics and cubic fourfolds containing a plane over a base field $\mathbf{k}$ of arbitrary characteristic $p \geq 0$. We phrase the results in terms of their bounded derived categories $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}$ of coherent sheaves; that our results imply corresponding results in this setting may be deduced by adapting the arguments of [BS20, Theorem 6.2]. Both examples are classical and well-studied, but our results allow us to give uniform statements and proofs for general $\mathbf{k}$, some of which are new even for $p = 0$.

8.1. Discriminant algebra and cover. — Following [Knu91, IV.4.8], associated with any quadric bundle $\rho\colon Q \to S$ is a graded locally free commutative $\mathscr{O}_S$-algebra $\mathscr{D}isc(\mathscr{E},q)$ called its *discriminant algebra*. This comes with a canonical $\mathscr{O}_S$-algebra morphism

$$\psi\colon \mathscr{D}isc(\mathscr{E},q) \to \mathcal{C}\ell(\mathscr{E},q).$$

For the universal quadric in $\mathbf{P}^n_{\mathbf{Z}}$, this is constructed as the commutant of the Clifford algebra relative to its even subalgebra. In general, $\mathscr{D}isc(\mathscr{E},q)$ is pulled back from the universal case, using the fact that the pair $(\mathscr{D}isc(\mathscr{E},q),\psi)$ is functorial for isometries of quadratic forms over S: see [Knu91, IV.4.4.8.10].

The discriminant algebra behaves differently depending on the parity of $\mathrm{rank}_{\mathscr{O}_S}(\mathscr{E})$. We focus on the case $\mathscr{E}$ has even rank $2\ell+2$, wherein [Knu91, IV.4.4.8.5] implies that the 0-th graded component is a locally free $\mathscr{O}_S$-module of rank 2 fitting into a short exact sequence

$$0 \to \mathscr{O}_S \to \mathscr{D}isc_0(\mathscr{E},q) \to \det(\mathscr{E}) \otimes \mathscr{L}^{\vee,\otimes \ell+1} \to 0.$$

The morphism $\psi_0 \colon \mathscr{D}isc_0(\mathscr{E},q) \to C\ell_0(\mathscr{E},q)$ factors through the centre $Z(C\ell_0(\mathscr{E},q))$ of the 0-th Clifford algebra: compare with [Knu91, IV.4.4.8.1] and [BK94, Theorem 3.7(ii)]. Furthermore, when S is regular and $(\mathscr{E},q)$ is generically nonsingular, then ψ_0 provides an isomorphism $\mathscr{D}isc_0(\mathscr{E},q) \cong Z(C\ell_0(\mathscr{E},q))$: see [Knu91, IV.4.4.8.3]. In any case, the spectrum

$$\mathbf{Disc}(Q/S) := \operatorname{Spec} \mathscr{D}isc_0(\mathscr{E},q)$$

shall be called the *discriminant cover* of S associated with $\rho \colon Q \to S$.

8.2. — An explicit presentation of $\mathscr{D}isc_0(\mathscr{E},q)$ is available locally on S. When the short exact sequence in 8.1 splits, there is a generator z of $\mathscr{D}isc_0(\mathscr{E},q)$ satisfying

$$\mathscr{D}isc_0(\mathscr{E},q) \cong \mathscr{O}_S[z]/(z^2 - \beta z + \gamma)$$

for some sections $\beta \colon \mathscr{O}_S \to \det(\mathscr{E}^\vee) \otimes \mathscr{L}^{\otimes \ell+1}$ and $\gamma \colon \mathscr{O}_S \to \det(\mathscr{E}^\vee)^{\otimes 2} \otimes \mathscr{L}^{\otimes 2\ell+2}$. These sections may be explicitly expressed when $\mathscr{E} \cong \bigoplus_{i=0}^{2\ell+1} \mathscr{E}_i$ furthermore splits into a sum of line bundles: Decompose the section $s_q \colon \mathscr{O}_S \to \operatorname{Sym}^2(\mathscr{E}^\vee) \otimes \mathscr{L}$ corresponding to the equation of Q in $\mathbf{P}\mathscr{E}$ from 4.1 as

$$s_q = \sum\nolimits_{i=0}^{n} \xi_i + \sum\nolimits_{0 \leq i < j \leq n} \xi_{i,j}$$

where $\xi_i \colon \mathscr{O}_S \to \mathscr{E}_i^{\vee,\otimes 2} \otimes \mathscr{L}$ and $\xi_{i,j} \colon \mathscr{O}_S \to \mathscr{E}_i^\vee \otimes \mathscr{E}_j^\vee \otimes \mathscr{L}$. Let T be the set of fix-point free order 2 permutations on the set $\{0,1,\ldots,2\ell+1\}$ and, for each element $\tau = (t_0,t_1)\cdots(t_{2\ell},t_{2\ell+1})$ written as a product of disjoint transposition with $t_{2i} < t_{2i+1}$, write $\xi_\tau := \xi_{t_0,t_1} \cdots \xi_{t_{2\ell},t_{2\ell+1}}$. With this notation, [Knu91, IV.4.4.8.5] gives

$$\beta = \sum\nolimits_{\tau \in T} \operatorname{sign}(\tau)\xi_\tau \quad \text{and} \quad \gamma = \Delta(\xi_0,\ldots,\xi_{2\ell,2\ell+1})$$

where Δ is a universal integral polynomial characterized by the equality $\beta^2 - 4\Delta = (-1)^\ell \det(b_q)$; observe that β is the Pfaffian of the antisymmetric matrix determined by the $\xi_{i,j}$. For example, $\beta = -\xi_{0,1}$ and $\Delta = \xi_0\xi_1$ when $\ell = 1$, and $\beta = \xi_{0,1}\xi_{2,3} - \xi_{0,2}\xi_{1,3} + \xi_{0,3}\xi_{1,2}$ and

$$\begin{aligned}\Delta = -4\xi_0\xi_1\xi_2\xi_3 + \xi_0\xi_1\xi_{2,3}^2 + \xi_0\xi_2\xi_{1,3}^2 + \xi_0\xi_3\xi_{1,2}^2 + \xi_1\xi_2\xi_{0,3}^2 + \xi_1\xi_3\xi_{0,2}^2 + \xi_2\xi_3\xi_{0,1}^2 \\ - \xi_0\xi_{1,2}\xi_{1,3}\xi_{2,3} - \xi_1\xi_{0,2}\xi_{2,3}\xi_{2,3} - \xi_2\xi_{0,1}\xi_{0,3}\xi_{1,3} - \xi_3\xi_{0,1}\xi_{0,2}\xi_{1,2} + \xi_{0,1}\xi_{0,3}\xi_{1,2}\xi_{2,3}\end{aligned}$$

when $\ell = 2$. When 2 is invertible on S, the coordinate change $z \mapsto z + \frac{1}{2}\beta$ identifies $\mathbf{Disc}(Q/S)$ with the usual cyclic double cover of S branched along the discriminant divisor of $\rho \colon Q \to S$.

The discriminant cover is intimately related to the moduli of spinor sheaves as in §7. The following two statements generalize [ABB14, Proposition B.5]:

8.3. Lemma. — *Let $\rho \colon Q \to S$ be a quadric 2ℓ-fold bundle with $S_3 = \varnothing$. There are canonical S-morphisms $\mathscr{M}_d(Q/S) \to \mathrm{M}_d(Q/S) \to \mathbf{Disc}(Q/S)$.*

Proof. Construct an S-morphism $\mathscr{M}_d(Q/S) \to \mathbf{Disc}(Q/S)$ as follows: Let $\tilde{\mu} \colon \mathscr{M}_d(Q/S) \to S$ be the structure morphism and $\mathscr{S}_d^\vee$ the universal spinor on $Q \times_S \mathscr{M}_d(Q/S)$. We claim that the morphism ψ_0 provides $\mathscr{S}_d^\vee$ with a $\mathrm{pr}_2^*\tilde{\mu}^* \mathscr{D}isc_0(\mathscr{E},q)$-module structure. For this, it suffices to show that left multiplication by the discriminant preserves the image of the Clifford multiplication map ϕ_d appearing in the defining presentation of the spinor sheaf in 5.3. This follows from the universal formula for image $z_0 := \psi_0(z)$ in $Z(C\ell_0(\mathscr{E},q))$ of the nontrivial generator of the discriminant algebra together with the formulae in [Knu91, IV.4.8.5 and IV.2.3.2], which imply that for any local section x of $\mathscr{E}$,

$z_0 \cdot x = x \cdot (b - z_0)$ for some local section b of $\mathcal{C}\ell_0(\mathcal{E}, q)$. This module structure now provides a morphism of sheaves of algebras

$$\mathrm{pr}_2^* \widetilde{\mu}^* \mathcal{D}isc_0(\mathcal{E}, q) \to \mathcal{H}om_{\mathcal{O}_{Q \times_S \mathcal{M}_d(Q/S)}}(\mathcal{S}_d^\vee, \mathcal{S}_d^\vee).$$

Since spinors are simple here by 6.4, the target is the structure sheaf. Adjunction for pr_2 provides a map $\widetilde{\mu}^* \mathcal{D}isc_0(\mathcal{E}, q) \to \mathcal{O}_{\mathcal{M}_d(Q/S)}$ of algebras and thus an S-morphism $\mathcal{M}_d(Q/S) \to \mathbf{Disc}(Q/S)$. The universal property of the coarse moduli space map then provides $\mathrm{M}_d(Q/S) \to \mathbf{Disc}(Q/S)$. ∎

The coarse moduli space of spinor sheaves matches with the discriminant cover whenever the quadric bundle $\rho : Q \to S$ has at worst corank 1 fibres:

8.4. Lemma. — *If $S_2 = \varnothing$, then the map $\mathrm{M}_d(Q/S) \to \mathbf{Disc}(Q/S)$ is an isomorphism.*

Proof. Having constructed a canonical S-morphism $\mathrm{M}_d(Q/S) \to \mathbf{Disc}(Q/S)$ in 8.3, whether it is an isomorphism is a question local on S. Replacing S by an fppf cover, we may assume that

$$Q = \Big\{(T_0 : T_1 : \cdots : T_{2\ell+1}) \in \mathbf{P}_S^{2\ell+1} : \gamma_0 T_0^2 - \beta T_0 T_1 + \gamma_1 T_1^2 + \sum\nolimits_{i=1}^{\ell} T_{2i} T_{2i+1} = 0\Big\}$$

for some $\beta, \gamma_0, \gamma_1 \in \mathrm{H}^0(S, \mathcal{O}_S)$. Maximal hyperbolic reduction along $\mathbf{P}W := \mathrm{V}(T_3, T_5, \ldots, T_{2\ell+1})$ may be identified with the quadric bundle

$$M = \{(T_0 : T_1) \in \mathbf{P}_S^1 : \gamma_0 T_0^2 - \beta T_0 T_1 + \gamma_1 T_1^2 = 0\}.$$

Since $S_2 = \varnothing$, $\mathrm{M}_d(Q/S) \cong M$ by 7.14, and so it remains to show that the resulting morphism $M \to \mathbf{Disc}(Q/S)$ is an isomorphism.

For each $0 \leq i \leq 2\ell + 1$, let e_i be the i-th basis vector of $\mathcal{E} \cong \mathcal{O}_S^{\oplus 2\ell+2}$ dual to the coordinate T_i. By [Knu91, IV.4.4.8.5], the generator z of $\mathcal{D}isc_0(\mathcal{E}, q)$ maps, up to a sign, to

$$z \mapsto e_0 e_1 - \beta\Big(\sum\nolimits_{i=1}^{\ell} e_{2i} e_{2i+1}\Big) + \Xi \in \mathcal{C}\ell_0(\mathcal{E}, q)$$

where every term in Ξ involves ≥ 4 basis vectors and contains e_{2i} for some $1 \leq i \leq \ell$. By its construction in 8.3, the map $M \to \mathbf{Disc}(Q/S)$ is determined by the action of z on the universal sheaf $\mathcal{S}_d^\vee$ on $M \times_S Q$ corresponding to the tautological ℓ-plane

$$\mathbf{P}\mathcal{W} := \big\{((T_0 : T_1), (T_0 : T_1 : T_2 : 0 : \cdots : T_{2\ell} : 0)) \in M \times_S Q : \gamma_0 T_0^2 - \beta T_0 T_1 + \gamma_1 T_1^2 = 0\big\}.$$

Since the image of z lies in the centre of $\mathcal{C}\ell_0(\mathcal{E}, q)$, its action on $\mathcal{S}_d^\vee$ is determined by its action on the generator $\det \mathcal{W}$ of the corresponding Clifford ideal, as in 5.2. Then

$$\begin{aligned} z \cdot \det \mathcal{W} = z \cdot (T_0 e_0 + T_1 e_1) \wedge \det W &= e_0 e_1 \cdot (T_0 e_0 + T_1 e_1) \wedge \det W \\ &= \big((\gamma_1 T_1 - \beta T_0) e_0 - \gamma_0 T_0 e_1\big) \wedge \det W \end{aligned}$$

Since the equation of M implies $-\gamma_0 T_0^2 = (\gamma_1 T_1 - \beta T_0) T_1$, the action of z on $\det \mathcal{W}$ may be locally expressed on the standard open cover $\mathrm{D}_+(T_1) \cup \mathrm{D}_+(T_0)$ of $M \subset \mathbf{P}_S^1$ as

$$z \cdot \det \mathcal{W} = \begin{cases} -\gamma_0 T_0 T_1^{-1} \det \mathcal{W} & \text{on } \mathrm{D}_+(T_1), \text{ and} \\ (\gamma_1 T_1 + \beta T_0) T_0^{-1} T_1 \det \mathcal{W} & \text{on } \mathrm{D}_+(T_0). \end{cases}$$

By 8.5, these scalars locally represent the generator of $\mu_* \mathcal{O}_M$. Therefore the map $\mathcal{D}isc_0(\mathcal{E}, q) \to \mu_* \mathcal{O}_M$ between locally free rank 2 $\mathcal{O}_S$-algebras matches the generators, so it is an isomorphism. ∎

The following gives an explicit description of the relative coordinate algebra of a family of quadrics of relative dimension 0. Note it holds even when there may be some corank 2 fibres.

8.5. Lemma. — *Let $\mu\colon M \to S$ be cut out in $\mathbf{P}^1_S$ by a regular section $\gamma_0 T_0^2 - \beta T_0 T_1 + \gamma_1 T_1^2$. Then*

$$\mu_* \mathcal{O}_M \cong \mathcal{O}_S[z]/(z^2 - \beta z + \gamma_0\gamma_1)$$

as a free rank 2 $\mathcal{O}_S$-algebra. A global section $\tilde{z}\colon \mathcal{O}_M \to \mathcal{O}_M$ representing z may be locally given by

$$\tilde{z}|_{\mathrm{D}_+(T_i)} = \begin{cases} \gamma_0 T_0 T_1^{-1} & \text{on } \mathrm{D}_+(T_1), \text{ and} \\ (\beta T_0 - \gamma_1 T_1) T_1 T_0^{-1} & \text{on } \mathrm{D}_+(T_0). \end{cases}$$

Proof. Compute $\mu_*\mathcal{O}_M$ using the Čech complex for the standard open cover $\mathbf{P}^1_S = \mathrm{D}_+(T_1) \cup \mathrm{D}_+(T_0)$:

$$\mathcal{O}_S[s]/(\gamma_0 s^2 - \beta s + \gamma_1) \times \mathcal{O}_S[t]/(\gamma_0 - \beta t + \gamma_1 t^2) \to \mathcal{O}_S[s, s^{-1}]/(\gamma_0 s^2 - \beta s + \gamma_1)$$

where $s := T_0 T_1^{-1}$, $t := T_1 T_0^{-1}$, and the differential is $(f(s), g(t)) \mapsto f(s) - g(s^{-1})$. The element $\tilde{z} := (\gamma_0 s, \beta - \gamma_1 t)$ lies in the kernel, so defines a section $z\colon \mathcal{O}_S \to \mu_*\mathcal{O}_M$. A direct computation shows that $\tilde{z}^2 - \beta\tilde{z} + \gamma_0\gamma_1 = 0$, and so z satisfies the same relation.

It remains to see that 1 and z freely generate $\mu_*\mathcal{O}_M$ as an $\mathcal{O}_S$-module. By the short exact sequence

$$0 \to \mathcal{O}_S \to \mu_*\mathcal{O}_M \to \mathrm{R}^1\pi_*\mathcal{O}_{\mathbf{P}^1_S}(-2) \to 0,$$

we must show that z maps to a generator of $\mathrm{R}^1\pi_*\mathcal{O}_{\mathbf{P}^1_S}(-2)$. The boundary map is computed by viewing $\tilde{z} = (\gamma_0 s, \beta - \gamma_1 t)$ as a local section of $\mathcal{O}_{\mathbf{P}^1_S}$, applying the Čech differential to obtain the element

$$\gamma_0 s - \beta + \gamma_1 t = (\gamma_0 T_0^2 - \beta T_0 T_1 + \gamma_1 T_1^2) \cdot \frac{1}{T_0 T_1},$$

and dividing out the equation $\gamma_0 T_0^2 - \beta T_0 T_1 + \gamma_1 T_1^2$ of M in $\mathbf{P}^1_S$. This means that z maps to the generator $1/T_0 T_1$ of $\mathrm{R}^1\pi_*\mathcal{O}_{\mathbf{P}^1_S}$, as desired. ■

8.6. Intersection of quadrics. — As a first application of our results, consider a complete intersection $X \subset \mathbf{P}^{2\ell+1}$ of m even-dimensional quadrics over an arbitrary field $\mathbf{k}$. Write $\rho\colon Q \to \mathbf{P}^{m-1}$ for the corresponding family of quadrics over the associated linear system. Whenever $n + 2 \geq 2m$, Kuznetsov gives in [Kuz08, Theorem 5.5] a semiorthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) = \langle \mathrm{Ku}(Q), \mathcal{O}_X(1), \mathcal{O}_X(2), \ldots, \mathcal{O}_X(n+2-2m)\rangle$$

where $\mathrm{Ku}(Q)$ is the Kuznetsov component of the quadric bundle $\rho\colon Q \to \mathbf{P}^{m-1}$ as in the Introduction. Combined with 7.25, this shows that the Kuznetsov component of the intersection of at most 4 quadrics is twisted geometric:

8.7. Proposition. — *Let $X \subset \mathbf{P}^{2\ell+1}$ be a complete intersection of $m \leq \ell + 1$ quadrics with corresponding quadric 2ℓ-fold bundle $\rho\colon Q \to \mathbf{P}^{m-1}$. Assume that X and Q are smooth over $\mathbf{k}$.*

(i) *If $m = 2$, then there is a semiorthogonal decomposition*

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) = \langle \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(C, \alpha), \mathcal{O}_X(1), \mathcal{O}_X(2), \ldots, \mathcal{O}_X(2\ell - 2)\rangle$$

where C is a smooth curve with a finite flat map $C \to \mathbf{P}^1$ of degree 2 and $\alpha \in \mathrm{Br}(C)$.

Assume henceforth that $\mathbf{k}$ is algebraically closed and $\rho\colon Q \to \mathbf{P}^{m-1}$ has finitely many corank 2 fibres.

(ii) *If $m = 3$, then there is a semiorthogonal decomposition*

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) = \langle \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(S, \alpha), \mathcal{O}_X(1), \mathcal{O}_X(2), \ldots, \mathcal{O}_X(2\ell - 4)\rangle$$

where $S \to \mathbf{P}^2$ is a proper generically finite morphism of degree 2 from a smooth projective surface and $\alpha \in \mathrm{Br}(S)$.

(iii) *If $m = 4$ and $\rho: Q \to \mathbf{P}^3$ has no corank 3 fibres, then there is a semiorthogonal decomposition*

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) = \langle \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(M,\alpha), \mathcal{O}_X(1), \mathcal{O}_X(2), \ldots, \mathcal{O}_X(2\ell-6)\rangle$$

where $M \to \mathbf{P}^3$ is a proper generically finite morphism of degree 2 from a smooth proper algebraic space and $\alpha \in \mathrm{Br}(M)$.

Proof. Smoothness of X implies that the corank $\geq m$ locus of $\rho: Q \to \mathbf{P}^{m-1}$ is empty. Combined with the assumptions, it is straightforward to see that the hypotheses of 7.25 are satisfied, whereupon the decompositions in each case follow from the discussion of 8.6.

Regarding the geometric properties: Smoothness of Q implies smoothness of each of C, S, and M. When $m = 1$, $C \to \mathbf{P}^1$ is flat because $\rho: Q \to \mathbf{P}^1$ has no corank ≥ 2 fibres or simply because C and $\mathbf{P}^1$ are smooth curves, C is a scheme by [Stacks, 0ADD]. Finally, when $m = 2$, S is a scheme by a result of Artin in [Art73, Théorème 4.7]. ■

8.8. Remarks. — A few remarks regarding this are in order:

- Bertini's theorem shows that the geometric hypotheses in 8.7 hold for general $\rho: Q \to \mathbf{P}^{m-1}$.
- When $\mathrm{char}\,\mathbf{k} \neq 2$, smoothness of Q implies that the general fibre of $\rho: Q \to \mathbf{P}^{m-1}$ is smooth. In particular, each cover of $\mathbf{P}^{m-1}$ is generically étale. However, when $\mathrm{char}\,\mathbf{k} = 2$, it is possible that every fibre of $\rho: Q \to \mathbf{P}^{m-1}$ has corank ≥ 1, in which case the covers of $\mathbf{P}^{m-1}$ would be generically purely inseparable. Examples of this nature do occur.
- 8.7(ii) improves on [Xie23, Proposition 6.5] and verifies expectations remarked there.
- 8.7(iii) generalizes [Add09] from the complex numbers to any algebraically closed field.
- The algebraic space M appearing in 8.7(iii) may sometimes be non-schematic, and this can sometimes be verified explicitly: see [CPZ25, Proposition 3.12] for an example.
- The methods here may be adapted to study families of complete intersections of quadrics, as considered already in [ABB14, §5].

8.9. Cubic fourfolds. — As a second application of our results, consider a smooth cubic fourfold $X \subset \mathbf{P}^5$ over an algebraically closed field $\mathbf{k}$ of arbitrary characteristic. As mentioned in the Introduction, the derived category of X has a well-known semiorthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2)\rangle$$

where the residual component $\mathcal{A}_X$ looks like the derived category of a K3 surface

At least when $\mathrm{char}\,\mathbf{k} \neq 2$, a particularly simple and well-studied case in which $\mathcal{A}_X$ is equivalent to the twisted derived category of a K3 surface is when X contains a plane. The following completes the picture for arbitrary base field characteristics:

8.10. Theorem. — *Let $X \subset \mathbf{P}^5$ be a smooth cubic fourfold over an algebraically closed field $\mathbf{k}$. Assume that X contains a plane P. Then there is a semiorthogonal decomposition*

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) = \langle \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(S,\alpha), \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2)\rangle$$

where S is a K3 surface of degree 2 and $\alpha \in \mathrm{Br}(S)$.

Proof. Fix a plane $P \subset X$. Linear projection centred at P is resolved on the blowup $b: \widetilde{X} \to X$ along P. The resulting morphism $a: \widetilde{X} \to \mathbf{P}^2$ is a quadric surface bundle. The Fano scheme $\mathbf{F}_2(X)$ is a finite set of reduced points by an unpublished result of Debarre in [Deb03, Lemma 3]—see also Starr's treatment in [BHB06, Appendix]—and so $a: \widetilde{X} \to \mathbf{P}^2$ has only finitely many corank 2 fibres and no corank 3 fibres. Since $\mathbf{k}$ is algebraically closed, the étale cover over the corank 2 locus from 4.12 has a section. The hypotheses of 7.25 thus are satisfied, and so there is a semiorthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\widetilde{X}) = \langle \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(S,\alpha), La^*\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathbf{P}^2)\otimes^L \mathscr{O}_a, La^*\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathbf{P}^2)\otimes^L \mathscr{O}_a(1)\rangle$$

where $S \to \mathbf{P}^2$ is an open in the moduli space of spinor sheaves along $a\colon \widetilde{X} \to \mathbf{P}^2$ and $\alpha \in \mathrm{Br}(S)$. The mutation argument in [Kuz10, pp. 228–230] remains valid here and yields an equivalence

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(S,\alpha) \simeq \mathscr{A}_X.$$

Finally, remark that S is a K3 surface: it is smooth since $\widetilde{X}$ is; $\omega_S \cong \mathscr{O}_S$ since $\mathscr{A}_X$ is a K3 category; and $\mathrm{H}^1(S,\mathscr{O}_S) = 0$ thanks to Noether's formula $12\chi(S,\mathscr{O}_S) = c_2(S)^2$ and the fact that the Hochschild homology of $\mathscr{A}_X$ recovers the Betti numbers of a K3 surface. ■

The final few paragraphs give explicit models for the K3 surface S. Without further assumptions on X, the discriminant cover $S_0 := \mathbf{Disc}(\widetilde{X}/\mathbf{P}^2)$ of $a\colon \widetilde{X} \to \mathbf{P}^2$ from 8.1 provides perhaps the most convenient model for S. Note that 8.3 and 8.4 give a canonical birational morphism $S \to S_0$ over $\mathbf{P}^2$ which is an isomorphism away from from the corank 2 locus of $a\colon \widetilde{X} \to \mathbf{P}^2$.

8.11. — To give an equation for S_0, choose projective coordinates $(x_0 : x_1 : x_2 : y_0 : y_1 : y_2)$ for $\mathbf{P}^5$ such that the plane in X is given by $P = \mathrm{V}(y_0,y_1,y_2)$, and identify $(y_0 : y_1 : y_2)$ as the coordinates on the $\mathbf{P}^2$ at the base of linear projection $a\colon \widetilde{X} \to \mathbf{P}^2$. The equation of X takes the form

$$f = g(y_0,y_1,y_2) + \sum\nolimits_{i=0}^{2} q_i(y_0,y_1,y_2)x_i + \sum\nolimits_{0\le i\le j\le 2} \ell_{i,j}(y_0,y_1,y_2)x_ix_j$$

where $g \in \mathrm{H}^0(\mathbf{P}^2,\mathscr{O}_{\mathbf{P}^2}(3))$, $q_i \in \mathrm{H}^0(\mathbf{P}^2,\mathscr{O}_{\mathbf{P}^2}(2))$, and $\ell_{i,j} \in \mathrm{H}^0(\mathbf{P}^2,\mathscr{O}_{\mathbf{P}^2}(1))$. The quadratic form defining $\widetilde{X}$ may be defined by the non-symmetric bilinear form

$$\begin{pmatrix} \ell_{0,0} & \ell_{0,1} & \ell_{0,2} & q_0 \\ & \ell_{1,1} & \ell_{1,2} & q_1 \\ & & \ell_{2,2} & q_2 \\ & & & g \end{pmatrix} : \mathscr{O}_{\mathbf{P}^2}^{\oplus 3} \oplus \mathscr{O}_{\mathbf{P}^2}(-1) \to \mathscr{O}_{\mathbf{P}^2}(1)^{\oplus 3} \oplus \mathscr{O}_{\mathbf{P}^2}(2).$$

By 8.2, $S_0 \to \mathbf{P}^2$ is the flat double cover given by $z^2 - (\ell_{0,1}q_2 - \ell_{0,2}q_1 + \ell_{1,2}q_0)z + \gamma = 0$ where

$$\begin{aligned}\gamma := (\ell_{0,0}\ell_{1,2}^2 + \ell_{1,1}\ell_{0,2}^2 + \ell_{2,2}\ell_{0,1}^2 - \ell_{0,1}\ell_{0,2}\ell_{1,2} - 4\ell_{0,0}\ell_{1,1}\ell_{2,2})g + \ell_{0,1}\ell_{1,2}q_0q_2 \\ + \ell_{1,1}\ell_{2,2}q_0^2 + \ell_{0,0}\ell_{2,2}q_1^2 + \ell_{0,0}\ell_{1,1}q_2^2 - \ell_{0,0}\ell_{1,2}q_1q_2 - \ell_{1,1}\ell_{0,2}q_0q_2 - \ell_{2,2}\ell_{0,1}q_0q_1.\end{aligned}$$

8.12. — One particularly pleasant situation in which global equations for S can be given is when X contains a second plane P' disjoint from P. Choose coordinates as in 8.11 such that $P = \mathrm{V}(y_0,y_1,y_2)$ and, furthermore, $P' = \mathrm{V}(x_0,x_1,x_2)$. The equation of X may then be written as the sum

$$f = f_{1,2}(x_0,x_1,x_2;y_0,y_1,y_2) + f_{2,1}(x_0,x_1,x_2;y_0,y_1,y_2)$$

where $f_{1,2}$ and $f_{2,1}$ are bihomogeneous of bidegrees $(1,2)$ and $(2,1)$, respectively. Viewed as functions on $\mathbf{P}^2 \times \mathbf{P}^2$, these cut out S: see [Has16, §1.2].

8.13. — As an explicit example, consider the cubic fourfold

$$X := \{(x_0 : x_1 : x_2 : y_0 : y_1 : y_2) \in \mathbf{P}^5 : x_0^2y_0 + x_1^2y_1 + x_2^2y_2 + x_0y_0^2 + x_1y_1^2 + x_2y_2^2 = 0\}.$$

By 8.11, the discriminant cover $S_0 \to \mathbf{P}^2$ is given by $z^2 - y_0y_1y_2(y_0^3 + y_1^3 + y_2^3) = 0$. In particular, S_0 is singular above points of intersection amongst the various components of the divisor appearing. By 8.12, S_0 is resolved by the K3 surface S may be given as

$$S := \{((x_0 : x_1 : x_2),(y_0 : y_1 : y_2)) \in \mathbf{P}^2 \times \mathbf{P}^2 : x_0^2y_0 + x_1^2y_1 + x_2^2y_2 = x_0y_0^2 + x_1y_1^2 + x_2y_2^2 = 0\}.$$

When $\mathbf{k}$ has characteristic 2, both [DK03, Theorem 1.1(vi) and (viii)] show that S is the supersingular K3 surface of Artin invariant 1. Since X here is a smooth q-bic fourfold with $q = 2$ in the sense of [Che25], this is isomorphic to the Fermat cubic fourfold. This completes the proof of Theorem E. ■

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