Geometry of *q*-bic Hypersurfaces

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Abstract

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Traditional algebraic geometric invariants lose some of their potency in positive characteristic. For instance, smooth projective hypersurfaces may be covered by lines despite being of arbitrarily high degree. The purpose of this dissertation is to define a class of hypersurfaces that exhibits such classically unexpected properties, and to offer a perspective with which to conceptualize such phenomena.

Specifically, this dissertation proposes an analogy between the eponymous *q-bic hypersurfaces*—special hypersurfaces of degree $q+1$, with q any power of the ground field characteristic, a familiar example given by the corresponding Fermat hypersurface—and low degree hypersurfaces, especially quadrics and cubics. This analogy is substantiated by concrete results such as: q-bic hypersurfaces are moduli spaces of isotropic vectors for a bilinear form; the Fano schemes of linear spaces contained in a smooth q-bic hypersurface are smooth, irreducible, and carry structures similar to orthogonal Grassmannian; and the intermediate Jacobian of a q-bic threefold is purely inseparably isogenous to the Albanese variety of its smooth Fano surface of lines.

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Introduction

Space over a field of positive characteristic is curved and arranged in surprising ways. Imagine the projective plane. Over the complex numbers, it is impossible to find a set of 7 points such that every triple is collinear. Yet, over a field of characteristic 2, the Fano plane is just this: see Figure 1. More generally, for any power q of a prime p, the $\mathbf{F}_q$-rational points and lines in the projective plane over a field of characteristic p provide a configuration with more collinearities than permitted in Euclidean space.

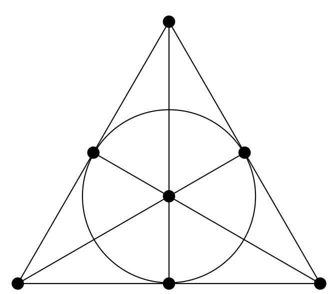

FIGURE 1. This is a depiction of the *Fano plane*, namely, the arrangement of $\mathbf{F}_2$-rational points and lines in the projective plane $\mathbf{P}^2$ over a field of characteristic 2. The arcs indicate the lines and the dots their intersection points. Curiously, the three midpoints of the triangle are collinear and their common line is disjoint from the three bisectors.

Fundamentally, these and other exotic shapes arise because systems of polynomial equations are, in a sense, easier to solve in positive characteristic than any potential complex counterpart. This leads to a degree of flexibility in geometric possibilities that is sometimes startling to classical intuition. Notoriously, first-order differential invariants lose some of their potency, leading to new possibilities such as inseparable morphisms, fibrations with singular general fibre, and general type varieties which are unirational.

Perhaps this is a glimmer to a glittering Atlantis. But in seeking treasure, one is immediately confronted with two questions: where to look, and what to look for? With charts of the seascape ahead partial at best, and instruments on hand designed for other purposes, one navigates by star and by instinct. With some luck, one returns a bit older but, hopefully, a bit wiser, with a map on which Lilliput is freshly inked, and maybe—just maybe—with a tiny pearl in hand.

This dissertation is the humble result of one such expedition.

Main results

The purpose of this dissertation is to systematically study the geometry of a class of hypersurfaces in positive characteristic which I call *q-bic hypersurfaces*. The contributions of this work come in two flavours:

First is a systematic development of an intrinsic algebraic and global geometric theory with which to study q-bic hypersurfaces. As shall be indicated below, q-bic hypersurfaces are of recurring interest, and I hope the theory developed here brings some unity and clarity to the fascinating discoveries made by authors before, and provides foundations for studying these and related objects for authors to come.

Second is the beginnings of an analogy between the projective geometry of q-bic hypersurfaces with that of quadric and cubic hypersurfaces. In the past, the distinctive properties of q-bic hypersurfaces were often villainized as "pathological", but I hope that this perspective offers a new light with which their idiosyncrasies may be conceptualized and appreciated. Looking forward, I hope this can inform the type of questions asked of and answers sought from the geometry of q-bic hypersurfaces and other similar objects.

The remainder of this Section details some concrete statements. But frankly, the true thesis in this manuscript is that q-bic hypersurfaces deserve their name: that they ought to be singled out and viewed as hypersurfaces belonging to one family; that they are reminiscent of $q$uadrics; and that the homophony is justified. To the extent I have proven this point is for you, dear reader, to judge.

Throughout, q is a power of a prime p, and $\mathbf{k}$ is a field of characteristic p.

q-bic forms. — The first contribution is an intrinsic algebraic theory of objects I call *q-bic forms*: over a field $\mathbf{k}$, these are bilinear forms

$$\beta\colon \mathrm{Fr}^*(V) \otimes_{\mathbf{k}} V \to \mathbf{k}$$

between a finite-dimensional vector space V and its q-power Frobenius twist $\mathrm{Fr}^*(V) := \mathbf{k} \otimes_{\mathrm{Fr},\mathbf{k}} V$; equivalently, q-bic forms are determined by sesquilinear forms on V which are q-linear in the first variable.

These forms do not enjoy any simple symmetry property; after all, the pairing relates distinct vector spaces. When the base field is $\mathbf{F}_{q^2}$, however, the q-power Frobenius is an involution and β may be asked to be *Hermitian* in the sense

$$\beta(v^{(q)}, w) = \beta(w^{(q)}, v)^q \quad \text{for all } v, w \in V$$

where $v^{(q)} := 1 \otimes v \in \mathrm{Fr}^*(V)$. For general $\mathbf{k}$, as will be explained in the text, a q-bic form nevertheless determines a Hermitian form on a $\mathbf{F}_{q^2}$-subvector space in V. This serves as a powerful arithmetic invariant of q-bic forms.

What does distinguish a q-bic form from a general bilinear form is the canonical q-linear map $(-)^{(q)}\colon V \to \mathrm{Fr}^*(V)$. This severely constrains the structure of q-bic forms, and makes it possible to give a classification over algebraically closed fields:

Theorem A. — *Let (V, β) be a q-bic form over an algebraically closed field. Then there exists a basis $V = \langle e_0, \ldots, e_n \rangle$ such that*

$$\mathrm{Gram}(\beta; e_0, \ldots, e_n) = \mathbf{N}_1^{\oplus a_1} \oplus \cdots \oplus \mathbf{N}_m^{\oplus a_m} \oplus \mathbf{1}^{\oplus b}$$

for some nonnegative integers m, and $a_1, \ldots, a_m, b$ satisfying $b + \sum_{k=1}^m k a_k = n + 1$.

See 1.4.1, and 1.1.12 and 1.1.15 for the notation. In particular, over an algebraically closed field, there are only finitely many isomorphism classes of q-bic forms of a given dimension.

q-bic hypersurfaces. — The subscheme X of $\mathbf{P}V$ parameterizing isotropic vectors for a nonzero q-bic form $\beta \colon \mathrm{Fr}^*(V) \otimes_{\mathbf{k}} V \to \mathbf{k}$ is a hypersurface of degree $q+1$: this is the *q-bic hypersurface* associated with β. An equation for X is obtained by pairing the coordinates of $\mathbf{P}V$ according to β. For example, if there were a basis $V = \langle e_0, \ldots, e_n \rangle$ such that

$$\beta(e_i^{(q)}, e_j) = \begin{cases} 1 & \text{if } i = j, \text{ and} \\ 0 & \text{if } i \neq j, \end{cases}$$

then in the corresponding coordinates $(x_0 : \cdots : x_n)$ of $\mathbf{P}V = \mathbf{P}^n$, the associated q-bic hypersurface is the Fermat hypersurface of degree $q+1$:

$$X = \mathrm{V}(x_0^{q+1} + x_1^{q+1} + \cdots + x_n^{q+1}) \subset \mathbf{P}^n.$$

Defined in this way, q-bic hypersurfaces are clearly similar to quadric hypersurfaces. This perspective is not entirely new: authors have perennially observed that q-bic hypersurfaces may be specified by something like a bilinear form and that certain geometric features are encoded by the algebra of the form. The novelty here, and what forms the second contribution of this work, is a systematic development of this perspective, in which global geometric properties of the hypersurface X are intrinsically expressed through the underlying form β. This brings scheme-theoretic methods to bear and, I believe, clarifies some of the geometric curiosities previously found about q-bics. For instance, these methods give an improvement and two new geometric proofs of Shioda's old observation from [Shi74] that smooth q-bic surfaces are unirational:

Theorem B. — *A smooth q-bic hypersurface X of dimension at least 2 with a $\mathbf{k}$-rational line admits a purely inseparable unirational parameterization of degree q.*

See 2.5.1; see 2.5.2 for a geometric view on Shioda's construction; and see 2.5.11 for the new construction.

From this perpsective, the Fano schemes $\mathbf{F}_r(X)$ classifying projective r-planes contained in X acquire an alternative moduli interpretation: $\mathbf{F}_r(X)$ parameterizes $(r+1)$-dimensional subspaces of V which are isotropic for β. These Fano schemes therefore find analogy with orthogonal Grassmannians. The third contribution of this work is to describe some of the basic geometric features of these schemes, a summary of which is as follows:

Theorem C. — *Let X be any q-bic hypersurface of dimension $(n-1)$. Then for each $0 < r < \frac{n}{2}$, its Fano scheme $\mathbf{F}_r(X)$ of r-planes*

- (i) *is nonempty,*
- (ii) *has dimension at least $(r+1)(n-2r-1)$,*
- (iii) *is connected whenever $n \geq 2r+2$, and*
- (iv) *is smooth of dimension $(r+1)(n-2r-1)$ at a point corresponding to an r-plane $\mathbf{P}U \subset X$ contained in the smooth locus of X.*

See 2.7.3, 2.7.4, 2.7.7, and 2.7.15. This is simplest when X is smooth:

Theorem D. — *Let X be a smooth q-bic hypersurface of dimension $(n-1)$. Then its Fano scheme $\mathbf{F}_r(X)$ of r-planes is smooth, irreducible, of dimension*

$$\dim \mathbf{F}_r(X) = (r+1)(n-2r-1) \quad \textit{whenever } 0 < r < \tfrac{n}{2}$$

and empty otherwise, and has canonical bundle

$$\omega_{\mathbf{F}_r(X)} \cong \mathcal{O}_{\mathbf{F}_r(X)}\big((q+1)(r+1)-(n+1)\big).$$

See 2.7.16. Here, $\mathcal{O}_{\mathbf{F}_r(X)}(1)$ is the Plücker line bundle restricted from the ambient Grassmannian. This gives an interesting collection of smooth projective varieties in positive characteristic depending on the three discrete parameters q, r, and n. Perhaps a particularly interesting sequence occurs upon fixing the prime power q

and positive integer r, and taking $n+1 = (q+1)(r+1)$, yielding smooth projective varieties of dimension $(r+1)^2(q-1)$ with trivial canonical bundle.

q-bic threefolds. — Whereas the coarse global geometry of q-bic hypersurfaces resembles that of quadrics, finer aspects are more closely analogous to that of cubic hypersurfaces. This is most striking in its geometry of lines. For example, a smooth q-bic surface contains $(q+1)(q^3+1)$ lines; when $q=2$, a q-bic surface is indeed a cubic surface, and this recovers the storied 27 lines.

The final contribution of this work is to elaborate on this analogy in the case X is a q-bic threefold. The main result to this effect is an analogue of the theorem of Clemens and Griffiths from [CG72, Theorem 11.19] which says that, over the complex numbers, the intermediate Jacobian of a cubic threefold is isomorphic to the Albanese of its Fano surface of lines. To formulate the result, observe that, as in the case of a cubic threefold, the Fano scheme $S := \mathbf{F}_1(X)$ of lines in X is a smooth projective surface. By the work of Murre in [Mur83, Mur85], there is an abelian variety $\mathbf{Ab}^2_X$ which parameterizes algebraically trivial 1-cycles in X and which captures the algebraic part of the intermediate Jacobian. Pulling and pushing algebraically trivial 0-cycles along the incidence correspondence $S \leftarrow \mathbf{L} \rightarrow X$ induces a natural morphism $\mathbf{L}_* \colon \mathbf{Alb}_S \rightarrow \mathbf{Ab}^2_X$. The result is:

Theorem E. — *The incidence correspondence* $\mathbf{L}$ *induces a purely inseparable isogeny*

$$\mathbf{L}_* \colon \mathbf{Alb}_S \rightarrow \mathbf{Ab}^2_X$$

from the Albanese variety of S to the intermediate Jacobian of X. Furthermore, every choice of Hermitian line $\ell \subset X$ induces a purely inseparable isogeny

$$\prod\nolimits_{i=0}^{q^2} \mathbf{Jac}_{C_i} \xrightarrow{\nu_*} \mathbf{Alb}_S \xrightarrow{\mathbf{L}_*} \mathbf{Ab}^2_X$$

from a product of Jacobians of smooth q-bic curves.

See 4.7.21. A Hermitian line a special line contained in X. For example, in the case X is the Fermat q-bic, a Hermitian line is the same as a $\mathbf{F}_{q^2}$-rational line. I do not yet know whether $\mathbf{L}_*$ is actually an isomorphism.

The geometry of the surface S is rich. Some of its basic properties and numerical invariants are as follows:

Theorem F. — *The Fano surface S is of general type with $\omega_S \cong \mathcal{O}_S(2q-3)$. Moreover:*

(i) *If $q > 2$, then S does not lift to $\mathrm{W}_2(\mathbf{k})$.*

(ii) *The ℓ-adic Betti numbers of S are given by*

$$\dim_{\mathbf{Q}_\ell} \mathrm{H}^i_{\text{ét}}(S, \mathbf{Q}_\ell) = \begin{cases} 1 & \text{if } i = 0 \text{ or } i = 4, \\ q(q-1)(q^2+1) & \text{if } i = 1 \text{ or } i = 3, \text{ and} \\ (q^4 - q^3 + 1)(q^2+1) & \text{if } i = 2. \end{cases}$$

(iii) *If $q = p$, the coherent Betti numbers of S are given by*

$$\dim_{\mathbf{k}} \mathrm{H}^i(S, \mathcal{O}_S) = \begin{cases} 1 & \text{if } i = 0, \\ p(p-1)(p^2+1)/2 & \text{if } i = 1, \text{ and} \\ p(p-1)(5p^4 - 2p^2 - 5p - 2)/12 & \text{if } i = 2. \end{cases}$$

See 4.7.1, 4.7.3, 4.7.7, and 4.7.28. Étale cohomology is computed by carefully relating S to a Deligne–Lusztig variety for the finite unitary group $\mathrm{U}_5(q)$. Coherent cohomology is computed by degenerating S to the Fano scheme S_0 of lines on a mildly singular q-bic threefold, computing cohomology on the singular surface with help from the modular representation theory of $\mathrm{U}_3(q)$, and then relating the cohomology of S_0 with S along the specially chosen degeneration.

Comments on the literature

Often in the guise of special equations, q-bic hypersurfaces have attracted the attention of mathematicians time and time again. These hypersurfaces tend to be distinguished via extremal geometric or arithmetic properties. The following is a sampling of works, roughly organized by topic, featuring q-bic hypersurfaces in some way. I have tried to be comprehensive, but the collection is surely incomplete: I apologize for the inevitable omission.

Projective duality. — One classical source of interest stems from the theory of projective duality: since Wallace observed in [Wal56, §7] that every point of a smooth q-bics curve is a flex, authors have characterized q-bic hypersurfaces amongst all projective hypersurfaces via exceptional properties of their duals: see [Par86, Propostion 3.7], [Hom87], [Hom89, Corollary 2.5], and [Hef89, Corollary 7.20] for the case of curves, and [KP91, Theorem 14] for the case of surfaces, and [Nom95, Theorem] for the case of general type hypersurfaces. See also [HK85]. In a related vein, smooth q-bics are characterized amongst all hypersurfaces as those whose moduli of smooth hyperplane sections is constant: see [Bea90].

Automorphisms and unitary groups. — The relationship between q-bic hypersurfaces and the finite unitary groups has been another major source of interest. Perhaps most famously, Tate used the large automorphism group to verify his conjecture on algebraic cycles for even-dimensional smooth q-bics: see [Tat65] and also [HM78, SK79]. Reciprocally, smooth q-bics give, via ℓ-adic cohomology, geometric realizations of representations of finite unitary groups: see [Lus76, Lemma 30] for an explicit mention; this has been systematized by Deligne–Lusztig theory, see [DL76]. See also [Han92a, §3] for some specifics on the relationship between Deligne–Lusztig theory, finite unitary groups, and smooth q-bics.

In the specific case of a smooth q-bic curve, its automorphism group, the finite projective unitary group $\mathrm{PU}_3(q)$, has order $q^3(q^3+1)(q^2-1)$, and this much exceeds the classical Hurwitz bound $84(g-1)$. In fact, smooth q-bic curves nonetheless have exceptionally large automorphism groups even amongst curves in positive characteristic: Stichtenoth proves in [Sti73a, Hauptsatz] that unless C is a smooth q-bic curve, then

$$\#\,\mathrm{Aut}(C) \leq 16g(C)^4.$$

See also [Sin74, Hen78, Sti73b, Nak87] for related results and refinements. For more on the automorphisms of smooth q-bic curves, see [Dum95, Section 8], [GSY15, Section 3.5], [HJ90], and [Bon11].

Finite fields and Hermitian hypersurfaces. — Another source of classical interest stems from the geometry of Hermitian q-bic hypersurfaces over finite fields. The classical references are [**Seg65**, **BC66**]; see [**HT16**, Chapter 2] for a more modern one, and see [**Hir79**, Section 7.3]. From the perspective of this thesis, these classical works study the configuration and arrangement of the Hermitian subspaces of general q-bic hypersurfaces. This leads to characterizations of q-bics in terms of combinatorial data: see [**HSTV91**, **Tha92**], for instance.

The Hermitian q-bic hypersurfaces tend to be characterized by having the maximal number of low-degree rational points amongst hypersurfaces in projective space. For example, Hermitian q-bic curves have the maximal number of $\mathbf{F}_{q^2}$-rational points as permitted by the Hasse–Weil bound, and this property characterizes these curves: see [**RS94**]. Other curves with many points are often related to the Hermitian q-bic curves: see [**FGT97**, **GSX00**, **GT08**, **CKT00**]. Likewise, Hermitian q-bic surfaces are characterized amongst surfaces over $\mathbf{F}_q$ that do not contain a plane as those with the maximal possible number of $\mathbf{F}_q$-rational points: see [**HK15**, **HK16**]. See also [**Agu19**, **AP20**] for threefolds, and [**HK13**, **Tir17**, **HK17**] for higher dimensions.

After the work of Goppa [**Gop81**], varieties with many rational points over finite fields are very interesting from the perspective of constructing error-correcting codes. So, in view of the results above, Hermitian q-bic hypersurfaces often feature in this field: see [**GS95**, **Han92b**, **Sti09**, **TVN07**, **BDH21**] for example.

From the perspective of finite projective geometries, special arrangements of points and linear spaces in Hermitian q-bic hypersurfaces are often of interest. For example, q-bic curves may be combinatorially characterized via special configurations of points: see [**HSTV91**, **Tha92**], [**Hir79**, Section 7.3], and [**HKT08**]. Special configurations of points on q-bic surfaces are studied in [**CK03**, **GK03**], for example; the configuration of lines is studied in [**Hir85**, Chapter 19] and [**EH99**, **BPRS21**], for example.

q-bics and bilinear forms. — This venerable connection between q-bic and Hermitian hypersurfaces means that many authors have expressed the equation of q-bic hypersurfaces in terms of a bilinear form: see [**BC66**, §4], [**Hef85**, p.69], [**Bea90**, p.125], [Shi01, §2], [HH16], and [KKP+21, §5]. Of particular note are the works [Shi01] and [KKP+21]: the former begins to develop methods akin to the theory of q-bic forms, and the latter gives a classification of q-bic forms. The latter work is also very interesting in that it distinguishes q-bic hypersurfaces as being extremal from the point of view of F-singularity theory: reduced q-bic hypersurfaces have the smallest F-pure threshold amongst hypersurfaces of the same degree. This perspective may perhaps help to explain the curious geometry of q-bics.

Unirationality and rational curves. — From the classification of algebraic varieties, q-bics are notorious for defying traditional geometric expectations. For instance, Shioda showed in [Shi74] that smooth q-bics hypersurfaces of dimension at least 2 are unirational. In particular, such q-bics are always rationally connected, and it has long been known that they, in fact, contain many lines: see [Col79, Example 1.27], [Kol15, Example 35], [Kol96, Example 4.6.3], [DLR17], and [Deb01, §2.15]; this last source even observes that the Fano scheme of lines of a smooth q-bic is smooth. Such properties indicate that the geometry of rational curves lying on q-bics may be quite subtle. This is already the case with q-bic surfaces: see [Oji19]. In another direction, it is not clear when a Fano smooth q-bic hypersurface is separably rationally connected: see [Con06, She12, BDE+13]; though contrast with [Zhu11, CZ14, Tia15, CR19] for the general results known.

Supersingular varieties. — Positive characteristic invariants of q-bic hypersurfaces tend to be very special. For instance, the Jacobians of q-bic curves are supersingular abelian varieties which, furthermore in the case $q = p$, are even superspecial, see [Eke87, p.172], [PW15], and [AP15, §5.3]. Using the term supersingular in a slightly different manner, Shioda's unirationality construction implies that q-bic hypersurfaces in general are supersingular; see also [SK79], and [Shi77a, Shi77b] for related results in the case of surfaces.

Singular q-bic hypersurfaces. — Singular q-bics have not occurred with as much prominence as their smooth counterparts, and most occurrences tend to feature cones or else the singular q-bic curves which are not cones. Notably, the non-linear components are rational unicuspidal curves, wherein the cusp is, in a sense, more singular than usual. For instance, the F-threshold of this cuspidal singularity is lower than it would have in other characteristics; see [MTW05, Example 4.3] for the case $q = 2$ and also [KKP+21]. These curves appear as generalizations of quasi-elliptic fibrations: compare [BM76, §1] and [Shi92, Tak91, IIL20].

Outline

What follows are four chapters and two appendices. Chapter 1 develops the algebraic theory of q-bic forms. Chapter 2 applies the theory to study the geometry of q-bic hypersurfaces. Chapter 3 illustrates this in low dimensions, computing examples for q-bic points, curves, and surfaces. Chapter 4, the core of the work, studies the geometry of q-bic threefolds in detail. Appendix A collects some generalities on projective geometry. Appendix B collects some representation theoretic facts and computations.

Notation and conventions

Throughout this manuscript, q is a positive integer power of a prime p, and $\mathbf{k}$ is a field of characteristic p. For a vector space V over $\mathbf{k}$, let $\mathbf{P}V$ denote the projective space of lines in V, so that

$$\mathrm{H}^0(\mathbf{P}V, \mathcal{O}_{\mathbf{P}V}(1)) = V^\vee.$$

Given a scheme X over $\mathbf{k}$, write $V_X := V \otimes_{\mathbf{k}} \mathcal{O}_X$. Let

$$\mathrm{eu}_{\mathbf{P}V} \colon \mathcal{O}_{\mathbf{P}V}(-1) \to V_{\mathbf{P}V}$$

denote the *Euler section* on $\mathbf{P}V$: the tautological section dual to the evaluation map for global sections. The projective subspace corresponding to a $(r+1)$-dimensional linear subspace $U \subseteq V$ will be written $\mathbf{P}U \subseteq \mathbf{P}V$ and referred to as the associated *r-plane*. For subspaces $U_1, U_2 \subseteq V$, write $\langle U_1, U_2 \rangle$ for the subspace spanned by U_1 and U_2 in V.

Chapter 1

q-bic Forms

A *q-bic form* on a module M over a $\mathbf{F}_{q^2}$-algebra R is an R-linear map

$$\beta \colon \mathrm{Fr}^*(M) \otimes_R M \to R$$

where $\mathrm{Fr}^*(M) := R \otimes_{\mathrm{Fr},R} M$ is the q-power Frobenius twist of M. This Chapter develops the theory of q-bic forms in the spirit of the theory of bilinear forms. Since q-bic forms pair different modules, there is no simple symmetry condition available. The feature that distinguishes q-bic forms amongst general bilinear forms between two modules is the canonical Fr-linear map $M \to \mathrm{Fr}^*(M)$ given by $m \mapsto 1 \otimes m$.

Basic definitions and constructions are given in 1.1. The relationship between q-bic forms and Hermitian forms over $\mathbf{F}_{q^2}$ is explained in 1.2; in particular, 1.2.14 shows that any nonsingular q-bic form over a separably closed field admits an orthonormal basis. Automorphisms are discussed in 1.3. The Chapter closes in 1.4 with comments on the classification and moduli of q-bic forms over an algebraically closed field.

1.1. Basic notions

This Section contains the basic definitions and properties of q-bic forms. Throughout, let R be a $\mathbf{F}_{q^2}$-algebra and write $\mathrm{Fr} \colon R \to R$ for the q-power Frobenius morphism. For an R-module M, write $M^\vee := \mathrm{Hom}_R(M,R)$. The functor $M \mapsto M^\vee$ restricts to a duality on the category of finite projective R-modules. In the case that M

is finite projective, $M^{\vee,\vee}$ will be tacitly identified with M via the canonical isomorphism $\mathrm{can}_M \colon M \to M^{\vee,\vee}$ which sends $m \in M$ to the evaluation functional $\mathrm{ev}_m \colon \mathrm{Hom}_R(M,R) \to R$.

1.1.1. q-linearization. — For any R-module M, let $\mathrm{Fr}^*(M) := R \otimes_{\mathrm{Fr},R} M$ be the (left) R-module with the action of R twisted by the q-power Frobenius. Elements of $\mathrm{Fr}^*(M)$ are R-linear combinations elements of the form $r \cdot m^{(q)} := r \otimes m$ where $r \in R$ and $m \in M$. Moreover, if $m = r'm'$ for some $r' \in R$ and $m' \in M$, then

$$r \cdot m^{(q)} = r \cdot (r'm')^{(q)} = r \otimes r'm' = rr'^q \otimes m' = (rr'^q) \cdot m'^{(q)}.$$

The map $m \mapsto m^{(q)}$ is the universal Fr-linear additive map $(-)^{(q)} \colon M \to \mathrm{Fr}^*(M)$ out of M. As a piece of notation, given a submodule $N \subseteq \mathrm{Fr}^*(M)$, write

$$\mathrm{Fr}^{-1}(N) := \left\{ m \in M \;\middle|\; m^{(q)} \in N \right\}.$$

1.1.2. q-bic forms. — A *q-bic form* over R is a pair (M,β) consisting of an R-module M and an R-linear map $\beta \colon \mathrm{Fr}^*(M) \otimes_R M \to R$. A morphism $\varphi \colon (M_1,\beta_1) \to (M_2,\beta_2)$ between two q-bic forms is a morphism $\varphi \colon M_1 \to M_2$ of R-modules such that

$$\beta_1(m',m) = \beta_2(\mathrm{Fr}^*(\varphi)(m'), \varphi(m)) \quad \text{for every } m' \in \mathrm{Fr}^*(M_1) \text{ and } m \in M_1,$$

where $\mathrm{Fr}^*(\varphi) \colon \mathrm{Fr}^*(M_1) \to \mathrm{Fr}^*(M_2)$ is the Fr-twist of φ. The morphism φ is an *isomorphism* if the underlying module map is an isomorphism.

Adjunction induces two R-linear maps which, by an abuse of notation, are denoted

$$\beta \colon M \to \mathrm{Fr}^*(M)^\vee \quad \text{and} \quad \beta^\vee \colon \mathrm{Fr}^*(M) \to M^\vee.$$

The two adjoints are dual to one another via $\mathrm{Hom}_R(-,R)$ in the sense that

$$\beta = \mathrm{Hom}_R(\beta^\vee, R) \circ \mathrm{can}_M \quad \text{and} \quad \beta^\vee = \mathrm{Hom}_R(\beta, R) \circ \mathrm{can}_{\mathrm{Fr}^*(M)}.$$

The form (M,β) is said to be *nondegenerate* if the adjoint map $\beta \colon M \to \mathrm{Fr}^*(M)$ is injective, and *nonsingular* if the adjoint map is an isomorphism.

1.1.3. Isotropicity. — Let (M,β) be a q-bic form over R. An element $m \in M$ is said to be *isotropic* if $\beta(m^{(q)},m)=0$. A submodule $M' \subseteq M$ is said to be *isotropic* if every element of M' is isotropic, and it is *totally isotropic* if the restriction of β to M' is the zero form. A morphism $\varphi \colon N \to M$ of R-modules is said to be *isotropic*, respectively *totally isotropic*, if the image of φ is an isotropic submodule, respectively totally isotropic submodule.

Since R is an algebra over $\mathbf{F}_{q^2}$, the two notions of isotropic subspaces agree:

1.1.4. Lemma. — *Let (M,β) be a q-bic form over R. Then a submodule $M' \subseteq M$ is isotropic if and only if it is totally isotropic.*

Proof. — It suffices to show that an isotropic submodule is totally isotropic. So consider a pair $m_1, m_2 \in M'$. Isotropicity gives, for any $\lambda \in R^\times$,

$$0 = \beta((\lambda m_1 + m_2)^{(q)}, \lambda m_1 + m_2) = \lambda^q \beta(m_1^{(q)}, m_2) + \lambda \beta(m_2^{(q)}, m_1).$$

Thus $\beta(m_2^{(q)}, m_1) = -\lambda^{q-1}\beta(m_1^{(q)}, m_2)$ for every $\lambda \in R^\times$. Since $\mathbf{F}_{q^2}^\times \subseteq R^\times$, the λ^{q-1} take on at least two distinct values, whence $\beta(m_2^{(q)}, m_1) = \beta(m_1^{(q)}, m_2) = 0$. Since every element of $\mathrm{Fr}^*(M')$ is an R-linear combination of elements of the form $m_1^{(q)}$, this shows that M' is totally isotropic. ■

1.1.5. Example. — If R were simply a $\mathbf{F}_q$-algebra, say, then the two notions of isotropic subspaces may diverge. For instance, a q-bic form over a field $\mathbf{k}$ contained in $\mathbf{F}_q$ is simply a bilinear form $\beta \colon V \otimes_{\mathbf{k}} V \to \mathbf{k}$. Take $V = \mathbf{k}^{\oplus 2}$ the standard 2-dimensional vector space and consider the bilinear form with Gram matrix

$$\mathrm{Gram}(\beta; e_1, e_2) = \begin{pmatrix} 0 & 1 \\ q-1 & 0 \end{pmatrix}.$$

Then V is isotropic but not totally isotropic.

1.1.6. Orthogonals. — Given submodules $N_1 \subseteq \mathrm{Fr}^*(M)$ and $N_2 \subseteq M$, write

$$N_1^\perp := \ker(M \xrightarrow{\beta} \mathrm{Fr}^*(M)^\vee \to N_1^\vee),$$

$$N_2^\perp := \ker(\mathrm{Fr}^*(M) \xrightarrow{\beta^\vee} M^\vee \to N_2^\vee).$$

These are the *orthogonals*, with respect to β, of N_1 and N_2, respectively. The orthogonals $\mathrm{Fr}^*(M)^\perp \subseteq M$ and $M^\perp \subseteq \mathrm{Fr}^*(M)$ are referred to as the *kernels* of β (and $\beta^\vee$). The *radical* of β is the submodule

$$\mathrm{rad}(\beta) := \left\{ m \in M \;\middle|\; \beta(m^{(q)}, n_1) = \beta(n_2, m) = 0 \text{ for all } n_1 \in M, n_2 \in \mathrm{Fr}^*(M) \right\}.$$

More succinctly, $\mathrm{rad}(\beta) = \mathrm{Fr}^*(M)^\perp \cap \mathrm{Fr}^{-1}(M^\perp)$.

1.1.7. Lemma. — *Let (M, β) be a q-bic form over R. The orthogonals of nested submodules $M_1 \subseteq M_2$ of* $\mathrm{Fr}^*(M)$ *or M satisfy $M_1^\perp \supseteq M_2^\perp$ as submodules of M or* $\mathrm{Fr}^*(M)$*.*

Proof. — This follows from the definitions in **1.1.6** since the restriction map to M_1 factors through $M_2^\vee \to M_1^\vee$. For instance, if given submodules of $\mathrm{Fr}^*(M)$,

$$M_2^\perp = \ker(M \to \mathrm{Fr}^*(M)^\vee \to M_2^\vee) \subseteq \ker(M \to \mathrm{Fr}^*(M)^\vee \to M_2^\vee \to M_1^\vee) = M_1^\perp,$$

and similarly if given submodules of M. ■

1.1.8. Lemma. — *Let (M, β) be a q-bic form over R. If M and* $\mathrm{coker}(\beta)$ *are finite projective, then there is an exact sequence of R-modules*

$$0 \to \mathrm{Fr}^*(M)^\perp \to M \xrightarrow{\beta} \mathrm{Fr}^*(M)^\vee \to M^{\perp,\vee} \to 0.$$

Proof. — It remains to identify $\mathrm{coker}(\beta)$ with $M^{\perp,\vee}$. The assumptions imply that each of $\mathrm{Fr}^*(M)^\perp$, M, $\mathrm{Fr}^*(M)^\vee$, and $\mathrm{coker}(\beta)$ are finite projective. Applying $\mathrm{Hom}_R(-, R)$ and making identifications using the canonical maps as in **1.1.2** gives an exact sequence

$$0 \to \mathrm{coker}(\beta)^\vee \to \mathrm{Fr}^*(M) \xrightarrow{\beta^\vee} M^\vee \to \mathrm{Fr}^*(M)^{\perp,\vee} \to 0.$$

This identifies $\mathrm{coker}(\beta)^\vee = \ker(\beta^\vee)$ as $M^\perp$. Reflexivity shows $\mathrm{coker}(\beta) = M^{\perp,\vee}$. ■

1.1.9. Lemma. — *Let (M, β) be a q-bic form over R. If M and* $\mathrm{coker}(\beta)$ *are finite projective, then*

$$M_1^{\perp,\perp} = M^\perp + M_1 \quad \text{and} \quad M_2^{\perp,\perp} = \mathrm{Fr}^*(M)^\perp + M_2$$

for any submodules $M_1 \subseteq$ $\mathrm{Fr}^*(M)$ *and $M_2 \subseteq M$ which are locally direct summands.*

Proof. — Argue only for M_1, the argument for M_2 being completely analogous. By the definitions of 1.1.6,

$$M_1^{\perp,\perp} = \ker(\mathrm{Fr}^*(M) \xrightarrow{\beta^\vee} M^\vee \to M_1^{\perp,\vee}) = M^\perp + \beta^{\vee,-1}(\ker(M^\vee \to M_1^{\perp,\vee})).$$

Since $M_1^\perp = \ker(M \to \mathrm{Fr}^*(M)^\vee \to M_1^\vee)$, that M_1 and $\mathrm{Fr}^*(M)$ are reflexive R-modules together with the exact sequence 1.1.8 implies $M_1^{\perp,\vee} = \mathrm{coker}(M_1 \to \mathrm{Fr}^*(M) \to M^\vee)$. Thus the kernel of $M^\vee \to M_1^{\perp,\vee}$ is the image of M_1 under $\beta^\vee$, giving the result. ■

1.1.10. Orthogonal complements. — Let (M, β) be a q-bic form over R. An *orthogonal complement* of a submodule $M' \subseteq M$ is another submodule $M'' \subseteq M$ such that $\beta(m'^{(q)}, m'') = \beta(m''^{(q)}, m') = 0$ for every $m' \in M'$ and $m'' \in M''$, and $M = M' \oplus M''$ as R-modules. This situation is signified by

$$(M, \beta) = (M', \beta_{M'}) \perp (M'', \beta_{M''})$$

where $\beta_{M'}$ and $\beta_{M''}$ denotes the restriction of β to M' and M'', respectively.

Orthogonal complements need not exist, and when they do, need not be unique. For instance, it follows from definitions that an orthogonal complement M'' of M' must satisfy

$$M'' \subseteq \mathrm{Fr}^*(M')^\perp \cap \mathrm{Fr}^{-1}(M'^\perp).$$

This may be reformulated into a numerical criterion over a field:

1.1.11. Lemma. — *Let (V, β) be a q-bic form over a field $\mathbf{k}$ on a finite-dimensional vector space V. Then a subspace $V' \subseteq V$ has an orthogonal complement if and only if*

$$\dim_{\mathbf{k}} V - \dim_{\mathbf{k}} V' = \dim_{\mathbf{k}} W - \dim_{\mathbf{k}} W \cap V'$$

where $W := \mathrm{Fr}^*(V')^\perp \cap \mathrm{Fr}^{-1}(V'^\perp)$. It is unique if and only if $W \cap V' = \{0\}$.

Proof. — By 1.1.10, V' has an orthogonal complement if and only if there is a linear subspace of W with complementary dimension and which is linearly disjoint with $W \cap V'$; taking dimensions gives the criterion in the statement. The complementary subspace $V'' \subseteq W$ is unique if and only if $W \cap V' = \{0\}$. ■

1.1.12. Gram matrix. — Let (M,β) be a q-bic form over R. If M is a finite free module, then a choice of basis renders β quite explicit. Let $\varphi\colon \bigoplus_{i=0}^n R\cdot e_i \cong M$ be a basis. The *Gram matrix* of β with respect to the basis $\langle e_0,\ldots,e_n\rangle$ is the matrix

$$\mathrm{Gram}(\beta;e_0,\ldots,e_n) := \big(\beta(e_i^{(q)},e_j)\big)_{i,j=0}^n \in \mathbf{Mat}_{(n+1)\times(n+1)}(R).$$

In other words, this is the matrix of the adjoint map $\beta\colon M \to \mathrm{Fr}^*(M)^\vee$ with respect to the basis $\langle e_0,\ldots,e_n\rangle$ in the source, and $\langle e_0^{(q)},\ldots,e_n^{(q)}\rangle$ in the target:

$$\mathrm{Fr}^*(\varphi)^\vee\circ\beta\circ\varphi\colon \bigoplus\nolimits_{i=0}^n R\cdot e_i \cong M \to \mathrm{Fr}^*(M)^\vee \cong \bigoplus\nolimits_{i=0}^n R\cdot e_i^{\vee,(q)}.$$

where $e_i^\vee$ is the dual basis element to e_i.

1.1.13. Change of basis. — Suppose $\varphi'\colon \bigoplus_{i=0}^n R\cdot e_i' \cong M$ is another choice of basis. Consider the change of basis isomorphism

$$A := \varphi^{-1}\circ\varphi'\colon \bigoplus\nolimits_{i=0}^n R\cdot e_i' \xrightarrow{\cong} \bigoplus\nolimits_{i=0}^n R\cdot e_i$$

viewed as an invertible matrix $A\in\mathbf{GL}_{n+1}(R)$. Its Fr-twist

$$\mathrm{Fr}^*(A) := \mathrm{Fr}^*(\varphi)^{-1}\circ\mathrm{Fr}^*(\varphi')\colon \bigoplus\nolimits_{i=0}^n R\cdot e_i'^{(q)} \xrightarrow{\cong} \bigoplus\nolimits_{i=0}^n R\cdot e_i^{(q)}$$

is the matrix obtained of A by taking q-powers entrywise. Then there is an identity

$$\mathrm{Gram}(\beta;e_0',\ldots,e_n') = \mathrm{Fr}^*(A)^\vee\cdot\mathrm{Gram}(\beta;e_0,\ldots,e_n)\cdot A.$$

Nonsingularity of β takes a familiar meaning in terms of the Gram matrix:

1.1.14. Lemma. — *Let (M,β) be a q-bic form over R on a finite free module M. The following are equivalent:*

(i) *β is nonsingular;*

(ii) *$\mathrm{Gram}(\beta;e_0,\ldots,e_n)$ is invertible for some basis $\langle e_0,\ldots,e_n\rangle$ of M; and*

(iii) *$\mathrm{Gram}(\beta;e_0,\ldots,e_n)$ is invertible for any basis $\langle e_0,\ldots,e_n\rangle$ of M.*

Proof. — That (ii) $\Leftrightarrow$ (iii) follows immediately from the discussion of **1.1.13**. By the definitions in **1.1.2**, β is nonsingular if and only if the adjoint map $\beta\colon M\to\mathrm{Fr}^*(M)$ is an isomorphism. By **1.1.12**, the matrix of this map, with respect to some basis $\langle e_0,\ldots,e_n\rangle$, is the Gram matrix $\mathrm{Gram}(\beta;e_0,\ldots,e_n)$, thereby yielding (i) $\Leftrightarrow$ (ii). ■

Gram matrices make it easy to give examples of q-bic forms. Given an $(n+1)$-by-$(n+1)$ matrix B over R, write $(R^{\oplus n+1}, B)$ for the unique q-bic form with Gram matrix B in the given basis. Particularly important examples are given in the following; they appear prominently in the classification of q-bic forms, see 1.4.1.

1.1.15. Standard forms. — For each positive integer k, write

$$\mathbf{N}_k := \begin{pmatrix} 0 & 1 & \cdots & & \\ & 0 & \cdots & & \\ & & \cdots & & \\ & & \cdots & 0 & 1 \\ & & \cdots & & 0 \end{pmatrix}$$

for the k-by-k Jordan block with 0 on the diagonals, and write $\mathbf{1}$ for the 1-by-1 identity matrix; $\mathbf{N}_1$ is the 1-by-1 zero matrix, and is often written $\mathbf{0}$. Given two matrices B_1 and B_2, write $B_1 \oplus B_2$ for their block diagonal sum. Let m and $a_1, \ldots, a_m, b$ be non-negative integers such that $n+1 = b + \sum_{k=1}^m k a_k$. The q-bic form

$$(R^{\oplus n+1}, \mathbf{N}_1^{\oplus a_1} \oplus \cdots \oplus \mathbf{N}_m^{\oplus a_m} \oplus \mathbf{1}^{\oplus b})$$

is called a *standard q-bic form* and the Gram matrix is called its *type*.

Associated with a q-bic form (M, β) over R are two natural sequences of R-modules, both obtained by successively taking orthogonals with respect to β. The simpler of the two produces a filtration of M, and is described in 1.1.16. The second produces a sequence of filtrations on the Frobenius twists $\mathrm{Fr}^{i,*}(M)$, see 1.1.20.

1.1.16. $\perp$-filtration. — Construct an increasing filtration $M_\bullet$ on M using β as follows: set $M_i = \{0\}$ for $i < 0$, $M_0 := \mathrm{rad}(\beta)$, and for each $i \geq 1$, inductively define

$$M_i := \mathrm{Fr}^*(\mathrm{Fr}^*(M_{i-1})^\perp)^\perp.$$

This is an increasing filtration in which each submodule M_i is isotropic, as can be seen by induction and 1.1.7: twisting by Fr and taking orthogonals twice yields

$$\begin{aligned} M_{i-2} \subseteq M_{i-1} \quad &\Rightarrow \quad \mathrm{Fr}^*(M_{i-2})^\perp \supseteq \mathrm{Fr}^*(M_{i-1})^\perp \quad \Rightarrow \quad M_{i-1} \subseteq M_i, \\ M_{i-1} \subseteq \mathrm{Fr}^*(M_{i-1})^\perp \quad &\Rightarrow \quad \mathrm{Fr}^*(M_{i-1})^\perp \supseteq M_i \quad \Rightarrow \quad M_i \subseteq \mathrm{Fr}^*(M_i)^\perp. \end{aligned}$$

Therefore the filtration $M_\bullet$ may be extended to a filtration

$$\mathrm{rad}(\beta) = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq \mathrm{Fr}^*(M_2)^\perp \subseteq \mathrm{Fr}^*(M_1)^\perp \subseteq \mathrm{Fr}^*(M_0)^\perp = M$$

called the $\perp$-*filtration* of (M, β).

In good cases, M_i is the radical of β restricted to $\mathrm{Fr}^*(M_i)^\perp$; in fact, the $\perp$-filtration can be characterized as the maximal filtration on M with this property. This implies that this filtration may be constructed recursively upon replacing M by $\mathrm{Fr}^*(M)^\perp$.

1.1.17. Lemma. — *Let (M, β) be a q-bic form over R. If M and* $\mathrm{coker}(\beta)$ *are finite projective, then M_i is the radical of β restricted to* $\mathrm{Fr}^*(M_i)^\perp$ *for each $i \geq 0$.*

Proof. — Indeed, by 1.1.9,

$$\begin{aligned}\mathrm{rad}(\beta_{\mathrm{Fr}^*(M_i)^\perp}) &= \mathrm{Fr}^*(\mathrm{Fr}^*(M_i)^\perp)^\perp \cap \mathrm{Fr}^{-1}(\mathrm{Fr}^*(M_i)^{\perp,\perp}) \cap \mathrm{Fr}^*(M_i)^\perp \\ &= M_{i+1} \cap M_i \cap \mathrm{Fr}^*(M_i)^\perp.\end{aligned}$$

Since $M_i \subseteq M_{i+1} \subseteq \mathrm{Fr}^*(M_i)^\perp$ by 1.1.16, this gives the result. ■

1.1.18. Frobenius-twisted form. — Given a q-bic form (M, β) over R, applying the functor Fr^* yields another q-bic form $(\mathrm{Fr}^*(M), \mathrm{Fr}^*(\beta))$ over R, where

$$\mathrm{Fr}^*(\beta) \colon \mathrm{Fr}^{2,*}(M) \otimes_R \mathrm{Fr}^*(M) \to R$$

is the bilinear pairing between $\mathrm{Fr}^{2,*}(M) = \mathrm{Fr}^*(M) \otimes_{\mathrm{Fr},R} R$ and $\mathrm{Fr}^*(M) = M \otimes_{\mathrm{Fr},R} R$ obtained as the R-linear extension of

$$\mathrm{Fr}^*(\beta)(m_1^{(q)}, m_2^{(q)}) = \beta(m_1, m_2)^q \quad \text{for every } m_1 \in \mathrm{Fr}^*(M) \text{ and } m_2 \in M,$$

notation as in 1.1.1. In particular, if M is a free R-module of rank $n+1$, then the Gram matrices of $\mathrm{Fr}^*(\beta)$ and β are related by

$$\mathrm{Gram}(\mathrm{Fr}^*(\beta); e_0^{(q)}, \ldots, e_n^{(q)}) = \mathrm{Gram}(\beta; e_0, \ldots, e_n)^{(q)}$$

for any basis $M \cong \bigoplus_{i=0}^n R \cdot e_i$; here, $(-)^{(q)}$ denotes taking q-powers entrywise.

1.1.19. Frobenius-twisted orthogonal. — Let (M, β) be a q-bic form over R. Given a submodule $N \subseteq \mathrm{Fr}^*(M)$, let

$$N^{\mathrm{Fr}^*(\perp)} := \ker\big(\mathrm{Fr}^*(\beta)^\vee \colon \mathrm{Fr}^{2,*}(M) \to \mathrm{Fr}^*(M)^\vee \to N^\vee\big)$$

be the submodule of $\mathrm{Fr}^{2,*}(M)$ obtained by taking orthogonals with respect to the Frobenius-twisted form $(\mathrm{Fr}^*(M), \mathrm{Fr}^*(\beta))$, see 1.1.18. This is referred to as the *Frobenius-twisted orthogonal of* N with respect to β. Since this operation is but taking orthogonals upon moving to another q-bic form, formal properties such as 1.1.7 and 1.1.9 continue to hold.

1.1.20. $\mathrm{Fr}^*(\perp)$-filtration. — Construct submodules $W_i \subseteq \mathrm{Fr}^{2i-1,*}(M)$ using M and β as follows: set $\mathrm{Fr}^{-1,*}(M) := W_0 := \mathrm{rad}(\beta)$, and for each $i \geq 1$, inductively define

$$W_i := W_{i-1}^{\mathrm{Fr}^*(\perp),\mathrm{Fr}^*(\perp)} \subseteq \mathrm{Fr}^{2i-1,*}(M).$$

Expanding the definition of the Frobenius-twisted orthogonal from 1.1.19, this is

$$\begin{aligned} W_{i-1}^{\mathrm{Fr}^*(\perp)} &:= \ker\big(\mathrm{Fr}^{2i-2,*}(\beta)^\vee \colon \mathrm{Fr}^{2i-2,*}(M) \to \mathrm{Fr}^{2i-3,*}(M)^\vee \to W_i^\vee\big), \\ W_i &:= \ker\big(\mathrm{Fr}^{2i-1,*}(\beta)^\vee \colon \mathrm{Fr}^{2i-1,*}(M) \to \mathrm{Fr}^{2i-2,*}(M)^\vee \to W_{i-1}^{\mathrm{Fr}^*(\perp),\vee}\big). \end{aligned}$$

For instance, $W_1 = M^\perp$ as a submodule of $\mathrm{Fr}^*(M)$.

As in 1.1.16, these modules are, in a sense, increasing and totally isotropic for β. Taking Frobenius-twisted orthogonals and applying 1.1.7 twice gives:

$$\begin{array}{lllll} \mathrm{Fr}^{2,*}(W_{i-2}) \subseteq W_{i-1} & \Rightarrow & \mathrm{Fr}^{2,*}(W_{i-2})^{\mathrm{Fr}^*(\perp)} \supseteq W_{i-1}^{\mathrm{Fr}^*(\perp)} & \Rightarrow & \mathrm{Fr}^{2,*}(W_{i-1}) \subseteq W_i, \\ \mathrm{Fr}^{*}(W_{i-1}) \subseteq W_{i-1}^{\mathrm{Fr}^*(\perp)} & \Rightarrow & \mathrm{Fr}^{*}(W_{i-1})^{\mathrm{Fr}^*(\perp)} \supseteq W_i & \Rightarrow & \mathrm{Fr}^{*}(W_i) \subseteq W_i^{\mathrm{Fr}^*(\perp)}. \end{array}$$

This means that, for each $i \geq 0$, this gives an increasing filtration of $\mathrm{Fr}^{2i-1,*}(M)$:

$$\mathrm{Fr}^{2i-2,*}(W_1) \subseteq \cdots \subseteq \mathrm{Fr}^{2,*}(W_{i-1}) \subseteq W_i \subseteq \mathrm{Fr}^*(W_{i-1}^{\mathrm{Fr}^*(\perp)}) \subseteq \cdots \subseteq \mathrm{Fr}^{2i-3,*}(W_1^{\mathrm{Fr}^*(\perp)}).$$

1.1.21. Rank and corank. — A sequence of numerical invariants may be extracted from the $\perp$- and $\mathrm{Fr}^*(\perp)$-filtrations for a q-bic form (V,β) on a finite-dimensional vector space over a field $\mathbf{k}$. The simplest, determined by the first step of either filtration, are its *rank* and *corank*:

$$\mathrm{rank}(V,\beta) := \dim_{\mathbf{k}} V/\mathrm{Fr}^*(V)^\perp = \dim_{\mathbf{k}} \mathrm{Fr}^*(V)/V^\perp,$$

$$\mathrm{corank}(V,\beta) := \dim_{\mathbf{k}} V - \mathrm{rank}(V,\beta) = \dim_{\mathbf{k}} \mathrm{Fr}^*(V)^\perp = \dim_{\mathbf{k}} V^\perp.$$

The identifications come from the exact sequence of 1.1.8. The rank of a q-bic form (V,β) is the same as the rank of any Gram matrix as in 1.1.12.

The corank changes within a bounded range upon restriction to subspaces:

1.1.22. Lemma. — *Let (V,β) be a q-bic form over a field $\mathbf{k}$. If (U,β_U) is its restriction to a subspace $U \subseteq V$, then*

$$\operatorname{corank}(V,\beta) - \operatorname{codim}(U \subseteq V) \leq \operatorname{corank}(U,\beta_U) \leq \operatorname{corank}(V,\beta) + \operatorname{codim}(U \subseteq V).$$

Proof. — In fact, the orthogonal sequence 1.1.8 induces a short exact sequence

$$0 \to \mathrm{Fr}^*(V)^{\perp_\beta} \cap U \to \mathrm{Fr}^*(U)^{\perp_{\beta_U}} \to \beta(U) \cap \mathrm{Fr}^*(V/U)^\vee \to 0.$$

Taking dimensions and using the final comments of 1.1.21 give the upper bound:

$$\begin{aligned}
\operatorname{corank}(U,\beta_U) &= \dim_{\mathbf{k}}(\mathrm{Fr}^*(V)^{\perp_\beta} \cap U) + \dim_{\mathbf{k}}(\beta(U) \cap \mathrm{Fr}^*(V/U)^\vee) \\
&\leq \dim_{\mathbf{k}} \mathrm{Fr}^*(V)^{\perp_\beta} + \dim_{\mathbf{k}} \mathrm{Fr}^*(V/U)^\vee \\
&= \operatorname{corank}(V,\beta) + \operatorname{codim}(U \subset V).
\end{aligned}$$

The lower bound corank is obtained from:

$$\operatorname{corank}(U,\beta_U) \geq \dim_{\mathbf{k}}(\mathrm{Fr}^*(V)^{\perp_\beta} \cap U) \geq \dim_{\mathbf{k}} \mathrm{Fr}^*(V)^{\perp_\beta} - \operatorname{codim}(U \subset V)$$

from which the statement follows upon consulting 1.1.21 again. ∎

1.2. q-bic and Hermitian forms

By definition, q-bic forms linearize a biadditive map $V \times V \to \mathbf{k}$ that is linear in the second variable, but only q-linear in the first. In the case that $\mathbf{k} \subseteq \mathbf{F}_{q^2}$, such forms are sesquilinear with respect the q-power Frobenius, and a notion of symmetry is given by that of Hermitian forms; see [Lan02, Chapter XIII, §7] for generalities and definitions on sesquilinear linear algebra. Over a general field $\mathbf{k}$, the Hermitian condition does not make sense. Nevertheless, associated with any q-bic form is a natural Hermitian form over $\mathbf{F}_{q^2}$: see 1.2.2. This associated Hermitian form elucidates special features and yields a powerful invariant of the q-bic form.

Throughout, (V,β) is a q-bic form of dimension $n+1$ over a field $\mathbf{k}$.

1.2.1. Hermitian subspaces. — A vector $v \in V$ is said to be *Hermitian* if

$$\beta(w^{(q)}, v) = \beta(v^{(q)}, w)^q \quad \text{for all } w \in V.$$

A subspace $U \subseteq V$ is said to be *Hermitian* if it is spanned by Hermitian vectors. Let

$$V_{\mathrm{Herm}} := \{ v \in V \text{ a Hermitian vector} \}.$$

The following gives some basic properties of the subset of Hermitian vectors; notably, it shows that the restriction of β therein gives rise to a canonical Hermitian form for the quadratic field extension $\mathbf{F}_{q^2}/\mathbf{F}_q$:

1.2.2. Lemma. — *The set V_{Herm} is a vector space over $\mathbf{F}_{q^2}$ and satisfies*

$$\beta(v_1^{(q)}, v_2) \in \mathbf{F}_{q^2} \quad \textit{for every } v_1, v_2 \in V_{\mathrm{Herm}}.$$

Thus the form $\beta_{\mathrm{Herm}} \colon \mathrm{Fr}^*(V_{\mathrm{Herm}}) \otimes_{\mathbf{F}_{q^2}} V_{\mathrm{Herm}} \to \mathbf{F}_{q^2}$ induced by β is a Hermitian form.

Proof. — Let $u, v \in V_{\mathrm{Herm}}$. For any $\lambda \in \mathbf{F}_{q^2}$ and any $w \in V$,

$$\begin{aligned} \beta(w^{(q)}, u + \lambda v) &= \beta(u^{(q)}, w)^q + \lambda \beta(v^{(q)}, w)^q \\ &= \big(\beta(u^{(q)}, w) + \beta(\lambda^{1/q} v^{(q)}, w)\big)^q = \beta((u+\lambda v)^{(q)}, w)^q \end{aligned}$$

showing that $u + \lambda v \in V_{\mathrm{Herm}}$, whence the first statement. Applying the defining property of Hermitian vectors twice shows

$$\beta(u^{(q)}, v) = \beta(v^{(q)}, u)^q = \beta(u^{(q)}, v)^{q^2}$$

proving the second statement. The final statement now follows. ∎

The last statement of 1.2.2 partially justifies the nomenclature:

1.2.3. Corollary. — *Suppose there is a basis $V = \langle v_0, \ldots, v_n \rangle$ consisting of Hermitian vectors. Then the associated Gram matrix is a Hermitian matrix over $\mathbf{F}_{q^2}$; that is,*

$$\mathrm{Gram}(\beta; v_0, \ldots, v_n)^\vee = \mathrm{Gram}(\beta; v_0, \ldots, v_n)^{(q)}. \quad ∎$$

The Hermitian form $(V_{\mathrm{Herm}}, \beta_{\mathrm{Herm}})$ associated is generally quite large; for instance, the next statement shows that V_{Herm} contains the radical of β:

1.2.4. Lemma. — $\mathrm{Fr}^*(V)^\perp \cap \mathrm{Fr}^{-1}(V^\perp) = V_{\mathrm{Herm}} \cap \mathrm{Fr}^*(V)^\perp = V_{\mathrm{Herm}} \cap \mathrm{Fr}^{-1}(V^\perp)$.

Proof. — It follows directly from definitions that the radical $\mathrm{Fr}^*(V)^\perp \cap \mathrm{Fr}^{-1}(V^\perp)$ of β is contained in V_{Herm}, so it remains to show the second equality in the statement. For $v \in V_{\mathrm{Herm}}$, the defining property of Hermitian vectors shows that, for any $w \in V$,

$$\beta(w^{(q)}, v) = 0 \quad \Longleftrightarrow \quad \beta(v^{(q)}, w) = 0$$

meaning $v \in \mathrm{Fr}^*(V)^\perp$ if and only of $v \in \mathrm{Fr}^{-1}(V^\perp)$, giving the second equality. ∎

Taking Hermitian vectors is compatible with orthogonal decompositions:

1.2.5. Lemma. — *An orthogonal decomposition* $(V, \beta) = (V', \beta') \perp (V'', \beta'')$ *induces an orthogonal decomposition of Hermitian spaces*

$$(V_{\mathrm{Herm}}, \beta_{\mathrm{Herm}}) = (V'_{\mathrm{Herm}}, \beta'_{\mathrm{Herm}}) \perp (V''_{\mathrm{Herm}}, \beta''_{\mathrm{Herm}}).$$

Proof. — Let $v \in V_{\mathrm{Herm}}$, and let $v = v' + v''$ be its unique decomposition with $v' \in V'$ and $v'' \in V''$. Then it suffices to show that

$$v' \in V'_{\mathrm{Herm}} := \left\{ u \in V' \;\middle|\; \beta'(w^{(q)}, u) = \beta'(u^{(q)}, w)^q \text{ for every } w \in V' \right\}$$

and similarly for v'' and V''. Since V' and V'' are orthogonal, for every $w \in V'$,

$$\beta'(w^{(q)}, v') = \beta(w^{(q)}, v' + v'') = \beta((v' + v'')^{(q)}, w)^q = \beta'(v'^{(q)}, w)^q.$$

Therefore $v' \in V'_{\mathrm{Herm}}$. An analogous argument shows $v'' \in V''_{\mathrm{Herm}}$. ∎

The notion of an orthogonal to a Hermitian subspace is unambiguous:

1.2.6. Lemma. — *If $U \subseteq V$ is a Hermitian subspace, then* $\mathrm{Fr}^*(U)^\perp = \mathrm{Fr}^{-1}(U^\perp)$.

Proof. — Choose a basis $U = \langle u_1, \ldots, u_m \rangle$ of Hermitian vectors. Then

$$\mathrm{Fr}^*(U)^\perp = \bigcap\nolimits_{i=1}^m \mathrm{Fr}^*(\langle u_i \rangle)^\perp \quad \text{and} \quad \mathrm{Fr}^{-1}(U^\perp) = \bigcap\nolimits_{i=1}^m \mathrm{Fr}^{-1}(\langle u_i \rangle^\perp).$$

Thus it suffices to consider the case when $U = \langle u \rangle$ is spanned by a single Hermitian vector. By the definition of a Hermitian vector,

$$\mathrm{Fr}^*(U)^\perp = \{ w \in V \mid \beta(u^{(q)}, w) = 0 \} = \{ w \in V \mid \beta(w^{(q)}, u) = 0 \} = \mathrm{Fr}^{-1}(U^\perp)$$

since β takes values in the field $\mathbf{k}$. ■

1.2.7. Corollary. — *Let $U \subseteq V$ be a Hermitian subspace such that β_U has no radical. Then U has a unique orthogonal complement.*

Proof. — By 1.2.6, it makes sense to set $W := \mathrm{Fr}^*(U)^\perp = \mathrm{Fr}^{-1}(U^\perp)$. Since β_U has no radical, $U \cap W = \{0\}$, yielding the first inequality in

$$\dim_{\mathbf{k}} V - \dim_{\mathbf{k}} U \geq \dim_{\mathbf{k}} W \geq \dim_{\mathbf{k}} V - \dim_{\mathbf{k}} U,$$

and the second inequality is because $W = \ker(V \to \mathrm{Fr}^*(U)^\vee)$. So equality holds, and 1.1.11 implies W is the unique orthogonal complement to U. ■

Symmetry of orthogonals in 1.2.6 has a partial converse for 1-dimensional spaces:

1.2.8. Lemma. — *Assume $\mathbf{k}$ is separably closed. If $L \subseteq V$ is a 1-dimensional subspace such that $\mathrm{Fr}^*(L)^\perp = \mathrm{Fr}^{-1}(L^\perp)$, then L is Hermitian.*

Proof. — Let $W := \mathrm{Fr}^*(L)^\perp = \mathrm{Fr}^{-1}(L^\perp)$. If $W = V$, then L lies in the radical of β and so any nonzero vector is Hermitian by 1.2.4. So suppose that W is of codimension 1 in V. Let $v \in L$ be any nonzero vector. Then, for any $\lambda \in \mathbf{k}^\times$,

$$\beta(-,\lambda v) - \beta((\lambda v)^{(q)},-)^q : V \to \mathbf{k}$$

is a q-linear functional that is identically zero if and only if λv is Hermitian. In any case, it vanishes on W, and hence passes to the quotient V/W; fix a nonzero $\bar{w} \in V/W$ and observe that, for any $\mu \in \mathbf{k}$,

$$\beta((\mu\bar{w})^{(q)},\lambda v) - \beta((\lambda v)^{(q)},\mu\bar{w})^q = \lambda\mu^q\big(\beta(\bar{w}^{(q)},v) - \lambda^{q^2-1}\beta(v^{(q)},\bar{w})\big).$$

Since W the orthogonal of L, neither $\beta(\bar{w}^{(q)},v)$ nor $\beta(v^{(q)},\bar{w})$ vanish. Thus λ may be taken to be any (q^2-1)-st root of $\beta(\bar{w}^{(q)},v)/\beta(v^{(q)},\bar{w})$. Then λv is Hermitian. ■

1.2.9. Equations for Hermitan vectors. — The subgroup V_{Herm} of Hermitian vectors of V may be endowed with the structure of a closed subgroup scheme of the affine space $\mathbf{A}V := \operatorname{Spec}(\operatorname{Sym}(V^\vee))$ on V, viewed as an additive group scheme. Consider the two additive maps

$$(v \mapsto \beta(-,v))\colon V \to \operatorname{Fr}^*(V)^\vee \quad \text{and} \quad (v \mapsto \beta(v^{(q)},-)^q)\colon V \to \operatorname{Fr}^*(V)^\vee.$$

The first map is linear over $\mathbf{k}$ and the second map is q^2-linear over $\mathbf{k}$. Their duals induce ring homomorphisms

$$\beta(-,v), \beta(v^{(q)},-)^q\colon \operatorname{Sym}(\operatorname{Fr}^*(V)) \to \operatorname{Sym}(V^\vee)$$

and hence morphisms of affine schemes $\mathbf{A}V \to \mathbf{A}\operatorname{Fr}^*(V)^\vee$. Their difference gives the equations for the subscheme of Hermitian points:

$$\mathbf{A}V_{\mathrm{Herm}} = \mathrm{V}\big(\beta(-,v) - \beta(v^{(q)},-)^q\big).$$

Explicitly, choose a basis $V = \langle v_0, \ldots, v_n \rangle$, let $B := \operatorname{Gram}(\beta; v_0, \ldots, v_n)$ be the corresponding Gram matrix, and let $\mathbf{x}^\vee := (x_0, \ldots, x_n)$ be the corresponding coordinates for $\mathbf{A}V \cong \mathbf{A}^{n+1}$. Then the equations for $\mathbf{A}V_{\mathrm{Herm}}$ may be expressed matricially as

$$B\mathbf{x} - B^{(q),\vee}\mathbf{x}^{(q^2)} = 0.$$

1.2.10. Examples. — Here are examples illustrating the structure of the scheme of Hermitian vectors. Notation for q-bic forms is as in 1.1.15.

(i) Let $(V,\beta) = (\mathbf{k}^{\oplus n+1}, \mathbf{1}^{\oplus n+1})$ be the q-bic form with Gram matrix given by the identity matrix. Then the group of Hermitian vectors

$$\mathbf{A}V_{\mathrm{Herm}} = \operatorname{Spec}\big(\mathbf{k}[x_0, \ldots, x_n]/(x_0 - x_0^{q^2}, \ldots, x_n - x_n^{q^2})\big)$$

is isomorphic to the étale group scheme $\mathbf{F}_{q^2}^{\oplus n+1}$.

(ii) Let k be any field and let $\mathbf{k} := k(t)$. Let $V = \langle v \rangle$ be a 1-dimensional vector space over $\mathbf{k}$, and let $\beta\colon \operatorname{Fr}^*(V) \otimes_{\mathbf{k}} V \to \mathbf{k}$ be the q-bic form determined by $\beta(v^{(q)}, v) = -t$. The subscheme of Hermitan vectors is given by

$$\mathbf{A}V_{\mathrm{Herm}} = \operatorname{Spec}\big(\mathbf{k}[x]/(t^{q-1}x - x^{q^2})\big).$$

This is a form of $\mathbf{F}_{q^2}$ which splits along the extension $k(t) \subset k(t^{1/(q+1)})$.

(iii) Let $(V,\beta) = (\mathbf{k}^{\oplus n+1}, \mathbf{N}_{n+1})$ be the q-bic form of dimension $n+1$ with Gram matrix given by a Jordan block of size $n+1$ with 0 on the diagonals. Then the subscheme of Hermitian points is given by

$$\mathbf{A}V_{\mathrm{Herm}} = \operatorname{Spec}\big(\mathbf{k}[x_0,\ldots,x_n]/(x_1, x_2 - x_0^{q^2}, \ldots, x_n - x_{n-2}^{q^2}, x_{n-1}^{q^2})\big).$$

The structure of this scheme depends on the parity of n:

$$\frac{\mathbf{k}[x_0,\ldots,x_n]}{(x_1, x_2 - x_0^{q^2}, \ldots, x_n - x_{n-2}^{q^2}, x_{n-1}^{q^2})} \cong \begin{cases} \mathbf{k}[x_0]/(x_0^{q^{2m}}) & \text{if } n = 2m-1, \text{ and} \\ \mathbf{k}[x_0] & \text{if } n = 2m. \end{cases}$$

Indeed, in both cases, the ideal gives the equations

$$x_1 = x_3 = \cdots = x_{2m-1} = 0 \quad \text{and} \quad x_{2m-2} = x_{2m-4}^{q^2} = \cdots = x_0^{q^{2m-2}}.$$

The difference is the equation $x_{n-1}^{q^2} = 0$: when $n = 2m$, this is implied by the vanishing of the odd-indexed variables; when $n = 2m-1$, this shows $0 = x_{2m-2}^{q^2} = x_0^{q^{2m}}$.

1.2.11. Hermitian vectors of nonsingular forms. — The remainder of this Section is devoted to the study of Hermitian vectors and subspaces for a nonsingular form (V,β). In this setting, these notions were already isolated by Shimada in [Shi01, Definition 2.11]; their identification as fixed points of a map, as will be done in 1.2.19, was also observed in [Shi01, Remark 2.13].

Examples 1.2.10(i) and 1.2.10(ii) indicate that the structure of the subgroup of Hermitian vectors for a nonsingular q-bic form is quite simple.

1.2.12. Lemma. — *If (V,β) is nonsingular, then $\mathbf{A}V_{\mathrm{Herm}}$ is an étale group scheme of degree $q^2(n+1)$ over $\mathbf{k}$, geometrically isomorphic to $\mathbf{F}_{q^2}^{\oplus n+1}$.*

Proof. — Consider the equations given for $\mathbf{A}V_{\mathrm{Herm}}$ in 1.2.9. Since β is nonsingular, its Gram matrix is invertible and the equations may be expressed as

$$\mathbf{x}^{(q^2)} = B^{(q),\vee,-1}B\mathbf{x}.$$

This is a system of $n+1$ equations in $n+1$ variables. The Jacobian of this system of equations is given by $B^{(q),\vee,-1}B$; this is of full rank since β is nonsingular. Therefore

$\mathbf{A}V_{\mathrm{Herm}}$ is étale of degree $q^2(n+1)$ over $\mathbf{k}$. By 1.2.2, $\mathbf{A}V_{\mathrm{Herm}}$ is a vector space over $\mathbf{F}_{q^2}$, so it must be geometrically isomorphic to $\mathbf{F}_{q^2}^{\oplus n+1}$. ■

The following shows that if (V,β) is nonsingular, then V is spanned by its Hermitian vectors after a separable field extension; in other words, after a base change, V itself is a Hermitian subspace. This is not true for general forms: 1.2.10(iii) shows that a form of type $\mathbf{N}_{2m-1}$ has no nonzero Hermitian vectors.

1.2.13. Proposition. — *Assume that* $\mathbf{k}$ *is separably closed. If* (V,β) *is nonsingular, then the natural map* $V_{\mathrm{Herm}} \otimes_{\mathbf{F}_{q^2}} \mathbf{k} \to V$ *is an isomorphism.*

Proof. — The two $\mathbf{k}$-vector spaces have the same dimension by 1.2.12, so it suffices to show that the map $V_{\mathrm{Herm}} \otimes_{\mathbf{F}_{q^2}} \mathbf{k} \to V$ is injective. Suppose for sake of contradiction that it is not injective. Then there is a linear relation in V of the form

$$v_{k+1} = a_0 v_0 + \cdots + a_k v_k \quad \text{with } k \geq 0,\ v_i \in V_{\mathrm{Herm}}, \text{ and } a_i \in \mathbf{k}.$$

Choose such a relation with k minimal. Minimality implies that the $v_0,\ldots,v_k$ are linearly independent in V. Since β is nonsingular, there exists $w \in V$ such that $\beta(w^{(q)}, v_i) = 0$ for $0 \leq i \leq k-1$, and $\beta(w^{(q)}, v_k) \neq 0$ which, up to scaling v_k, may be taken to be 1. Since each of the vectors $v_0,\ldots,v_{k+1}$ are Hermitian,

$$\begin{aligned} a_k = \beta(w^{(q)}, v_{k+1}) &= \beta(v_{k+1}^{(q)}, w)^q \\ &= \sum\nolimits_{i=0}^{k} a_i^{q^2} \beta(v_i^{(q)}, w)^q = \sum\nolimits_{i=0}^{k} a_i^{q^2} \beta(w^{(q)}, v_i) = a_k^{q^2} \end{aligned}$$

and so $a_k \in \mathbf{F}_{q^2}$. But then $v_k' := v_{k+1} - a_k v_k$ lies in V_{Herm} by 1.2.2 and there is a linear relation in V given by

$$v_k' = a_0 v_0 + \cdots + a_{k-1} v_{k-1}.$$

This contradicts the minimality of the original linear relation, thereby showing the injectivity of $V_{\mathrm{Herm}} \otimes_{\mathbf{F}_{q^2}} \mathbf{k} \to V$. ■

This implies that there is always a basis for which the Gram matrix of a nonsingular form is the identity matrix; compare with 1.2.3.

1.2.14. Corollary. — *A nonsingular q-bic form (V,β) over a separably closed field has an orthonormal basis: there exists a basis $V = \langle v_0, \ldots, v_n\rangle$ such that for $0 \leq i,j \leq n$,*

$$\beta(v_i^{(q)}, v_j) = \begin{cases} 0 & \text{if } i \neq j, \text{ and} \\ 1 & \text{if } i = j. \end{cases}$$

Proof. — By 1.2.13, it suffices to show that the Hermitian form $(V_{\mathrm{Herm}}, \beta_{\mathrm{Herm}})$ from 1.2.2 admits such a basis. So replace V by V_{Herm} and work over $\mathbf{F}_{q^2}$.

Choose a basis $V = \langle u_0, \ldots, u_n\rangle$ such that $\beta(u_i^{(q)}, u_j) \neq 0$ if and only if $i = j$. This can be achieved, for instance, by successively choosing a nonisotropic vector and then taking its orthogonal complement, possible by 1.2.7. Since β is Hermitian,

$$\beta_i := \beta(u_i^{(q)}, u_i) = \beta(u_i^{(q)}, u_i)^q \quad \text{for each } 0 \leq i \leq n.$$

Therefore $\beta_i \in \mathbf{F}_q$. The image of the $(q+1)$-power map $\mathbf{F}_{q^2}^\times \to \mathbf{F}_{q^2}^\times$ is the cyclic subgroup of order $q-1$ given by $\mathbf{F}_q^\times$, so every element of $\mathbf{F}_q$ has a $(q+1)$-st root in $\mathbf{F}_{q^2}$. Then taking

$$v_i := \beta_i^{-1/(q+1)} u_i \quad \text{for each } 0 \leq i \leq n$$

gives the desired basis. ■

Questions about Hermitian subspaces in all of V may be reduced to questions about subspaces of the associated Hermitian form V_{Herm}. For instance, isotropic Hermitian subspaces are always contained in a maximal such:

1.2.15. Corollary. — *Assume that* $\mathbf{k}$ *is separably closed. If (V,β) is nonsingular, then any isotropic Hermitian subspace $U \subseteq V$ is contained in an isotropic Hermitian subspace of dimension $\lfloor \frac{n+1}{2} \rfloor$.*

Proof. — Set $U_{\mathrm{Herm}} := U \cap V_{\mathrm{Herm}}$. Then 1.2.13 implies that $U_{\mathrm{Herm}} \otimes_{\mathbf{F}_{q^2}} \mathbf{k} \to U$ is an isomorphism, and so it suffices to see that U_{Herm} is contained in a maximal isotropic subspace of V_{Herm}. This is standard: U_{Herm} can be completed to a sum of hyperbolic planes, at which point 1.2.7 gives an orthogonal complement; then induction applied to the zero subspace of the complement gives the desired maximal isotropic subspace. ■

1.2.16. Hermitian self-map. — The Hermitian vectors of a nonsingular q-bic form (V,β) can be understood in a different way via a canonical self-map of V induced by β. Let $(\mathrm{Fr}^*(V),\mathrm{Fr}^*(\beta))$ be the Frobenius twist of (V,β), see 1.1.18. Consider the q^2-linear map

$$\phi := (\beta^{-1}\circ \mathrm{Fr}^*(\beta)^\vee)\circ(-)^{(q^2)} \colon V \to \mathrm{Fr}^{2,*}(V) \to V$$

obtained by composing the q^2-linear map $(-)^{(q^2)}\colon V \to \mathrm{Fr}^{2,*}(V)$ with the linear isomorphism $\beta^{-1}\circ\mathrm{Fr}^*(\beta)^\vee\colon \mathrm{Fr}^{2,*}(V)\to V$. This is a canonical self-map of V which expresses a certain symmetry property of the form:

1.2.17. Lemma. — *The q^2-linear map $\phi\colon V\to V$ satisfies*

$$\beta(w,\phi(v)) = \mathrm{Fr}^*(\beta)(v^{(q^2)},w) \quad \textit{for all } v\in V \textit{ and } w \in \mathrm{Fr}^*(V).$$

Proof. — Indeed, compute using the definition of ϕ:

$$\begin{aligned}\beta(w,\phi(v)) &= w^\vee\circ\beta\circ(\beta^{-1}\circ\mathrm{Fr}^*(\beta)^\vee\circ v^{(q^2)}) \\ &= w^\vee\circ\mathrm{Fr}^*(\beta)^\vee\circ v^{(q^2)} = \mathrm{Fr}^*(\beta)(v^{(q^2)},w).\end{aligned}$$ ■

1.2.18. Corollary. — *Let (V,β) be a nonsingular q-bic form. Then for every $v,w\in V$,*

$$\beta(\phi(v)^{(q)},\phi(w)) = \beta(v^{(q)},w)^{q^2}.$$

In particular, v is isotropic if and only of $\phi(v)$ is isotropic.

Proof. — Apply 1.2.17 twice and use that $\mathrm{Fr}^*(\beta)(x^{(q)},y^{(q)}) = \beta(x,y)^q$ from 1.1.18:

$$\beta(\phi(v)^{(q)},\phi(w)) = \beta(w^{(q)},\phi(v))^q = \beta(v^{(q)},w)^{q^2}$$ ■

The following shows that linear subspaces fixed by the canonical self-map ϕ are precisely the Hermitian subspaces, as defined in 1.2.2:

1.2.19. Corollary. — *Let (V,β) be a nonsingular q-bic form. Then*

(i) *a vector $v\in V$ is Hermitian if and only if $\phi(v)=v$; and*

(ii) *a subspace $U\subseteq V$ is Hermitian if and only if $\phi(U)=U$.*

Proof. — If v is Hermitian, then by its definition in **1.2.1**,

$$\beta(w^{(q)}, v) = \beta(v^{(q)}, w)^q = \mathrm{Fr}^*(\beta)(v^{(q^2)}, w^{(q)}) = \beta(w^{(q)}, \phi(v))$$

for all $w \in V$; here, the second equality is due to the definition of $\mathrm{Fr}^*(\beta)$ as in **1.1.18**, and the third equality is due to **1.2.17**. Since β is nonsingular, this implies that $\phi(v) = v$. Reversing the argument shows that if $\phi(v) = v$, then v is Hermitian. This proves (i).

Consider (ii). If $U \subseteq V$ is Hermitian, then U admits a basis $\langle v_1, \ldots, v_m \rangle$ consisting of Hermitian vectors. Thus

$$\phi(U) = \langle \phi(v_1), \ldots, \phi(v_m) \rangle = \langle v_1, \ldots, v_m \rangle = U$$

by (i). Conversely, if $\phi(U) = U$, then ϕ restricts to a map $U \to U$. This is a bijective q^2-linear map, so there is a basis of U consisting of fixed vectors, see [**SGA7$_{\mathrm{II}}$**, Exposé XXII, 1.1]. Thus U is Hermitian by (i). ■

This gives a way of deciding when an arbitrary vector is contained in a Hermitian subspace of a particular dimension:

1.2.20. Corollary. — *Let (V, β) be a nonsingular q-bic form. Then for any $v \in V$,*

$$\min\{\dim_{\mathbf{k}} U \mid U \textit{ a Hermitian subspace containing } v\} = \dim_{\mathbf{k}} \langle \phi^i(v) \mid i \geq 0 \rangle.$$

Proof. — If v is contained in a Hermitian subspace U, then **1.2.19**(ii) implies $\phi^i(v) \in U$ for every $i \geq 0$. This shows the inequality "$\geq$" between the quantities in the statement. On the other hand, the space $\langle \phi^i(v) \mid i \geq 0 \rangle$ is fixed under ϕ and so it is Hermitian by **1.2.19**(ii). This proves the inequality "$\leq$". ■

1.3. Automorphisms

An automorphism of a q-bic form (M, β) over a $\mathbf{F}_{q^2}$-algebra R is a self-isomorphism of (M, β), as defined in **1.1.2**; the set of automorphisms forms a group $\mathrm{Aut}(M, \beta)$. When M is a finite projective R-module, the group of automorphisms may be enriched to an R-group scheme $\mathbf{Aut}(M,\beta)$, see 1.3.1. This Section discusses a few basic properties of these group schemes, and provides a few simple examples.

1.3.1. Automorphism group schemes. — Let (M,β) be a q-bic form over a $\mathbf{F}_{q^2}$-algebra R. Consider the group-valued functor $\mathrm{Alg}_R \to \mathrm{Grps}$ on the category of R-algebras given by

$$S \mapsto \mathrm{Aut}(M \otimes_R S, \beta \otimes \mathrm{id}_S).$$

This is the subfunctor of the functor $\mathbf{GL}(M)$ of linear automorphisms of M, specified as the stabilizer of the element $\beta \in \mathrm{Hom}_R(\mathrm{Fr}^*(M) \otimes_R M, R)$. Thus when M is a finite projective R-module, this is representable by a closed subgroup scheme of $\mathbf{GL}(M)$, see [DG70, II.1.2.4 and II.1.2.6]. The representing scheme is denoted $\mathbf{Aut}(M,\beta)$ and is referred to as the *automorphism group scheme of* (M,β).

The automorphism group scheme is *a priori* contained in a much smaller closed subscheme of $\mathbf{GL}(M)$:

1.3.2. Lemma. — *Let (M,β) be a q-bic form over R. Assume M is finite projective. Then $\mathbf{Aut}(M,\beta)$ stabilizes the associated $\perp$- and $\mathrm{Fr}^*(\perp)$-filtrations.*

Proof. — By their description in 1.1.16 and 1.1.20, the formation of the two filtrations commutes with extension of scalars to any R-algebra S. Since the filtrations are constructed by twisting by Fr and by taking successive kernels with respect to β, it follows that they are preserved by the action of the automorphism groups $\mathrm{Aut}(M \otimes_R S, \beta \otimes \mathrm{id}_S)$. Thus the filtrations are stabilized by the entire automorphism group scheme. ■

Orthogonal decompositions give a simple construction of subgroup schemes:

1.3.3. Lemma. — *Let (M,β) be a q-bic form over R. An orthogonal decomposition $(M,\beta) = (M_1,\beta_1) \perp (M_2,\beta_2)$ induces an inclusion of group schemes*

$$\mathbf{Aut}(M_1,\beta_1) \times \mathbf{Aut}(M_2,\beta_2) \subseteq \mathbf{Aut}(M,\beta).$$

Proof. — The action of $\mathbf{Aut}(M_i, \beta_i)$ on M_i extends to one on M via the trivial action on the complement. ∎

The remainder of this Section is concerned with q-bic forms (V, β) over a field $\mathbf{k}$. The next statement identifies the tangent space to the identity of $\mathbf{Aut}(V, \beta)$:

1.3.4. Proposition. — *Let (V, β) be a q-bic form over a field $\mathbf{k}$. Then there is a canonical isomorphism of $\mathbf{k}$-vector spaces*

$$\mathrm{Lie}(\mathbf{Aut}(V, \beta)) \cong \mathrm{Hom}_{\mathbf{k}}(V, \mathrm{Fr}^*(V)^{\perp}).$$

Proof. — The Lie algebra of $\mathbf{Aut}(V, \beta)$ is the $\mathbf{k}$-vector space of isomorphisms

$$\varphi : V \otimes_{\mathbf{k}} \mathbf{k}[\epsilon]/(\epsilon^2) \to V \otimes_{\mathbf{k}} \mathbf{k}[\epsilon]/(\epsilon^2)$$

of $\mathbf{k}[\epsilon]/(\epsilon^2)$-modules which restrict to the identity upon setting $\epsilon = 0$, and which preserve β. The first condition means that φ is determined by the $\mathbf{k}$-linear map

$$\epsilon\bar{\varphi} := \varphi - \mathrm{id} : V \otimes_{\mathbf{k}} \mathbf{k}[\epsilon]/(\epsilon) \to V \otimes_{\mathbf{k}} (\epsilon)/(\epsilon^2).$$

Write $\varphi = \mathrm{id} + \epsilon\bar{\varphi}$. Then since $\epsilon^2 = 0$, that φ preserves β means

$$\beta(v, w) = \beta(\mathrm{Fr}^*(\mathrm{id} + \epsilon\bar{\varphi})(v), (\mathrm{id} + \epsilon\bar{\varphi})(w)) = \beta(v, w) + \beta(v, \epsilon\bar{\varphi}(\bar{w}))$$

for every $v \in \mathrm{Fr}^*(V \otimes_{\mathbf{k}} \mathbf{k}[\epsilon]/(\epsilon^2))$ and $w \in V \otimes_{\mathbf{k}} \mathbf{k}[\epsilon]/(\epsilon^2)$, and where $\bar{w} \in V \otimes_{\mathbf{k}} \mathbf{k}[\epsilon]/(\epsilon)$ is the image of w under the quotient map. Viewing $\bar{\varphi}$ as a linear map $V \to V$, this means that $\beta(v, \bar{\varphi}(w)) = 0$ for all $v \in \mathrm{Fr}^*(V)$ and $w \in V$. Thus $\bar{\varphi}$ factors through $\mathrm{Fr}^*(V)^{\perp}$ and the map $\varphi \mapsto \bar{\varphi}$ determines an isomorphism from $\mathrm{Lie}(\mathbf{Aut}(V, \beta))$ to $\mathrm{Hom}_{\mathbf{k}}(V, \mathrm{Fr}^*(V)^{\perp})$. ∎

1.3.5. Unitary groups. — In the case that (V, β) is a nonsingular q-bic form, it follows from 1.3.4 that its automorphism group scheme is reduced and finite. Write

$$\mathrm{U}(V, \beta) := \mathbf{Aut}(V, \beta)$$

and call it the *unitary group* of β. For the standard nonsingular form $(\mathbf{k}^{\oplus n+1}, \mathbf{1}^{\oplus n+1})$, this is the classical finite unitary group $\mathrm{U}_{n+1}(q)$, as in [CCN+85, §2.1].

To give another description of this group, observe that $g \in \mathbf{GL}(V)$ lies in $\mathrm{U}(V,\beta)$ if and only if the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "g"'] \arrow[d, "\beta"'] & V \\ \mathrm{Fr}^*(V)^\vee \arrow[r, "\mathrm{Fr}^*(g)^{\vee,-1}"] & \mathrm{Fr}^*(V)^\vee \arrow[u, "\beta^{-1}"'] \end{tikzcd}$$

In other words, letting $F\colon \mathbf{GL}(V) \to \mathbf{GL}(V)$ be the morphism of algebraic groups determined by $g \mapsto \beta^{-1} \circ \mathrm{Fr}^*(g)^{\vee,-1} \circ \beta$, the unitary group is the subgroup of fixed points of F:

$$\mathrm{U}(V,\beta) = \mathbf{GL}(V)^F.$$

Compare this description with [Ste16, Lecture 11].

1.3.6. Type $\mathbf{N}_2^{\oplus a} \oplus \mathbf{1}^{\oplus b}$. — Automorphism group schemes of singular q-bic forms are intricate in rather different ways. For example, let $a, b \geq 0$ be integers and consider a q-bic form (V,β) of type $\mathbf{N}_2^{\oplus a} \oplus \mathbf{1}^{\oplus b}$ over a perfect field $\mathbf{k}$, see 1.1.15 for notation. Let

$$U_- := \mathrm{Fr}^*(V)^\perp \quad \text{and} \quad U_+ := \mathrm{Fr}^{-1}(V^\perp)$$

be kernels of β; since $\mathbf{k}$ is perfect $(-)^{(q)}\colon V \to \mathrm{Fr}^*(V)$ is a bijection and so U_+ is a-dimensional. Set $U := U_- \oplus U_+$. Then the restricted q-bic form (U,β_U) is of type $\mathbf{N}_2^{\oplus a}$, and so induces an isomorphism $\beta_U|_{U_+}\colon U_+ \to \mathrm{Fr}^*(U_-)$. It follows from 1.1.11 that U fits into a unique orthogonal decomposition

$$(V,\beta) \cong (U,\beta_U) \perp (W,\beta_W)$$

where $W := \mathrm{Fr}^*(U)^\perp \cap \mathrm{Fr}^{-1}(U^\perp)$ and (W,β_W) is of type $\mathbf{1}^{\oplus b}$. The following computes the automorphism group scheme in general, see also 3.3.1, 3.6.1, 3.10.5, and 3.14.4 for explicit low dimension expressions. Notation: given a group scheme $\mathbf{G}$ over $\mathbf{k}$, write $\mathbf{G}[\mathrm{Fr}] := \ker(\mathrm{Fr}\colon \mathbf{G} \to \mathbf{G})$ for the subgroup scheme obtained as the kernel of the q-power absolute Frobenius homomorphism.

1.3.7. Proposition. — *Let (V,β) be a q-bic form of type $\mathbf{N}_2^{\oplus a} \oplus \mathbf{1}^{\oplus b}$ over a perfect field. Then $\mathbf{Aut}(V,\beta)$ is isomorphic to the a^2-dimensional closed subgroup scheme of $\mathbf{GL}_{2a+b}$*

consisting of

$$\left(\begin{array}{cc|c} A_- & B & \mathbf{y}^\vee \\ 0 & A_+ & 0 \\ \hline 0 & \mathbf{x} & C \end{array}\right)$$

where $A_\pm \in \mathbf{GL}(U_\pm)$*,* $B \in \mathbf{Hom}(U_+, U_-)[\mathrm{Fr}]$*,* $C \in \mathrm{U}_a(q)$*,* $\mathbf{x} \in \mathbf{Hom}(U_+, W)[\mathrm{Fr}]$*, and* $\mathbf{y} \in \mathbf{Hom}(U_-, W)[\mathrm{Fr}^2]$*, subject to the equations*

$$A_-^{(q),\vee} \cdot \beta_U|_{U_-} \cdot A_+ = \beta_U|_{U_-} \quad \textit{and} \quad C^{(q),\vee} \cdot \beta_W \cdot \mathbf{x} + \mathbf{y}^{(q)} \cdot \beta_U|_{U_-} \cdot A_- = 0.$$

Proof. — Let $(V, \beta) \cong (U, \beta_U) \perp (W, \beta_W)$ be the orthogonal decomposition provided by 1.3.6, using that the base field is perfect. The $\perp$-filtration is given by $U_- \subset U_- \oplus W$, whereas the first step of the $\mathrm{Fr}^*(\perp)$-filtration is $\mathrm{Fr}^*(U_+)$, see 1.1.16 and 1.1.20. Thus by 1.3.2, the automorphism group scheme is the closed subgroup scheme of $\mathbf{GL}(U_- \oplus U_+ \oplus W)$ consisting of block matrices that satisfy

$$\left(\begin{array}{cc|c} A_-^\vee & 0 & 0 \\ 0 & A_+^\vee & 0 \\ \hline \mathbf{y} & 0 & C^\vee \end{array}\right)^{(q)} \left(\begin{array}{cc|c} 0 & \beta_U|_{U_+} & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & \beta_W \end{array}\right) \left(\begin{array}{cc|c} A_- & B & \mathbf{y}^\vee \\ 0 & A_+ & 0 \\ \hline 0 & \mathbf{x} & C \end{array}\right) = \left(\begin{array}{cc|c} 0 & \beta_U|_{U_+} & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & \beta_W \end{array}\right)$$

such that $A_\pm \in \mathbf{GL}(U_\pm)$, $B \in \mathbf{Hom}(U_+, U_-)[\mathrm{Fr}]$, $C \in \mathbf{GL}(W)$, $\mathbf{x} \in \mathbf{Hom}(U_+, W)[\mathrm{Fr}]$, and $\mathbf{y} \in \mathbf{Hom}(U_-, W)$. Expanding shows that $C \in \mathrm{U}_b(q)$ and gives the 2 equations in the statement, completing the computation. ■

Another simple cases is when the form has a radical:

1.3.8. Lemma. — *Let* (V, β) *be a q-bic form. Let* $(\bar{V}, \beta_{\bar{V}})$ *be the q-bic form obtained by dividing out the radical* $L := \mathrm{rad}(\beta)$*. Then there is a block matrix decomposition*

$$\mathbf{Aut}(V, \beta) \cong \begin{pmatrix} \mathbf{GL}(L) & \mathbf{Hom}(L, \bar{V}) \\ & \mathbf{Aut}(\bar{V}, \beta_{\bar{V}}) \end{pmatrix}$$

so in particular, $\dim \mathbf{Aut}(V, \beta) = \dim \mathbf{Aut}(\bar{V}, \beta_{\bar{V}}) + \dim_{\mathbf{k}} L \cdot \dim_{\mathbf{k}} V$.

Proof. — This follows from a matrix calculation upon observing that $\mathbf{Aut}(V, \beta)$ must preserve the radical. ■

1.4. Classification of q-bic forms

Over an algebraically closed field, q-bic forms are classified up to isomorphism by the finitely many numerical invariants arising from its $\perp$- and $\mathrm{Fr}^*(\perp)$-filtrations; in short, this means that every q-bic form is a standard in the sense of 1.1.15:

1.4.1. Theorem. — *Let (V, β) be a q-bic form of dimension $n+1$ over an algebraically closed field $\mathbf{k}$. Then there exists a basis $V = \langle e_0, \ldots, e_n \rangle$ such that*

$$\mathrm{Gram}(\beta; e_0, \ldots, e_n) = \mathbf{N}_1^{\oplus a_1} \oplus \cdots \oplus \mathbf{N}_m^{\oplus a_m} \oplus \mathbf{1}^{\oplus b}$$

for $m, a_1, \ldots, a_m, b \in \mathbf{Z}_{\geq 0}$ such that $b + \sum_{k=1}^m k a_k = n+1$.

1.4.2. Remarks on the classification theorem. — A proof of the classification theorem using the abstract theory of q-bic forms will appear elsewhere. The observation is that a q-bic form is completely determined by the numerical characteristics of its $\perp$- and $\mathrm{Fr}^*(\perp)$-filtrations, as defined in 1.1.16 and 1.1.20, and a suitable basis can be constructed by examining how the two filtrations interact.

During the preparation of this work, another group has independently discovered this classification and given a proof via explicit matrix methods: see [KKP+21, Theorem 7.1]. Various partial cases of 1.4.1 have been known for much longer. When β is Hermitian, this classification was well-known to authors working in finite projective geometry, see [Seg65, BC66]. The case when β is nonsingular has been rediscovered several times, see [Hef85, Bea90]. The case when β is of corank 1 was established by Hoai Hoang in [HH16] using explicit matrix methods.

1.4.3. Remark regarding general fields. — The classification of q-bic forms over non-closed fields is decidedly more subtle. For instance, the two q-bic forms

$$\left(\mathbf{F}_{q^2}^{\oplus 3}, \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}\right) \quad \text{and} \quad \left(\mathbf{F}_{q^2}^{\oplus 3}, \begin{pmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 1 \end{pmatrix}\right)$$

are not isomorphic over $\mathbf{F}_{q^2}$, as can be seen by showing that the schemes of Hermitian vectors, as from 1.2.9, are not isomorphic over $\mathbf{F}_{q^2}$; see also [Bon11, Exercise 2.4].

Even for separably closed fields, the situation is quite complicated. For instance, let k be any field and let $\mathbf{k}$ be the separable closure of the function field $k(t)$.

Consider a 2-dimensional q-bic form over $\mathbf{k}$ given by

$$\left(\mathbf{k}^{\oplus 2}, \left(\begin{smallmatrix} 0 & f \\ 0 & g \end{smallmatrix}\right)\right) \quad \text{for some } f, g \in \mathbf{k}.$$

Whenever $f \neq 0$, this isomorphic to a form of type $\mathbf{N}_2$ upon passing to the perfect closure of $\mathbf{k}$. However, such an isomorphism is defined over $\mathbf{k}$ if and only if g is a q-power: Indeed, a direct computation shows that the change of coordinates bringing the above form to the standard $\mathbf{N}_2$ form is given by

$$\begin{pmatrix} 1 & -g^{1/q}/f \\ 0 & 1/f \end{pmatrix}$$

where this is viewed as a matrix over the perfect closure of $\mathbf{k}$.

1.4.4. Moduli of q-bic forms. — A parameter space for the set of q-bic forms on V is given by the $(n+1)^2$-dimensional affine space

$$q\text{-}\mathbf{bics}(V) := \mathbf{A}(\mathrm{Fr}^*(V)^\vee \otimes V^\vee).$$

This carries a universal q-bic form

$$\beta_{\mathrm{univ}} \colon \mathrm{Fr}^*(V) \otimes V \otimes \mathcal{O}_{q\text{-}\mathbf{bics}(V)} \to \mathbf{k}$$

such that the fibre over a point $[\beta] \in q\text{-}\mathbf{bics}(V)$ recovers β. A choice of basis $V = \langle e_0, \ldots, e_n \rangle$ yields an isomorphism $\mathrm{Gram} \colon q\text{-}\mathbf{bics}(V) \to \mathbf{Mat}_{(n+1)\times(n+1)}$ of affine spaces over $\mathbf{k}$, given by

$$[\beta] \mapsto \mathrm{Gram}(\beta; e_0, \ldots, e_n)$$

taking a q-bic form to its Gram matrix.

1.4.5. Rank stratification. — For each $0 \leq r \leq n+1$, let

$$q\text{-}\mathbf{bics}(V)_{\leq r} := \{ [\beta] \in q\text{-}\mathbf{bics}(V) \mid \mathrm{rank}(\beta) \leq r \}$$

be the closed subscheme consisting of q-bic forms whose rank, in the sense of 1.1.21, is at most r; equivalently, this is the locus of q-bic forms with corank at least $n+1-r$. This gives a filtration by closed subschemes

$$q\text{-}\mathbf{bics}(V) = q\text{-}\mathbf{bics}(V)_{\leq n+1} \supseteq q\text{-}\mathbf{bics}(V)_{\leq n} \supseteq \cdots \supseteq q\text{-}\mathbf{bics}(V)_{\leq 1} \supseteq q\text{-}\mathbf{bics}(V)_{\leq 0}.$$

The locally closed subschemes

$$q\text{-}\mathbf{bics}(V)_r := q\text{-}\mathbf{bics}(V)_{\leq r} \setminus q\text{-}\mathbf{bics}(V)_{\leq r-1}$$

together give the *rank stratification* of q-$\mathbf{bics}(V)$.

Since ranks are compatible with taking Gram matrices, q-$\mathbf{bics}(V)_{\leq r}$ is isomorphic to the locus of $(n+1)$-by-$(n+1)$ matrices of rank at most r. Thus q-$\mathbf{bics}(V)_{\leq r}$ is irreducible and has codimension $(n+1-r)^2$; see [Ful98, Chapter 14] for example. For instance, q-$\mathbf{bics}(V)_{\leq n}$ is a hypersurface and q-$\mathbf{bics}(V)_{\leq 0}$ is a point.

1.4.6. Type stratification. — A much finer stratification of q-$\mathbf{bics}(V)$ is afforded by the classification theorem 1.4.1. Namely, each q-bic form is isomorphic to a standard form of type λ, and the sets

$$q\text{-}\mathbf{bics}(V)_\lambda := \{\, [\beta] \in q\text{-}\mathbf{bics}(V) \mid \mathrm{type}(\beta) = \lambda \,\}$$

may be construed as locally closed subschemes of q-$\mathbf{bics}(V)$, which together give a stratification q-$\mathbf{bics}(V)$—the *type stratification*—refining the rank stratification. For instance, q-$\mathbf{bics}(V)_{n+1} = q$-$\mathbf{bics}(V)_{\mathbf{1}^{\oplus n+1}}$ and

$$q\text{-}\mathbf{bics}(V)_n = \coprod\nolimits_{k=1}^{n+1} q\text{-}\mathbf{bics}(V)_{\mathbf{N}_k \oplus \mathbf{1}^{\oplus n+1-k}}.$$

To describe the basic properties of these strata, let $\mathbf{Aut}(V,\lambda)$ denote the automorphism group scheme of any q-bic form of type λ.

1.4.7. Lemma. — *q-$\mathbf{bics}(V)_\lambda$ is irreducible and of codimension* $\dim \mathbf{Aut}(V,\lambda)$.

Proof. — Fix a q-bic form β on V of type λ. By 1.4.1, the map $\mathbf{GL}(V) \to q$-$\mathbf{bics}(V)$ given by $g \mapsto g \cdot \beta$ is a surjection onto q-$\mathbf{bics}(V)_\lambda$. Since the fibres are isomorphic to $\mathbf{Aut}(V,\beta)$, the result follows. ∎

1.4.8. Corollary. — *A general q-bic form of corank $a \leq \frac{n+1}{2}$ is of type $\mathbf{N}_2^{\oplus a} \oplus \mathbf{1}^{\oplus n+1-2a}$.*

Proof. — Comparing 1.4.5, 1.4.7, and 1.3.7 shows that both q-$\mathbf{bics}(V)_{n+1-a}$ and q-$\mathbf{bics}(V)_{\mathbf{N}_2^{\oplus a} \oplus \mathbf{1}^{\oplus n+1-a}}$ are irreducible of codimension a^2 in q-$\mathbf{bics}(V)$. Since the type stratum is contained in the rank stratum, the result follows. ∎

Chapter 2

q-bic Hypersurfaces

Given a q-bic form (V, β) over a field $\mathbf{k}$, the space

$$X := \left\{ [v] \in \mathbf{P}V \;\middle|\; \beta(v^{(q)}, v) = 0 \right\}$$

parameterizing isotropic vectors is a hypersurface of degree $q+1$ in the projective space $\mathbf{P}V$: this is the *q-bic hypersurface* associated with β. Defined in this way, q-bic hypersurfaces are akin to quadrics; the aim of this Chapter is to substantiate this analogy by systematically relating global geometric properties of q-bic hypersurfaces with algebraic properties of the underlying q-bic form.

The basic definitions are given in 2.1. Differential invariants of X are expressed in terms of β in 2.2; for instance, X is smooth smooth if and only if β is nonsingular, and the nonsmooth locus of X is canonically the q-fold linear space given by the kernel of $\beta^\vee$: see 2.2.3 and 2.2.7. Automorphisms of X are related with those of β in 2.3. Section 2.4 contains a basic study of cones over q-bic hypersurfaces: 2.4.1 shows that X is a cone if and only if β has a radical, and that the vertex of X is given by $\mathrm{rad}(\beta)$. More interestingly, cone points of q-bic hypersurfaces, defined in 2.4.5, may be characterized and the set of which carries a canonical scheme structure: see 2.4.7, 2.4.8, and 2.4.9. Section 2.5 explains two unirationality constructions for smooth q-bic hypersurfaces: see see 2.5.4 and 2.5.11. Some basic cohomological properties of q-bic hypersurfaces are collected in 2.6.

Section 2.7 initiates the study of the Fano schemes of linear spaces associated with q-bic hypersurfaces. Notably, the Fano schemes of a smooth q-bic are smooth, irreducible, and has dimension independent of q: see 2.7.16. The tautological incidence correspondence is studied in more detail in 2.8. Finally, 2.9 constructs a canonical filtration of smooth q-bic hypersurfaces by closed subschemes induced by the Hermitian self-map of 1.2.16.

Throughout this Chapter, $\mathbf{k}$ is a field containing $\mathbf{F}_{q^2}$, $\mathrm{Fr}\colon \mathbf{k} \to \mathbf{k}$ denotes the q-power Frobenius homomorphism, and V is a $\mathbf{k}$-vector space of dimension $n+1$.

2.1. Setup and basic properties

2.1.1. q-bic equations and hypersurfaces. — Let (V,β) be a q-bic form over a field $\mathbf{k}$, as defined in 1.1.2. Unless otherwise stated, the form β will be assumed to be nonzero. This induces a nonzero section f_β of $\mathcal{O}_{\mathbf{P}V}(q+1)$ via

$$f_\beta := \beta(\mathrm{eu}^{(q)}, \mathrm{eu})\colon \mathcal{O}_{\mathbf{P}V}(-q-1) \to \mathrm{Fr}^*(V)_{\mathbf{P}V} \otimes V_{\mathbf{P}V} \xrightarrow{\beta} \mathcal{O}_{\mathbf{P}V}$$

where $\mathrm{eu}\colon \mathcal{O}_{\mathbf{P}V}(-1) \to V_{\mathbf{P}V}$ is the tautological Euler section. In other words, f_β is the degree $q+1$ polynomial obtained by pairing the linear coordinates of $\mathbf{P}V$ with their q-powers according to β. This section is called the *q-bic equation* associated with (V,β). The degree $q+1$ hypersurface of $\mathbf{P}V$ given by

$$X = X_\beta := \mathrm{V}(f_\beta) \subset \mathbf{P}V$$

is the *q-bic hypersurface* associated with the q-bic form (V,β).

2.1.2. — Explicitly, let $V = \langle e_0, \ldots, e_n\rangle$ be a basis and let $\mathbf{x}^\vee := (x_0 : \cdots : x_n)$ be the corresponding projective coordinates on $\mathbf{P}V = \mathbf{P}^n$. For each $0 \leq i,j \leq n$, let $a_{ij} := \beta(e_i^{(q)}, e_j)$ be the (i,j)-entry of the Gram matrix of β with respect to the chosen basis, see 1.1.12. Then the q-bic equation f_β is the polynomial

$$f_\beta(x_0,\ldots,x_n) = \mathbf{x}^{\vee,(q)} \cdot \mathrm{Gram}(\beta; e_0,\ldots,e_n) \cdot \mathbf{x} = \sum\nolimits_{i,j=0}^n a_{ij} x_i^q x_j$$

and the associated q-bic hypersurface X is its vanishing locus in $\mathbf{P}^n$.

The simple yet fundamental observation is that q-bic hypersurfaces are moduli spaces of isotropic vectors for the q-bic form β. Thus they are akin to quadric hypersurfaces, and their geometry may be accessed via algebraic methods relating to the bilinear form β.

2.1.3. Proposition. — *The q-bic hypersurface X associated with a q-bic form (V,β) represents the functor* $\mathrm{Sch}_{\mathbf{k}}^{\mathrm{opp}} \to \mathrm{Set}$ *given by*

$$T \mapsto \left\{ \mathcal{V}' \subset V_T \;\middle|\; \mathcal{V}' \text{ rank } 1 \text{ subbundle isotropic for } \beta \right\}.$$

Proof. — This follows directly from the moduli description of projective space, see [**Stacks**, 01NE], together with the construction of X in **2.1.1**. ■

A simple application of this observation is to show that a linear section of a q-bic hypersurface is another q-bic hypersurface:

2.1.4. Lemma. — *Let X be the q-bic hypersurface associated with a q-bic form (V,β). Let $U \subseteq V$ be any linear subspace and let*

$$\beta_U \colon \mathrm{Fr}^*(U) \otimes U \subset \mathrm{Fr}^*(V) \otimes V \xrightarrow{\beta} \mathbf{k}$$

be the—possibly zero—q-bic form on U obtained by restricting β. Then $X \cap \mathbf{P}U$ is the q-bic hypersurface associated with the q-bic form (U,β_U).

Proof. — It follows from **2.1.3** that both $X \cap \mathbf{P}U$ and X_{β_U} represent the functor of lines in U isotropic for β_U. Alternatively, this can be seen directly in coordinates by examining **2.1.2**. ■

2.1.5. Terminology. — It is convenient to conflate certain notions regarding q-bic forms with their counterparts in the setting of q-bic hypersurfaces. Let X be the q-bic hypersurface associated with a q-bic form (V,β). If the base change of (V,β) to an algebraic closure of $\mathbf{k}$ is isomorphic to a standard form of type λ, see **1.1.15** and **1.4.1**, then X is said to be *of type* λ. The *(co)rank* of X is the (co)rank of (V,β) in the sense of **1.1.21**. Projective subspaces $\mathbf{P}U \subseteq X$ for which U is an isotropic Hermitian subspace of (V,β) in the sense of **1.2.1** are called *Hermitian subspaces* of X; in the case $\mathbf{P}U$ is a point, this is also called a *Hermitian point* of X.

2.2. Differential invariants

Differential invariants of a q-bic hypersurface, such as the (co)normal, (co)tangent, and embedded tangent sheaves, admit simple descriptions in terms of an underlying q-bic form. As such, smoothness of a q-bic hypersurface and even the schematic nonsmooth locus may be described in terms of the underlying form (V,β): see 2.2.3 and 2.2.7. Most interesting are the properties of the embedded tangent sheaf, see 2.2.8. This carries a q-bic form and its properties encapsulate many of the idiosyncrasies found in the projective geometry of q-bic hypersurfaces: see 2.2.10, 2.2.12, and 2.2.13.

2.2.1. Conormal map. — For any complete intersection $Y \subset \mathbf{P}V$ whose equations have degrees coprime to the characteristic, its conormal sequence is a short exact sequence

$$0 \to \mathscr{C}_{Y/\mathbf{P}V} \xrightarrow{\delta} \Omega^1_{\mathbf{P}V}|_Y \to \Omega^1_Y \to 0.$$

The conormal map δ maps local equations of Y to their differential. When Y is a hypersurface of degree d, a choice of equation f determines an isomorphism $f \colon \mathscr{O}_Y(-d) \to \mathscr{C}_{Y/\mathbf{P}V}$ sending a local generator of $\mathscr{O}_Y(-d)$ to f.

The conormal map of a q-bic hypersurface admits a neat description in terms of the q-bic form β:

2.2.2. Lemma. — *There exists a commutative diagram*

$$\begin{tikzcd}
\mathscr{O}_X(-q-1) \arrow[rr, "\mathrm{eu}^{(q)}"'] \arrow[d, "f_\beta"'] & & \mathrm{Fr}^*(V)_X(-1) \arrow[d, "\beta^\vee"] \\
\mathscr{C}_{X/\mathbf{P}V} \arrow[r, "\delta"] & \Omega^1_{\mathbf{P}V}|_X \arrow[r, hook] & V_X^\vee(-1)
\end{tikzcd}$$

Proof. — Observe that $\beta^\vee \circ \mathrm{eu}^{(q)} \colon \mathscr{O}_X(-q-1) \to V_X^\vee(-1)$ factors through $\Omega^1_{\mathbf{P}V}|_X$: this sheaf is the kernel of $\mathrm{eu}^\vee \colon V_X^\vee(-1) \to \mathscr{O}_X$ and

$$\mathrm{eu}^\vee \circ \beta^\vee \circ \mathrm{eu}^{(q)} = f_\beta$$

vanishes on X. Since $q+1 = 1$ in $\mathbf{k}$, together with Euler's formula, this shows that the diagram in question commutes. ■

Smoothness of q-bic hypersurfaces is characterized in terms of nonsingularity of the underlying q-bic form:

2.2.3. Lemma. — *Let X be the q-bic hypersurface associated with a q-bic form (V, β). Then the following are equivalent:*

(i) *X is smooth;*

(ii) *β is nonsingular; and*

(iii) *$\mathrm{Gram}(\beta; v_0, \ldots, v_n)$ is invertible for any basis $V = \langle v_0, \ldots, v_n \rangle$.*

Proof. — That (ii) $\Leftrightarrow$ (iii) are equivalent follows directly from **1.1.14**. To see that these conditions are equivalent to (i), recall from [**Har77**, Theorem II.8.17] that X is smooth if and only if the conormal sequence

$$0 \to \mathscr{C}_{X/\mathbf{P}V} \xrightarrow{\delta} \Omega^1_{\mathbf{P}V}|_X \to \Omega^1_X \to 0$$

is exact and Ω^1_X is locally free of rank $n-1$. By **2.2.2**, the conormal map δ coincides with $\beta^\vee \circ \mathrm{eu}^{(q)}$. So if β is nonsingular, then $\beta^\vee \circ \mathrm{eu}^{(q)}$ is injective on each fibre and X is smooth. Conversely, if β is singular, then up to passing to a purely inseparable extension of $\mathbf{k}$, there is a 1-dimensional subspace $L \subset V$ such that $\mathrm{Fr}^*(L) \subset \mathrm{Fr}^*(V)^\perp$. Then L is isotropic and $\beta^\vee \circ \mathrm{eu}^{(q)}$ is zero on the fibre over $x := \mathbf{P}L \in X$, showing that Ω^1_X is not locally free at x and so X is not smooth. ∎

Together with the classification of nonsingular q-bic forms, this implies that over a separably closed field there is only one smooth q-bic hypersurface of a given dimension up to projective equivalence. This statement has been rediscovered many times: see, for instance, [**Par86**, Proposition 3.7], [**Hef85**, Corollary 9.11], and [**Bea90**, Théorème].

2.2.4. Corollary. — *Over a separably closed field, all smooth q-bic hypersurfaces in $\mathbf{P}V$ are projectively equivalent.*

Proof. — Let X and X' be smooth q-bic hypersurfaces in $\mathbf{P}V$, and let β and β' be defining q-bic forms. By **2.2.3**, these forms are nonsingular. By **2.1.3**, it suffices to show that there is an isomorphism $(V, \beta) \cong (V, \beta')$ of q-bic forms. By **1.2.14**, there exists diagonalizing bases $\langle v_0, \ldots, v_n \rangle$ for β, and $\langle v'_0, \ldots, v'_n \rangle$ for β'. The isomorphism $V \to V$ determined by $v_i \mapsto v'_i$ then yields an isomorphism of the q-bic forms. ■

Another way of stating this is that, over a separably closed field, the only smooth q-bic hypersurface is the Fermat hypersurface of degree $q+1$:

2.2.5. Corollary. — *Let X be a smooth q-bic hypersurface over a separably closed field. Then there exists a choice of coordinates $(x_0 : \cdots : x_n)$ of $\mathbf{P}V = \mathbf{P}^n$ such that*

$$X = \mathrm{V}(x_0^{q+1} + \cdots + x_n^{q+1}) \subset \mathbf{P}^n. \qquad ■$$

2.2.6. Nonsmooth locus. — The proof of 2.2.3 indicates that the singular locus may become visible only after a purely inseparable extension of $\mathbf{k}$. This means that the scheme-theoretic singular locus is defined by q-power equations. To explain, recall that the *nonsmooth locus* of a morphism $f : Y \to S$ which is flat, locally of finite presentation, and such that the nonempty fibres of f are of dimension d, is the closed subscheme of Y defined by the d-th Fitting ideal

$$\mathrm{Sing}(f) := \mathrm{V}(\mathrm{Fitt}_d(\Omega_{Y/S})).$$

Importantly, this scheme is supported on the points where f is not smooth, and its formation commutes with arbitrary base change; see [**Stacks**, 0C3K and 0C3I]. When $S = \mathrm{Spec}(\mathbf{k})$, write $\mathrm{Sing}(Y) := \mathrm{Sing}(f)$.

The following statement identifies the nonsmooth locus of a q-bic hypersurface. The formula below means, in the case that $V^\perp = \mathrm{Fr}^*(U)$ for some linear subspace $U \subseteq V$, $\mathrm{Sing}(X)$ is the closed subscheme cut out by q-powers of the linear forms vanishing on U; in particular, $\mathrm{Sing}(X)$ would be supported on $\mathbf{P}U$.

2.2.7. Lemma. — *For a q-bic hypersurface X associated with a q-bic form (V, β),*

$$\mathrm{Sing}(X) := \mathrm{Fitt}_{n-1}(\Omega^1_X) = \mathrm{V}(\mathcal{O}_X(-q) \xrightarrow{\mathrm{eu}^{(q)}} \mathrm{Fr}^*(V)_X \twoheadrightarrow (\mathrm{Fr}^*(V)/V^\perp)_X).$$

Proof. — The conormal sequence in 2.2.1 yields a presentation of Ω^1_X. Since $\Omega^1_{\mathbf{P}V}|_X$ is of rank n, this gives the first equality in

$$\mathrm{Fitt}_{n-1}(\Omega^1_X) = \mathrm{V}(\delta : \mathscr{C}_{X/\mathbf{P}V} \to \Omega^1_{\mathbf{P}V}|_X) = \mathrm{V}(\beta^\vee \circ \mathrm{eu}^{(q)} : \mathcal{O}_X(-q) \to \mathrm{Fr}^*(V)_X \to V_X^\vee).$$

The second equality follows from 2.2.2. This implies the result upon noting that, by 1.1.8, $\beta^\vee$ factors through the surjection $\mathrm{Fr}^*(V) \to \mathrm{Fr}^*(V)/V^\perp$. ■

2.2.8. Embedded tangent sheaf. — Given a local complete intersection closed subscheme $Y \subseteq \mathbf{P}V$, the *embedded tangent sheaf* of Y in $\mathbf{P}V$ is the sheaf $\mathcal{T}_Y^{\mathrm{e}}$ defined via pullback in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{T}_Y(-1) \arrow[r] & \mathcal{T}_{\mathbf{P}V}(-1)|_Y \arrow[r, "\delta^\vee"] & \mathcal{N}_{Y/\mathbf{P}V}(-1) \\
0 \arrow[r] & \mathcal{T}_Y^{\mathrm{e}} \arrow[u, two heads] \arrow[r] & V_Y \arrow[u, two heads] \arrow[r] & \mathcal{N}_{Y/\mathbf{P}V}(-1) \arrow[u, equal] \\
& \mathcal{O}_Y(-1) \arrow[u, hook] \arrow[r, equal] & \mathcal{O}_Y(-1) \arrow[u, hook, "\mathrm{eu}"]
\end{tikzcd}$$

obtained by juxtaposing the tangent sequence and the restriction to Y of the Euler sequence for $\mathbf{P}V$. Then for every smooth point $y \in Y$, the fibre $\mathcal{T}_{Y,y}^{\mathrm{e}} \subseteq V$ is the linear subspace underlying the embedded tangent space $\mathbf{T}_{Y,y}$ of Y at y.

This property of the embedded tangent sheaf together with the computation of the conormal map from 2.2.2 yields a neat description of the linear subspaces underlying the embedded tangent spaces of a q-bic hypersurface:

2.2.9. Lemma. — *Let $x = \mathbf{P}L$ be a smooth* $\mathbf{k}$*-point of a q-bic hypersurface X. Then*

$$\mathbf{T}_{X,x} = \mathbf{P}\,\mathrm{Fr}^*(L)^\perp.$$

Proof. — The identification of the conormal map in 2.2.2 implies

$$\mathcal{T}_X^{\mathrm{e}} = \ker(\delta^\vee \colon V_X \to \mathcal{N}_{X/\mathbf{P}V}(-1)) = \ker(\mathrm{eu}^{(q),\vee} \circ \beta \colon V_X \to \mathrm{Fr}^*(V)_X \to \mathcal{O}_X(q)).$$

Thus the fibre of $\mathcal{T}_X^{\mathrm{e}}$ at a $x = \mathbf{P}L$ is, on the one hand, the linear subspace of V underlying $\mathbf{T}_{X,x}$ by 2.2.8, and on the other hand, the subspace $\mathrm{Fr}^*(L)^\perp$. ■

The following identifies the embedded tangent bundle of a smooth q-bic hypersurface. This was first observed in [She12, Equation (3)].

2.2.10. Lemma. — *The morphism $\beta \colon V_X \to \mathrm{Fr}^*(V)_X^\vee$ induces a morphism*

$$\mathcal{T}_X^{\mathrm{e}} \to \mathrm{Fr}^*(\Omega^1_{\mathbf{P}V}(1))|_X.$$

If β is nonsingular, then this is an isomorphism.

Proof. — The computation of 2.2.2 yields a commutative diagram of solid arrows with exact rows, given by

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{T}_X^{\mathrm{e}} \arrow[r] \arrow[d, dashed] & V_X \arrow[r] \arrow[d, "\beta"] & \mathcal{N}_{X/\mathbf{P}V}(-1) \arrow[d, "f_\beta"] \\ 0 \arrow[r] & \mathrm{Fr}^*(\Omega^1_{\mathbf{P}V}(1))|_X \arrow[r] & \mathrm{Fr}^*(V)_X^\vee \arrow[r, "\mathrm{eu}^{(q),\vee}"] & \mathcal{O}_X(q) \arrow[r] & 0\,. \end{tikzcd}$$

This implies that β restricts to a morphism $\mathcal{T}_X^{\mathrm{e}} \to \mathrm{Fr}^*(\Omega^1_{\mathbf{P}V}(1))|_X$ filling the dashed arrow in the diagram.

If β is nonsingular, then $\beta\colon V_X \to \mathrm{Fr}^*(V)_X^\vee$ is an isomorphism. Moreover, X is smooth in this case by 2.2.3 so $V_X \to \mathcal{N}_{X/\mathbf{P}V}(-1)$ is surjective. That the map $\mathcal{T}_X^{\mathrm{e}} \to \mathrm{Fr}^*(\Omega^1_{\mathbf{P}V}(1))|_X$ is an isomorphism now follows from the Five Lemma. ∎

2.2.11. — Composing the Euler section with the map of 2.2.10 yields a morphism

$$\beta \circ \mathrm{eu}\colon \mathcal{O}_X(-1) \hookrightarrow \mathcal{T}_X^{\mathrm{e}} \to \mathrm{Fr}^*(\Omega^1_{\mathbf{P}V}(1))|_X$$

which is nonzero whenever β is, inducing a rational map $X \dashrightarrow \mathbf{P}(\Omega^1_{\mathbf{P}V}(1)|_X)$. In general, this is not linear over $\mathbf{k}$, and may be linearized as follows: form the diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{Fr}_{X/\mathbf{k}}"] \arrow[rr, bend left, "\mathrm{Fr}"] \arrow[dr] & X^{(q)} \arrow[r, "\mathrm{pr}_X"] \arrow[d] & X \arrow[d] \\ & \mathrm{Spec}(\mathbf{k}) \arrow[r, "\mathrm{Fr}"] & \mathrm{Spec}(\mathbf{k}) \end{tikzcd}$$

in which $X^{(q)} := X \otimes_{\mathbf{k},\mathrm{Fr}} \mathbf{k}$ is the q-power Frobenius twist of X—so that X' is the q-bic hypersurface associated with $(\mathrm{Fr}^*(V), \mathrm{Fr}^*(\beta))$, see 1.1.18—and $\mathrm{Fr}_{X/\mathbf{k}}\colon X \to X^{(q)}$ is the $\mathbf{k}$-linear q-power Frobenius morphism. Then

$$\mathrm{Fr}^*(\Omega^1_{\mathbf{P}V}(1))|_X \cong \mathrm{Fr}^*_{X/\mathbf{k}}(\mathrm{pr}_X^*\,\Omega^1_{\mathbf{P}V}(1)|_X) \cong \mathrm{Fr}^*_{X/\mathbf{k}}(\Omega^1_{\mathbf{P}V^{(q)}}(1)|_{X^{(q)}})$$

where $\mathbf{P}V^{(q)} := \mathbf{P}\,\mathrm{Fr}^*(V) = \mathbf{P}V \otimes_{\mathbf{k},\mathrm{Fr}} \mathbf{k}$. Then $\beta \circ \mathrm{eu}$ induces a map of $\mathcal{O}_X$-modules

$$\sigma_X\colon \mathcal{O}_X(-1) \to \mathrm{Fr}^*_{X/\mathbf{k}}(\Omega^1_{\mathbf{P}V^{(q)}}(1)|_{X^{(q)}})$$

which, in turn, induces a rational map $X \dashrightarrow \mathbf{P}(\Omega^1_{\mathbf{P}V^{(q)}}(1)|_{X^{(q)}})$. The Euler sequence shows that the $\mathbf{k}$-points of the projective bundle of $\Omega^1_{\mathbf{P}V^{(q)}}(1)$ are

$$\mathbf{P}(\Omega^1_{\mathbf{P}V^{(q)}}(1)) = \{\, (L \subset W \subset \mathrm{Fr}^*(V)) \mid \dim_{\mathbf{k}} L = 1 \text{ and } \dim_{\mathbf{k}} W = n \,\}$$

flags consisting of a line and a hyperplane in $\mathrm{Fr}^*(V)$.

2.2.12. Lemma. — *There exists a commutative diagram of rational maps*

$$\begin{tikzcd} & \mathbf{P}(\Omega^1_{\mathbf{P}V^{(q)}}(1)|_{X^{(q)}}) \arrow[d, "\tau"] \\ X \arrow[ur, dashed, "\sigma_X"] \arrow[r, "\mathrm{Fr}_{X/\mathbf{k}}"] & X^{(q)} \end{tikzcd}$$

of schemes over $\mathbf{k}$ *such that* σ_X *is defined away from* $\mathbf{P}\,\mathrm{Fr}^*(V)^\perp$, *and*

$$\sigma_X(x) = (\mathrm{Fr}^*(L) \subset L^\perp \subset \mathrm{Fr}^*(V)) \quad \textit{for any } x = \mathbf{P}L \in (X \setminus \mathbf{P}\,\mathrm{Fr}^*(V)^\perp)(\mathbf{k}).$$

If X *is furthermore smooth, then* σ_X *is a morphism and* $\sigma_X^*\mathcal{O}_\tau(-1) = \mathcal{O}_X(-1)$.

Proof. — That the diagram exists follows from the comments of 2.2.11. It is indeterminate at the points $x = \mathbf{P}L$ in which

$$\beta \circ \mathrm{eu} \colon \mathcal{O}_X(-1) \hookrightarrow \mathcal{T}_X^{\mathrm{e}} \to \mathrm{Fr}^*(\Omega^1_{\mathbf{P}V}(1)|_X)$$

is not injective; this occurs precisely when $L \subseteq \ker(\beta \colon V \to \mathrm{Fr}^*(V)^\vee) = \mathrm{Fr}^*(V)^\perp$. Away from such points, $\sigma_X(x)$ determines the hyperplane in $\mathrm{Fr}^*(V)$ with equation $\beta(-, L)$. The final statement now follows from 2.2.3. ∎

2.2.13. Tangent form. — Since the embedded tangent sheaf $\mathcal{T}_X^{\mathrm{e}}$ is a subsheaf of V_X, the q-bic form β defining X restricts to a form

$$\beta_{\tan} \colon \mathrm{Fr}^*(\mathcal{T}_X^{\mathrm{e}}) \otimes \mathcal{T}_X^{\mathrm{e}} \subset \mathrm{Fr}^*(V)_X \otimes V_X \xrightarrow{\beta} \mathcal{O}_X,$$

called the *tangent q-bic form* of X. On the fibre at a smooth $\mathbf{k}$-point $x = \mathbf{P}L$, this is the restriction of β to the subspace $\mathrm{Fr}^*(L)^\perp$ underlying the tangent hyperplane $\mathbf{T}_{X,x}$.

When X is smooth, the basic properties of $\beta_{\tan}$ are as follows:

2.2.14. Proposition. — *Let X be the q-bic hypersurface associated with a q-bic form (V,β). If X is smooth, then its tangent form*

$$\beta_{\mathrm{tan}} \colon \mathrm{Fr}^*(\mathcal{T}_X^{\mathrm{e}}) \otimes \mathcal{T}_X^{\mathrm{e}} \to \mathcal{O}_X$$

is everywhere of corank 1 and it induces an exact sequence

$$0 \longrightarrow \mathrm{Fr}^*(\mathcal{N}_{X/\mathbf{P}V}(-1))^\vee \xrightarrow{\phi_X} \mathcal{T}_X^{\mathrm{e}} \xrightarrow{\beta_{\mathrm{tan}}} \mathrm{Fr}^*(\mathcal{T}_X^{\mathrm{e}})^\vee \xrightarrow{\mathrm{eu}^{(q),\vee}} \mathrm{Fr}^*(\mathcal{O}_X(-1))^\vee \longrightarrow 0$$

where the map ϕ_X is induced by $\beta^{-1} \circ \delta^{(q)}$.

Proof. — Note that β_{tan} has corank at most 1 by **1.1.22**; it has corank at least 1 by **2.2.3**, since the restriction of the tangent form at a point $x = \mathbf{P}L \in X$ defines, by **2.1.4**, the q-bic hypersurface $X \cap \mathbf{T}_{X,x}$ and this is singular at x. In fact, this implies by **2.2.7** that

$$\ker(\beta_{\mathrm{tan}}^\vee \colon \mathrm{Fr}^*(\mathcal{T}_X^{\mathrm{e}}) \to \mathcal{T}_X^{\mathrm{e},\vee}) = \mathrm{image}(\mathrm{eu}^{(q)} \colon \mathrm{Fr}^*(\mathcal{O}_X(-1)) \to \mathrm{Fr}^*(\mathcal{T}_X^{\mathrm{e}})\big),$$

thereby identifying the cokernel of β_{tan} in the above exact sequence. To identify the kernel, consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{T}_X^{\mathrm{e}} \arrow[r] \arrow[d, "\beta_{\mathrm{tan}}"] & V_X \arrow[r, "\delta^\vee"'] & \mathcal{N}_{X/\mathbf{P}V}(-1) \arrow[r] & 0 \\
0 & \mathrm{Fr}^*(\mathcal{T}_X^{\mathrm{e}})^\vee \arrow[l] & \mathrm{Fr}^*(V_X)^\vee \arrow[l] \arrow[u, "\beta^{-1}"] & \mathrm{Fr}^*(\mathcal{N}_{X/\mathbf{P}V}(-1))^\vee \arrow[l, "\delta^{(q)}"'] \arrow[u] & 0 \arrow[l]
\end{tikzcd}$$

in which the rows are the exact sequences defining the embedded tangent bundle from **2.2.8**. The map $\mathrm{Fr}^*(\mathcal{N}_{X/\mathbf{P}V}(-1))^\vee \to \mathcal{N}_{X/\mathbf{P}V}$ since, by **2.2.2**, it is

$$\delta^\vee \circ \beta^{-1} \circ \delta^{(q)} = (\mathrm{eu}^{(q),\vee} \circ \beta) \circ \beta^{-1} \circ (\beta^\vee \circ \mathrm{eu}^{(q)})^{(q)} = (\mathrm{eu}^{(q),\vee} \circ \beta \circ \mathrm{eu})^{(q),\vee}$$

the q-power of the equation of X. Thus there is a morphism

$$\phi_X := \beta^{-1} \circ \delta^{(q)} \colon \mathrm{Fr}^*(\mathcal{N}_{X/\mathbf{P}V}(-1))^\vee \to \mathcal{T}_X^{\mathrm{e}}$$

and the diagram now shows that this is an isomorphism onto the kernel of β_{tan}. ∎

2.2.15. Residual point of tangency. — Let x be a closed point of a smooth q-bic hypersurface X. The computation of 2.2.14 means that the intersection $X \cap \mathbf{T}_{X,x}$ of X with its embedded tangent space at x is a q-bic hypersurface of corank 1 which is singular at x. Besides the singular point, each q-bic of corank 1 has another distinguished point corresponding to $\mathrm{Fr}^*(\mathbf{T}_{X,x})^\perp$; this is given by the isotropic line subbundle $\mathrm{Fr}^*(\mathscr{C}_{X/\mathbf{P}V}(1))$ above. This point is called the *residual point of tangency* to x and is denoted by $\phi_X(x)$. See 2.9.9 for more.

2.3. Automorphisms

Let X be the q-bic hypersurface associated with a q-bic form (V,β). Then X has an automorphism group scheme, denoted $\mathbf{Aut}(X)$, which is a group scheme over $\mathbf{k}$, locally of finite type: see [MO67, Theorem 3.7]. This brief Section discusses a few properties of this scheme.

2.3.1. Linear automorphisms. — Let $\mathbf{Aut}(V,\beta)$ be the automorphism group scheme of the q-bic form (V,β), as introduced in 1.3.1. It acts linearly on $\mathbf{P}V$ preserving X, thereby inducing a natural morphism $\mathbf{Aut}(V,\beta) \to \mathbf{Aut}(X)$ of group schemes. Its image is the subgroup scheme consisting of linear automorphisms of X. Its kernel is the intersection

$$\mathbf{Aut}(V,\beta) \cap \mathbf{G}_m \cong \boldsymbol{\mu}_{q+1}$$

of $\mathbf{Aut}(V,\beta)$ with the central torus $\mathbf{G}_m \subset \mathbf{GL}(V)$. Examining its action on any $v \in \mathrm{Fr}^*(V)$ and $w \in V$ such that $\beta(v,w) \neq 0$ shows it consists only of $(q+1)$-st roots of unity. Therefore there is an exact sequence of group schemes

$$1 \to \boldsymbol{\mu}_{q+1} \to \mathbf{Aut}(V,\beta) \to \mathbf{Aut}(X).$$

This sequence is not right exact for general (V,β), see 3.6.2 for example. It is, however, usually surjective when (V,β) is nonsingular:

2.3.2. Lemma. — *If (V,β) is nonsingular and (n,q) is neither $(1,2)$ nor $(2,3)$, then the restriction morphism $\mathbf{Aut}(V,\beta) \to \mathbf{Aut}(X)$ is surjective and $\mathbf{Aut}(X) \cong \mathrm{PU}(V,\beta)$.*

Proof. — By 2.2.3, X is smooth. By 2.3.3 below, $\mathbf{Aut}(X)$ is reduced; see also [MO67, Example 5]. Then by [Cha78, Theorem 1] and [MM64, Theorem 2], all automorphisms of X are linear; see also [Poo05, Theorem 1.1]. This proves the statement. This can also be verified by a direct computation: see [Shi88]. ∎

The Lie algebra of $\mathbf{Aut}(X)$ can be determined as follows:

2.3.3. Lemma. — *There are canonical isomorphisms of* $\mathbf{k}$*-vector spaces:*

$$\mathrm{Lie}(\mathbf{Aut}(X)) \cong \mathrm{H}^0(X, \mathcal{T}_X) \cong \mathrm{Fr}^*(V)^\perp \otimes V^\vee.$$

Proof. — Its defining exact sequence from 2.2.8 together with the computation of the conormal map from 2.2.2 give a canonical identification

$$\mathrm{H}^0(X, \mathcal{T}_X^{\mathrm{e}}(1)) = \ker\big(V \otimes V^\vee \xrightarrow{\beta \otimes \mathrm{id}} \mathrm{Fr}^*(V)^\vee \otimes V^\vee \xrightarrow{\mathrm{mult}} \mathrm{Sym}^{q+1}(V^\vee)\big).$$

The kernel of the multiplication map $\mathrm{Fr}^*(V)^\vee \otimes V^\vee \to \mathrm{Sym}^{q+1}(V^\vee)$ consists of the q-bic polynomials vanishing on X. This is the 1-dimensional subspace spanned by the equation f_β of X and is the image under $\beta \otimes \mathrm{id}$ of the trace element of $V \otimes V^\vee$. Recalling that $\mathrm{Fr}^*(V)^\perp = \ker(\beta : V \to \mathrm{Fr}^*(V)^\vee)$, this means there is a canonical short exact sequence

$$0 \to \mathbf{k} \xrightarrow{\mathrm{tr}} \mathrm{H}^0(X, \mathcal{T}_X^{\mathrm{e}}(1)) \to \mathrm{Fr}^*(V)^\perp \otimes V^\vee \to 0.$$

The trace element is the image of the Euler section in

$$0 \to \mathcal{O}_X \xrightarrow{\mathrm{eu}} \mathcal{T}_X^{\mathrm{e}}(1) \to \mathcal{T}_X \to 0$$

so this gives the claimed identification of $\mathrm{H}^0(X, \mathcal{T}_X)$. ∎

Although $\mathbf{Aut}(V, \beta)/\boldsymbol{\mu}_{q+1}$ is not always isomorphic to $\mathbf{Aut}(X)$, their Lie algebras are nonetheless isomorphic:

2.3.4. Proposition. — *Then canonical morphism* $\mathbf{Aut}(V, \beta) \to \mathbf{Aut}(X)$ *induces an isomorphism on Lie algebras.*

Proof. — Taking Lie algebras of along the exact sequence of group schemes given in 2.3.1 shows that the map $\mathrm{Lie}(\mathbf{Aut}(V,\beta)) \to \mathrm{Lie}(\mathbf{Aut}(X))$ is injective. But these spaces have the same dimensions by 1.3.4 and 2.3.3, and so they are isomorphic. ■

2.4. Cones

Cones over q-bic hypersurfaces often intervene in inductive arguments and constructions, and the notion of cone points provides a manner with which to endow the set of Hermitian points of a q-bic hypersurface with a scheme structure. This Section develops some basic properties of cones, see especially 2.4.7, 2.4.8, and 2.4.9. An application of this material is to count the number of Hermitian points in a smooth q-bic hypersurface, and to show that maximal isotropic subspaces are Hermitian: see 2.4.13 and 2.4.16.

To fix terminology, consider first a projective variety $Y \subset \mathbf{P}V$ over an algebraically closed field. Then Y is said to be a *cone over a closed point* $v \in Y$ if for every closed point $y \in Y$, the line $\langle v, y\rangle$ spanned in $\mathbf{P}V$ by v and y is contained in Y. The locus of such points v form a linear subspace $\mathrm{Vert}(Y)$ called the *vertex of* Y. A *cone* is a projective variety with a nonempty vertex. A projective variety over a general field is called a *cone* if it is a cone upon base extension to an algebraic closure.

From its definition in 1.1.6, it follows that the radical of a q-bic form can only grow along purely inseparable extensions. This allows the vertex of a q-bic hypersurface to be described in terms of its q-bic form when the base field is perfect:

2.4.1. Proposition. — *Let X be the q-bic hypersurface associated with a q-bic form (V,β). If $\mathbf{k}$ is perfect, then the vertex of X is defined over $\mathbf{k}$ and is given by*

$$\mathrm{Vert}(X) = \mathbf{P}(\mathrm{rad}(\beta)) = \mathbf{P}\,\mathrm{Fr}^*(V)^{\perp} \cap \mathbf{P}\,\mathrm{Fr}^{-1}(V^{\perp}).$$

In particular, X is a cone if and only if β has a radical.

Proof. — Since the radical $W := \mathrm{rad}(\beta) = \mathrm{Fr}^*(V)^{\perp} \cap \mathrm{Fr}^{-1}(V^{\perp})$ of β is defined over $\mathbf{k}$, it suffices to check that this give the vertex of X upon passage to the algebraic closure. For the remainder of the proof, assume $\mathbf{k}$ is algebraically closed.

The inclusion $\mathbf{P}W \subseteq \mathrm{Vert}(X)$ follows easily from the definitions. For the reverse inclusion $\mathbf{P}W \supseteq \mathrm{Vert}(X)$, consider a closed point $z = \mathbf{P}\langle w\rangle$ of the vertex. The goal is to show $\beta(w^{(q)}, v) = \beta(v^{(q)}, w) = 0$ for all $v \in V$. If $y = \mathbf{P}\langle v\rangle$ were contained in X, then $\langle y, z\rangle = \mathbf{P}\langle v, w\rangle$ is contained in X, and the result follows from 1.1.4. If y were not contained in X, let $\lambda \in \mathbf{k}$ be arbitrary and consider

$$\beta((v+\lambda w)^{(q)}, v+\lambda w) = \beta(v^{(q)}, v) + \lambda\beta(v^{(q)}, w) + \lambda^q\beta(w^{(q)}, v).$$

That $y \notin X$ means $\beta(v^{(q)}, v) \neq 0$. Thus if $\beta(v^{(q)}, w) \neq 0$ or $\beta(w^{(q)}, v) \neq 0$, then there exists nonzero λ such that $v + \lambda w$ is isotropic. Then $y' := \mathbf{P}\langle v + \lambda w\rangle$ is a point of X and so

$$X \supseteq \langle y', z\rangle = \mathbf{P}\langle v + \lambda w, w\rangle = \mathbf{P}\langle v, w\rangle = \langle y, z\rangle,$$

contradicting $y \notin X$. Thus $\beta(v^{(q)}, w) = \beta(w^{(q)}, v) = 0$, whence $w \in W$. ■

As a consequence, a q-bic hypersurface X with high corank must be a cone:

2.4.2. Corollary. — *If $2\dim \mathrm{Sing}(X) \geq \dim X$, then X is either of type $\mathbf{N}_2^{\oplus m}$ or a cone.*

Proof. — Consider the contrapositive: assume X is not of type $\mathbf{N}_2^{\oplus m}$ and does not have a vertex. By 2.4.1, this means the q-bic form (V, β) underlying X does not have a radical. By the classification 1.4.1, this menas β is a sum of forms of type $\mathbf{1}$ and $\mathbf{N}_k$ with $k \geq 2$. Since the corank of β is maximized by a sum of forms of type $\mathbf{N}_2$,

$$2\,\mathrm{corank}(\beta) < 2\,\mathrm{corank}(\mathbf{N}_2^{\oplus\lfloor \dim_{\mathbf{k}} V/2\rfloor}) \leq \dim_{\mathbf{k}} V.$$

Then 2.2.7 implies $2\dim \mathrm{Sing}(X) < \dim X$, as required. ■

Cones over q-bic hypersurfaces frequently arise upon taking special linear sections of a given q-bic. The following characterizes when this happens:

2.4.3. Corollary. — *Let X be the q-bic hypersurface associated with a q-bic form (V, β). Let $\mathbf{P}U \subseteq X$ be a linear subspace. Then a linear section $X \cap \mathbf{P}W$ is a cone with vertex containing $\mathbf{P}U$ if and only if $W \subseteq \mathrm{Fr}^*(U)^\perp \cap \mathrm{Fr}^{-1}(U^\perp)$.*

Proof. — By 2.1.4, $X \cap \mathbf{P}W$ is the q-bic defined by the restriction β_W of β to W. Then, by 2.4.1, $X \cap \mathbf{P}W$ is a cone with vertex containing $\mathbf{P}U$ if and only if U lies in the radical of β_W. And this happens if and only if $W \subseteq \mathrm{Fr}^*(U)^\perp \cap \mathrm{Fr}^{-1}(U^\perp)$. ■

The following is a special case in which $\mathbf{P}W$ is furthermore contained in X. A particular form of this has been observed by Shimada in [Shi01, Proposition 2.10]. This refines the general fact that given a projective variety $Y \subseteq \mathbf{P}V$ and a $\mathbf{k}$-point $y \in Y$, any linear subspace $\mathbf{P}U \subseteq Y$ passing through y must be contained in the embedded tangent space of Y at y.

2.4.4. Corollary. — *Let $\mathbf{P}U \subseteq \mathbf{P}W \subset X$ be a nested pair of linear subspaces. Then*

$$\mathbf{P}W \subseteq X \cap \mathbf{P}\,\mathrm{Fr}^*(U)^\perp \cap \mathbf{P}\,\mathrm{Fr}^{-1}(U^\perp).$$

Proof. — This follows from 2.4.3, where $\mathbf{P}W$ is viewed as the q-bic in $\mathbf{P}W$ defined by the zero form, and noting that any linear subspace is a vertex. ■

2.4.5. Cone points. — A $\mathbf{k}$-point y of a projective variety $Y \subseteq \mathbf{P}V$ is said to be a *cone point* of Y if there exists a hyperplane $H \subset \mathbf{P}V$ such that $Y \cap H$ is a cone over y. A few remarks about this definition when $\mathbf{k}$ is algebraically closed:

(i) If Y is a linear or a quadric hypersurface, then every $\mathbf{k}$-point is a cone point.
(ii) Cone points of a smooth cubic hypersurface are its *Eckardt points*.
(iii) If a cone point $y \in Y$ is a smooth point, then the witnessing hyperplane H must contain the embedded tangent space of Y at y.
(iv) In the case that Y is a hypersurface and y is a smooth point, cone points are sometimes referred to as *total inflection points* or *star points*; see [CC10, CC11].

Cone points of q-bic hypersurfaces are analogous to those of quadrics as in 2.4.5(i) in that $X \cap H$ is itself a q-bic, see 2.1.4, which is a cone over a lower dimensional q-bic. Unlike quadrics, however, q-bics generally only have finitely many cone points, and are thus also analogous to the Eckardt points of cubic hypersurfaces as in 2.4.5(ii).

Cone points of a q-bic hypersurface may be characterized as follows:

2.4.6. Lemma. — *Let X be a q-bic hypersurface and let $x = \mathbf{P}L$ be a $\mathbf{k}$-point of X.*

(i) *Let* $\mathbf{P}W \subseteq \mathbf{P}V$ *be a linear subspace. Then* $X \cap \mathbf{P}W$ *is a cone over* x *if and only if*

$$W \subseteq \mathrm{Fr}^*(L)^\perp \cap \mathrm{Fr}^{-1}(L^\perp).$$

(ii) *The point* x *is a cone point of* X *if and only if*

$$\dim_{\mathbf{k}}(\mathrm{Fr}^*(L)^\perp \cap \mathrm{Fr}^{-1}(L^\perp)) \geq \dim_{\mathbf{k}} V - 1.$$

Proof. — Item (i) follows directly from **2.4.3**. Item (ii) now follows since x is a cone point of X if and only if there exists a hyperplane W of V contained in $\mathrm{Fr}^{-1}(L^\perp) \cap \mathrm{Fr}^*(L)^\perp$. ■

This criterion can be reformulated into a more geometric classification of cone points. The notion of Hermitian points of a q-bic hypersurface was defined in **2.1.5**.

2.4.7. Corollary. — *A* $\mathbf{k}$*-point* x *of* X *is a cone point if and only if either*

(i) x *is a singular point, or*

(ii) x *is a smooth point lying in* $\mathbf{P}\,\mathrm{Fr}^*(V)^\perp \subseteq X$*, or*

(iii) x *is a smooth Hermitian point.*

Proof. — Let $x = \mathbf{P}L$. Then by **2.4.6**(i), x is a cone point if and only there is a hyperplane lying in $\mathrm{Fr}^*(L)^\perp \cap \mathrm{Fr}^{-1}(L^\perp)$. This happens if and only if either

(i) $\mathrm{Fr}^*(L)^\perp = V$, or

(ii) $\mathrm{Fr}^{-1}(L^\perp) = V$, or

(iii) $\mathrm{Fr}^*(L) = \mathrm{Fr}^{-1}(L^\perp)$ are both hyperplanes.

The first condition occurs when x is a singular point, see **2.2.9**; if x is not a singular point, then the second condition occurs when $\beta^\vee \colon \mathrm{Fr}^*(V) \to V^\vee \twoheadrightarrow L^\vee$ vanishes; dually, this means $\beta \colon L \subset V \to \mathrm{Fr}^*(V)^\vee$ is zero, so $L \subseteq \mathrm{Fr}^*(V)^\perp$. Finally, the third condition occurs precisely when x is a smooth Hermitian point, see **1.2.8**. ■

The following recasts cone points of smooth q-bic surfaces in several ways:

2.4.8. Proposition. — *Let* X *be a smooth* q*-bic hypersurface. Then for a* $\mathbf{k}$*-point* x *of* X*, the following are equivalent:*

(i) $X \cap \mathbf{T}_{X,x}$ *is a cone over a smooth* q*-bic hypersurface of dimension* 2 *less;*

(ii) *x is a cone point of X;*

(iii) *x is a Hermitian point of X.*

Proof. — That (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) follows from definitions and **2.4.7**(iii). For (iii) $\Rightarrow$ (i), let (V,β) be a q-bic form underlying X and let $x = \mathbf{P}L$ with L a 1-dimensional Hermitian subspace of V. Then $\mathrm{Fr}^*(L)^\perp = \mathrm{Fr}^{-1}(L^\perp)$ by **1.2.6**, and this is the hyperplane underlying the embedded tangent space of X at x, see **2.2.9**. The restriction of β therein contains L in its radical and has corank at most 1 by **1.1.22**. Thus by **2.4.1**, $\mathbf{T}_{X,x} \cap X$ is a cone over a smooth q-bic of 2 dimensions less. ∎

2.4.9. Scheme of cone points. — The criterion of **2.4.6** may be used to produce global equations for the set of cone points in a q-bic hypersurface X, moreover endowing it with a natural scheme structure. Consider the morphism of $\mathcal{O}_X$-modules:

$$(\mathrm{eu}^\vee \circ \beta^\vee, \mathrm{eu}^{(q^2),\vee} \circ \beta^{(q)}) \colon \mathrm{Fr}^*(V)_X \to \mathcal{O}_X(1) \oplus \mathcal{O}_X(q^2).$$

Assume for the moment that $\mathbf{k}$ is algebraically closed. Let $x = \mathbf{P}L$ be a closed point of X and let $\kappa(x)$ be its residue field. Taking fibres at x of the kernels of the components gives:

$$\ker(\mathrm{eu}^\vee \circ \beta^\vee \colon \mathrm{Fr}^*(V)_X \to \mathcal{O}_X(1)) \otimes_{\mathcal{O}_X} \kappa(x) = L^\perp,$$
$$\ker(\mathrm{eu}^{(q^2),\vee} \circ \beta^{(q)} \colon \mathrm{Fr}^*(V)_X \to \mathcal{O}_X(q^2)) \otimes_{\mathcal{O}_X} \kappa(x) = \mathrm{Fr}^*(\mathrm{Fr}^*(L)^\perp).$$

Combined with **2.4.6**(ii), this implies that the degeneracy locus X_{cone} given by

$$\mathrm{rank}\big((\mathrm{eu}^\vee \circ \beta^\vee, \mathrm{eu}^{(q^2),\vee} \circ \beta^{(q)}) \colon \mathrm{Fr}^*(V)_X \to \mathcal{O}_X(1) \oplus \mathcal{O}_X(q^2)\big) \leq 1$$

is supported on the set of cone points of X.

This construction may be performed for any q-bic hypersurface X over any field $\mathbf{k}$ to yield the *scheme of cone points* X_{cone} of X. Its equations may be simplified slightly:

2.4.10. Lemma. — *The scheme X_{cone} of cone points is the top degeneracy locus of*

$$(\mathrm{eu}^\vee \circ \beta^\vee, \delta^{(q),\vee}) \colon \mathrm{Fr}^*(\mathcal{T}_{\mathbf{P}V}(-1))|_X \to \mathcal{O}_X(1) \oplus \mathrm{Fr}^*(\mathcal{N}_{X/\mathbf{P}V}(-1)).$$

On the smooth locus $X_{\mathrm{sm}} \subset X$, the scheme X_{cone} is defined by the vanishing of

$$\mathrm{eu}^\vee \circ \beta^\vee \colon \mathrm{Fr}^*(\mathcal{T}_X(-1))|_{X_{\mathrm{sm}}} \to \mathcal{O}_X(1)|_{X_{\mathrm{sm}}}.$$

Proof. — By the Euler sequence, $\mathrm{Fr}^*(\mathcal{T}_{\mathbf{P}V}(-1))|_X$ is the quotient of $\mathrm{Fr}^*(V)_X$ by the section $\mathrm{eu}^{(q)} \colon \mathcal{O}_X(-q) \to \mathrm{Fr}^*(V)_X$. The first statement follows from **2.4.9** upon observing that

$$\mathrm{eu}^\vee \circ \beta^\vee \circ \mathrm{eu}^{(q)} = 0 \quad \text{and} \quad \mathrm{eu}^{(q^2),\vee} \circ \beta^{(q)} \circ \mathrm{eu}^{(q)} = 0$$

as morphisms of line bundles on X. The identification of the second component with the normal map $\delta^{(q),\vee}$ follows from **2.2.2**.

For the second statement, the normal map $\delta^{(q),\vee}$ is surjective on the smooth locus of X, and thereon, the degeneracy locus coincides with the vanishing locus of the first component $\mathrm{eu}^\vee \circ \beta^\vee$ restricted to $\ker(\delta^{(q),\vee}) = \mathrm{Fr}^*(\mathcal{T}_X(-1))$. ■

The scheme X_{cone} is therefore the rank 1 degeneracy locus of a map of bundles of ranks n and 2. Thus it has expected dimension 0, see [**Ful98**, Chapter 14], for example. When it does have expected dimension, its degree is as follows:

2.4.11. Lemma. — *If the scheme of cone points is of expected dimension* 0*, then*

$$\deg(X_{\mathrm{cone}}) = \frac{(q^{n+1} - (-1)^{n+1})(q^n - (-1)^n)}{q^2 - 1}.$$

Proof. — Since X_{cone} is given as a degeneracy locus in **2.4.10**, this follows from the Thom–Porteous formula, see [**Ful98**, Theorem 14.4 and Example 14.4.1]. Namely, the class of X_{cone} in the Chow ring of X is given by the degree $n-1$ component of

$$\frac{c(\mathrm{Fr}^*(\Omega^1_{\mathbf{P}V}(1))|_X)}{c(\mathcal{O}_X(-1))c(\mathcal{O}_X(-q^2))} = \frac{1}{(q+1)^2}\Big(q \cdot \frac{1}{1+qh} - \frac{1}{q-1} \cdot \frac{1}{1-h} + \frac{q^3}{q-1} \cdot \frac{1}{1-q^2 h}\Big)$$

where $h := c_1(\mathcal{O}_X(1))$ and $c(\mathrm{Fr}^*(\Omega^1_{\mathbf{P}V}(1))|_X) = c(\mathcal{O}_X(q))^{-1}$ by the Euler sequence. Expanding the geometric series and taking the coefficient of h^{n-1} gives

$$[X_{\mathrm{cone}}] = \frac{(-1)^{n-1}q^n(q-1) - 1 + q^{2n+1}}{(q+1)^2(q-1)} \cdot h^{n-1} = \frac{(q^{n+1} - (-1)^{n+1})(q^n - (-1)^n)}{q^2 - 1} \cdot \frac{h^{n-1}}{q+1}$$

in $\mathrm{CH}_0(X)$. Since $\deg(h) = q+1$, the result follows. ■

For general X, the schematic structure of X_{cone} may be quite intricate even when it is of expected dimension: see 3.6.3, 3.10.3, and 3.10.4. When X is smooth, it is as simple as possible:

2.4.12. Lemma. — *If X is a smooth q-bic hypersurface, then X_{cone} is étale over* $\mathbf{k}$.

Proof. — The result will follow upon showing that the conormal morphism

$$\delta_{X_{\mathrm{cone}}/X} \colon \mathscr{C}_{X_{\mathrm{cone}}/X} \to \Omega^1_X|_{X_{\mathrm{cone}}}$$

is an isomorphism of sheaves on X_{cone}. By the second statement of 2.4.10, X_{cone} is the vanishing locus of $\mathrm{eu}^\vee \circ \beta^\vee \colon \mathrm{Fr}^*(\mathscr{T}_X(-1)) \to \mathscr{O}_X(1)$, so restriction gives a map

$$\delta_{X_{\mathrm{cone}}/X} \circ (\mathrm{eu}^\vee \circ \beta^\vee) \colon \mathrm{Fr}^*(\mathscr{T}_X(-1))|_{X_{\mathrm{cone}}} \to \Omega^1_X|_{X_{\mathrm{cone}}}$$

Locally, $\delta_{X_{\mathrm{cone}}/X}$ acts by differentiating the coordinates $\mathrm{eu}^\vee$. Thus this map is dual to the isomorphism induced by $\mathscr{T}_X^{\mathrm{e}} \cong \mathrm{Fr}^*(\Omega^1_{\mathbf{P}V}(1))|_X$ from 2.2.10 upon passing to the quotient by the Euler section. ■

This gives a geometric method to determine the number of Hermitian points of a smooth q-bic hypersurface. See also [BC66, Theorem 8.1] and [Seg65, n.32].

2.4.13. Corollary. — *The number of Hermitian points of a smooth q-bic $(n-1)$-fold over a separably closed field is*

$$\#X_{\mathrm{Herm}} = \frac{(q^{n+1}-(-1)^{n+1})(q^n-(-1)^n)}{q^2-1}.$$

Proof. — By 2.4.8, the Hermitian points of X are the cone points of X. Since the scheme of cone points is étale by 2.4.12, this implies that $\#X_{\mathrm{Herm}} = \deg(X_{\mathrm{cones}})$ and the result follows from 2.4.11. ■

Cone points are often incident with linear spaces contained in X, see 2.4.4. However, linear spaces in X that contain cone points are typically quite special. The remainder of this Section presents a basic study of this. The following gives equations for the subscheme of cone points contained in a given linear subspace:

2.4.14. Lemma. — *Let* $\mathbf{P}U \subset X$ *be a linear subspace. Then* $\mathbf{P}U \cap X_{\mathrm{cone}}$ *is the degeneracy locus of a map*

$$\mathrm{Fr}^*(\mathcal{N}_{\mathbf{P}U/\mathbf{P}V}(-1)) \to \mathcal{O}_{\mathbf{P}U}(1) \oplus \mathcal{O}_{\mathbf{P}U}(q^2).$$

If $\mathbf{P}U$ *is contained in the smooth locus, then* $\mathbf{P}U \cap X_{\mathrm{cone}}$ *is the vanishing locus of a map*

$$\mathrm{Fr}^*(\mathcal{N}_{\mathbf{P}U/X}(-1)) \to \mathcal{O}_{\mathbf{P}U}(1).$$

Proof. — Restrict the morphism from 2.4.9 defining X_{cone} to $\mathbf{P}U$. The kernel computation there shows that there is an inclusion

$$\mathrm{Fr}^*(U)_{\mathbf{P}U} \subset \ker\big((\mathrm{eu}^\vee \circ \beta^\vee, \mathrm{eu}^{(q^2),\vee} \circ \beta^{(q)}) \colon \mathrm{Fr}^*(V)_{\mathbf{P}U} \to \mathcal{O}_{\mathbf{P}U}(1) \oplus \mathcal{O}_{\mathbf{P}U}(q^2)\big).$$

Passing to the quotient by this and the q-power Euler section, as in the proof of 2.4.10, gives the first statement. If $\mathbf{P}U \subset X$ is contained in the smooth locus, argue as in the second part of 2.4.10 to see that the map $\mathrm{Fr}^*(\mathcal{N}_{\mathbf{P}U/\mathbf{P}V}(-1)) \to \mathcal{O}_{\mathbf{P}U}(q^2)$ is the natural surjection to $\mathrm{Fr}^*(\mathcal{N}_{X/\mathbf{P}V}(-1))|_{\mathbf{P}U}$. Passing to the kernel gives the result. ■

If X is of even dimension $2m$ and $\mathbf{P}U$ is an m-plane contained in X, 2.4.14 expresses $\mathbf{P}U \cap X_{\mathrm{cone}}$ as a rank 1 degeneracy locus between bundles of ranks $m+1$ and 2. Therefore, so long as the locus is nonempty, it has dimension at least 0, see [Ful98, Chapter 14]. The following verifies that the degeneracy locus is nonempty in this case using a normal bundle computation performed later in 2.7.6:

2.4.15. Corollary. — *Let* X *be a* q*-bic hypersurface of dimension* $2m$*. Any* m*-plane* $\mathbf{P}U$ *in* X *contains a cone point, and if* $\dim(\mathbf{P}U \cap X_{\mathrm{cone}}) = 0$*, then*

$$\deg(\mathbf{P}U \cap X_{\mathrm{cone}}) = \frac{q^{2m+2}-1}{q^2-1} = q^{2m} + q^{2m-2} + \cdots + q^2 + 1.$$

Proof. — If $\mathbf{P}U$ intersects the singular locus of X, then it contains a cone point, see 2.4.7. If $\mathbf{P}U$ is contained in the smooth locus of X, then the second statement of 2.4.14 shows that $\mathbf{P}U \cap X_{\mathrm{cone}}$ is the zero locus of a map

$$\mathrm{Fr}^*(\mathcal{N}_{\mathbf{P}U/X}(-1)) \to \mathcal{O}_{\mathbf{P}U}(1).$$

Since $\mathbf{P}U$ is maximal isotropic, 2.7.6 gives $\mathcal{N}_{\mathbf{P}U/X}(-1) \cong \mathrm{Fr}^*(\Omega^1_{\mathbf{P}U}(1))$. Thus the section above vanishes on $\mathbf{P}U$, showing it contains a cone point.

If $\dim(\mathbf{P}U \cap X_{\mathrm{cone}}) = 0$, the Thom–Porteous formula, see [Ful98, Theorem 14.4 and Example 14.4.1], shows that its class in the Chow ring is the degree m part of

$$\frac{c(\mathcal{O}_{\mathbf{P}U} \otimes \mathrm{Fr}^*(V/U)^\vee)}{c(\mathcal{O}_{\mathbf{P}U}(-1))c(\mathcal{O}_{\mathbf{P}U}(-q^2))} = \frac{1}{1-q^2}\left(\frac{1}{1-h} - \frac{q^2}{1-q^2h}\right) \in \mathrm{CH}(\mathbf{P}U)$$

where $h := c_1(\mathcal{O}_{\mathbf{P}U}(1))$. Expanding and taking degrees gives the result. ■

Consequently, any maximal linear space in an even dimensional q-bic hypersurface must be Hermitian:

2.4.16. Corollary. — *Let X be a smooth q-bic hypersurface of dimension $2m$ over a separably closed field. Then any m-plane contained in X is Hermitian.*

Proof. — Let $\mathbf{P}U$ be such an m-plane. Then it contains a cone point by 2.4.15. Since X_{cone} is étale over $\mathbf{k}$ by 2.4.12, the intersection $\mathbf{P}U \cap X_{\mathrm{cone}}$ in fact consists of

$$\deg(\mathbf{P}U \cap X_{\mathrm{cone}}) = \frac{q^{2m+2}-1}{q^2-1} = \#\mathbf{P}^m(\mathbf{F}_{q^2})$$

reduced points. But cone points of X are also its Hermitian points, see 2.4.8, and its Hermitian points arise from a $\mathbf{F}_{q^2}$-subspace that spans V, see 1.2.2 and 1.2.13. Therefore $\mathbf{P}U$ must be spanned by its Hermitian points. ■

2.5. Unirationality

Despite typically being of general type, a smooth q-bic hypersurface is typically unirational. This was first discovered by Shioda via an explicit coordinate computation, see [Shi74]. This Section describes two geometric unirationality constructions: the first, summarized in 2.5.4, refines and generalizes Shioda's construction, and is based on projecting from a $\mathbf{k}$-rational line; the second, see 2.5.11, gives a new construction that is based on examining tangent lines to the hypersurface based at a fixed $\mathbf{k}$-rational line. Both constructions should be compared with the standard unirationality constructions of cubic hypersurfaces: see [CG72, Appendix B], [Mur72, §§2 and 5], and [Bea77, Exemple 4.5.1].

A summary of the results of 2.5.4 and 2.5.11 is:

2.5.1. Proposition. — *Let X be a smooth q-bic hypersurface of dimension at least 2. If X contains a $\mathbf{k}$-rational line, then X admits a purely inseparable unirational parameterization of degree q over $\mathbf{k}$.* ■

2.5.2. Unirationality via projection from a line. — Let X be the smooth q-bic hypersurface of dimension $n-1 \geq 2$, and let (V,β) be an underlying q-bic form. The first construction takes a choice of $\mathbf{k}$-rational line $\ell = \mathbf{P}U$ contained in X and produces a purely inseparable parameterization $\mathbf{P}^{n-1} \dashrightarrow X$ of degree q defined over $\mathbf{k}$. A special case of this construction was first found by Shioda in [Shi74] via explicit coordinate computations.

Set $W := V/U$ and write $\psi \colon V \to W$ for the quotient map. Consider linear projection $\mathbf{P}V \dashrightarrow \mathbf{P}W$ with centre $\ell = \mathbf{P}U$. By A.2.4, this is resolved along the incidence correspondence

$$\mathbf{P}\psi := \left\{ ([V'],[W']) \in \mathbf{P}V \times \mathbf{P}W \;\middle|\; \psi(V') \subseteq W' \right\}.$$

The first projection $\mathbf{P}\psi \to \mathbf{P}V$ is isomorphic to the blowup of $\mathbf{P}V$ along ℓ, whereas the second projection $\mathbf{P}\psi \to \mathbf{P}W$ is isomorphic to the $\mathbf{P}^2$-bundle associated with the locally free $\mathcal{O}_{\mathbf{P}W}$-module $\mathcal{V}$ of rank 3 fitting into the pullback diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U_{\mathbf{P}W} & \longrightarrow & V_{\mathbf{P}W} & \longrightarrow & W_{\mathbf{P}W} & \longrightarrow & 0 \\
 & & \| & & \cup & & {\scriptstyle\mathrm{eu}}\uparrow & & \\
0 & \longrightarrow & U_{\mathbf{P}W} & \longrightarrow & \mathcal{V} & \longrightarrow & \mathcal{O}_{\mathbf{P}W}(-1) & \longrightarrow & 0\,.
\end{array}$$

Let $\beta_{\mathcal{V}} \colon \mathrm{Fr}^*(\mathcal{V}) \otimes \mathcal{V} \to \mathcal{O}_{\mathbf{P}W}$ be the q-bic form induced by β on $\mathcal{V}$. Let X' be the inverse image of X along the blowup $\mathbf{P}\psi \to \mathbf{P}V$. Then $X' \to \mathbf{P}W$ is the family of q-bic curves defined by the q-bic form $(\mathcal{V},\beta_{\mathcal{V}})$ over $\mathbf{P}W$. Each fibre contains the line ℓ and the type of each fibre determines a sequence of subschemes of $\mathbf{P}W$. In the following statement, let y be a point of $\mathbf{P}W$ corresponding to a 3-dimensional linear subspace $V' \subset V$ containing U, so that the fibre X'_y is a q-bic curve in the plane $\mathbf{P}V'$.

2.5.3. Lemma. — *The family $X' \to \mathbf{P}W$ of q-bic curves determines a filtration*

$$\mathbf{P}W =: D_0 \supseteq D_1 \supseteq D_2 \supseteq D_3 \supseteq D_4$$

of $\mathbf{P}W$ by closed subschemes such that the type of X'_y over $D_i^\circ := D_i \setminus D_{i-1}$ is given by

	D_0°	D_1°	D_2°	D_3°	D_4°
$\mathrm{type}(X'_y)$	$\mathbf{N}_3$	$\mathbf{0} \oplus \mathbf{1}^{\oplus 2}$	$\mathbf{0} \oplus \mathbf{N}_2$	$\mathbf{0}^{\oplus 2} \oplus \mathbf{1}$	$\mathbf{0}^{\oplus 3}$

and such that D_1 is a q^2-bic hypersurface.

Proof. — Let $\tilde{X}$ and E be the strict transform of X and the exceptional divisor along the blowup $\mathbf{P}\psi \to \mathbf{P}V$. Since X is irreducible and $\ell \subset X$ is an inclusion of smooth varieties, the decomposition of X' into irreducible components is given by

$$X' = \tilde{X} \cup E.$$

Therefore the fibre X'_y for a general point $y \in \mathbf{P}W$ is the union of ℓ along with an irreducible plane curve of degree q, so X'_y is of type $\mathbf{N}_3$ by the classification of q-bic curves: see 1.4.1, or 3.4.1 for the case of q-bic curves specifically. The filtration of $\mathbf{P}W$ now arises by locally pulling back the type stratification from the parameter space of q-bic forms on a 4-dimensional vector space: see 1.4.4, 1.4.6 and 3.4.3.

It remains to show that D_1 is a q^2-bic hypersurface. Consider the q-bic form $(\mathcal{V}, \beta_{\mathcal{V}})$ defining the family $X' \to \mathbf{P}W$. Since U is totally isotropic for β, there is a commutative diagram of $\mathcal{O}_{\mathbf{P}W}$-modules

$$\begin{tikzcd}
0 \arrow[r] & U_{\mathbf{P}W} \arrow[r] \arrow[d] & \mathcal{V} \arrow[r] \arrow[d, "\beta_{\mathcal{V}}"] & \mathcal{O}_{\mathbf{P}W}(-1) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbf{P}W}(q) \arrow[r] & \mathrm{Fr}^*(\mathcal{V})^\vee \arrow[r] & \mathrm{Fr}^*(U)^\vee_{\mathbf{P}W} \arrow[r] & 0
\end{tikzcd}$$

in which the rows are exact. Since $\beta_{\mathcal{V}}$ is generically of rank 2, the diagram implies that the vertical map $\mathcal{O}_{\mathbf{P}W}(-1) \to \mathrm{Fr}^*(U)^\vee_{\mathbf{P}W}$ on the right is injective. This shows that the kernels of $\beta_{\mathcal{V}}$ factor through the subbundle $U_{\mathbf{P}W} \subset \mathcal{V}$. Now D_1 is the locus over which the fibres of $X' \to \mathbf{P}W$ become cones; by 2.4.1, this is the locus over which the kernels of $\beta_{\mathcal{V}}$ coincide and yield a radical. This is given by the degeneracy locus of

$$\mathrm{Fr}^*(\beta_{\mathcal{V}}) \oplus \beta_{\mathcal{V}}^\vee \colon \mathrm{Fr}^*(U)_{\mathbf{P}W} \to \mathcal{O}_{\mathbf{P}W}(q^2) \oplus \mathcal{O}_{\mathbf{P}W}(1).$$

Since both the source and target are the same rank, taking determinants shows that D_1 is the vanishing locus of the morphism

$$\det(\mathrm{Fr}^*(\beta_{\mathcal{V}}) \oplus \beta_{\mathcal{V}}^\vee) \colon \mathcal{O}_{\mathbf{P}W} \to \mathrm{Fr}^*(\det(U))^\vee \otimes \mathcal{O}_{\mathbf{P}W}(q^2+1).$$

The construction of this section shows it is a product of linear and q^2-power combinations of the coordinates of $\mathbf{P}W$, and so D_1 is a q^2-bic hypersurface. ■

Let $\eta \in \mathbf{P}W$ be the generic point and consider the generic fibre $\tilde{X}_\eta$ of $\tilde{X} \to \mathbf{P}W$. On the one hand, 2.5.3 implies that $\tilde{X}_\eta$ is geometrically isomorphic to the degree q component of a q-bic curve of type $\mathbf{N}_3$, so it is geometrically a unicuspidal rational curve: see 3.4.2. On the other hand, X is regular so $\tilde{X}_\eta$ is regular and its cusp becomes visible only upon passing to some purely inseparable extension of the function field $\mathbf{k}(\mathbf{P}W)$. This extension is found by considering the exceptional divisor E_X of the blowup $\tilde{X} \to X$, which more globally gives:

2.5.4. Proposition. — *Let X be a smooth q-bic hypersurface of dimension at least 2. For every $\mathbf{k}$-rational line $\ell \subset X$, there exists a diagram*

$$\begin{tikzcd}
\ell \arrow[d, phantom, "\cap"] & E_X \arrow[l] \arrow[d, phantom, "\cap"] & E_X \times_{\mathbf{P}W} E_X \arrow[l] \arrow[d, phantom, "\cap"] \\
X & \tilde{X} \arrow[l] \arrow[d] & \tilde{X} \times_{\mathbf{P}W} E_X \arrow[l] \arrow[d] \\
& \mathbf{P}W & E_X \arrow[l]
\end{tikzcd}$$

in which

(i) *$\tilde{X} \to X$ is the blowup along ℓ with exceptional divisor E_X,*

(ii) *$E_X \to \mathbf{P}W$ is purely inseparable of degree q and finite away from $D_2 \subset \mathbf{P}W$, and*

(iii) *the fibre product $\tilde{X} \times_{\mathbf{P}W} E_X$ is $\mathbf{k}$-rational.*

Thus X admits a purely inseparable unirational parameterization of degree q over $\mathbf{k}$.

Proof. — Let $y \in \mathbf{P}W$ be a point corresponding to a 2-plane $\mathbf{P}V' \subset \mathbf{P}V$ containing the line ℓ. Then the fibre $E_{X,y}$ of the exceptional divisor is the intersection between ℓ and the degree q-curve $\tilde{X}_y$ in $\mathbf{P}V'$. If $y \notin D_2$, then X'_y has type either $\mathbf{N}_3$ or $\mathbf{1}^{\oplus 2} \oplus \mathbf{0}$ by **2.5.3**, and this is a scheme of length q supported on the unique singular point of $\tilde{X}_y$. This proves (ii) and implies (iii): E_X is rational, being a projective bundle over the line ℓ, and the generic fibre of $\tilde{X} \times_{\mathbf{P}W} E_X$ is a unicuspidal rational curve. ■

2.5.5. Discriminant divisor. — Before discussing the next unirationality construction, continue with the setting of **2.5.2** and consider the divisor $D_1 \subset \mathbf{P}W$ from **2.5.3** over which the fibre type of $X' \to \mathbf{P}W$ changes; in analogy with the theory of conic bundles, call this the *discriminant divisor* of the family of q-bic curves. Since D_1 itself is a q^2-bic hypersurface, the classification theorem **1.4.1** applies. The following determines the type of D_1 in properties of the line ℓ; since this is a geometric statement, the base field will be assumed to be algebraically closed.

2.5.6. Lemma. — *Assume* $\mathbf{k} = \bar{\mathbf{k}}$ *and let*

$$r := \operatorname{rank}\big(\beta \oplus \operatorname{Fr}^*(\beta^\vee) \colon U \oplus \operatorname{Fr}^{2,*}(U) \to \operatorname{Fr}^*(W)^\vee\big).$$

(i) *If* $\#\ell \cap X_{\mathrm{Herm}} = q^2+1$, *then* $r = 2$ *and* $\operatorname{type}(D_1) = \mathbf{0}^{\oplus n-3} \oplus \mathbf{1}^{\oplus 2}$.

(ii) *If* $\#\ell \cap X_{\mathrm{Herm}} = 1$, *then* $r = 3$ *and* $\operatorname{type}(D_1) = \mathbf{0}^{\oplus n-4} \oplus \mathbf{N}_3$.

(iii) *If* $\#\ell \cap X_{\mathrm{Herm}} = 0$, *then* $3 \leq r \leq 4$ *and*

$$\operatorname{type}(D_1) = \begin{cases} \mathbf{0}^{\oplus n-4} \oplus \mathbf{N}_2 \oplus \mathbf{1} & \textit{when } r = 3, \textit{ and} \\ \mathbf{0}^{\oplus n-4} \oplus \mathbf{1}^{\oplus 3} & \textit{when } r = 4. \end{cases}$$

Proof. — Both maps $\beta \colon U \to \operatorname{Fr}^*(W)^\vee$ and $\operatorname{Fr}^*(\beta^\vee) \colon \operatorname{Fr}^{2,*}(U) \to \operatorname{Fr}^*(W)^\vee$ are injective since β is nonsingular, and so $r \geq 2$. In fact, $r = 2$ if and only if ℓ is Hermitian as in (i): If ℓ is Hermitian, choose a Hermitian basis $U = \langle u_0, u_1 \rangle$ and observe that

$$\ker(\beta \oplus \operatorname{Fr}^*(\beta^\vee)) = \big\langle (u_0, -u_0^{(q^2)}), (u_1, -u_1^{(q^2)}) \big\rangle.$$

Conversely, if $r = 2$, then the first and second projections of the direct sum restrict to isomorphisms between $\ker(\beta \oplus \operatorname{Fr}^*(\beta^\vee))$ and U and $\operatorname{Fr}^{2,*}(U)$, respectively. Therefore the q^2-linear map

$$U \xrightarrow{\mathrm{Fr}^2} \mathrm{Fr}^{2,*}(U) \xrightarrow{\mathrm{pr}_2^{-1}} \ker(\beta \oplus \mathrm{Fr}^*(\beta^\vee)) \xrightarrow{\mathrm{pr}_1} U$$

is a bijection and [SGA7$_{\mathrm{II}}$, Exposé XXII, 1.1] gives a basis $U = \langle u_0, u_1 \rangle$ consisting of vectors fixed for this map. Comparing with the definition of Hermitian vectors from 1.2.1 shows that u_0 and u_1 are Hermitian.

To determine the type of D_1, observe that a choice of basis $U = \langle u_0, u_1 \rangle$ expresses the equation $\det(\mathrm{Fr}^*(\beta_{\mathcal{V}}) \oplus \beta_{\mathcal{V}})$ of D_1 constructed in 2.5.3 as

$$\det\begin{pmatrix} \beta(-,u_0)^q & \beta(-,u_1)^q \\ \beta(u_0^{(q)},-) & \beta(u_1^{(q)},-) \end{pmatrix} = \beta(-,u_0)^q \beta(u_1^{(q)},-) - \beta(u_0^{(q)},-)\beta(-,u_1)^q$$

where, for $u \in U$, the linear forms $\beta(-,u)\colon \mathrm{Fr}^*(W) \to \mathbf{k}$ and $\beta(u^{(q)},-)\colon W \to \mathbf{k}$ are identified with the corresponding sections in

$$\beta(-,u) \in \mathrm{Fr}^*(W)^\vee \subset \mathrm{H}^0(\mathbf{P}W, \mathcal{O}_{\mathbf{P}W}(q)) \;\text{ and }\; \beta(u^{(q)},-) \in W^\vee = \mathrm{H}^0(\mathbf{P}W, \mathcal{O}_{\mathbf{P}W}(1)).$$

Now consider each case in turn:

(i) Choose u_0 and u_1 to be Hermitian. Setting $y_i := \beta(u_i^{(q)},-)$ then gives

$$\beta(-,u_i) = \beta(u_i^{(q)},-)^q = y_i^q \quad \text{whence} \quad D_1 = \mathrm{V}(y_0^{q^2} y_1 - y_0 y_1^{q^2})$$

showing that it is of type $\mathbf{0}^{\oplus n-3} \oplus \mathbf{1}^{\oplus 2}$.

(ii) Choose u_0 to be Hermitian. Then $(u_0, -u_0^{(q^2)})$ lies in the kernel of $\beta \oplus \mathrm{Fr}^*(\beta^\vee)$, so $r = 3$. Thus there are coordinates y_0, y_1, and y_2 such that

$$\beta(-,u_0) = \beta(u_0^{(q)},-)^q = y_0^q, \quad \beta(u_1^{(q)},-) = y_1, \quad \beta(-,u_1) = y_2^q.$$

Therefore $D_1 = \mathrm{V}(y_0^{q^2} y_1 - y_0 y_2^{q^2})$ and it has type $\mathbf{0}^{\oplus n-4} \oplus \mathbf{N}_3$.

(iii) If $r = 3$, then there is a basis $U = \langle u_0, u_1 \rangle$ such that the vector $(u_0, -u_1^{(q^2)})$ lies in the kernel of $\beta \oplus \mathrm{Fr}^*(\beta^\vee)$. Thus there are coordinates y_0, y_1, and y_2 with

$$\beta(-,u_0) = \beta(u_1^{(q)},-) = y_0^q, \quad \beta(u_0^{(q)},-) = y_1, \quad \beta(-,u_1) = y_2^q.$$

and so $D_1 = \mathrm{V}(y_0^{q^2+1} - y_1 y_2^{q^2})$ has type $\mathbf{0}^{\oplus n-4} \oplus \mathbf{N}_2 \oplus \mathbf{1}$. Otherwise, $r = 4$ and there are coordinates y_0, y_1, y_2, and y_3 so that

$$\beta(-,u_0) = y_0^q, \quad \beta(u_1^{(q)},-) = y_1, \quad \beta(u_0^{(q)},-) = y_2, \quad \beta(-,u_1) = y_3^q.$$

Then $D_1 = \mathrm{V}(y_0^{q^2} y_1 - y_2 y_3^{q^2})$ and it is of type $\mathbf{0}^{\oplus n-4} \oplus \mathbf{1}^{\oplus 3}$. ∎

2.5.7. — The discriminant divisor D_1 is related to the $(n-3)$-dimensional $D_{1,\ell}$ scheme which parameterizes lines in X incident with ℓ; see 2.8.8 below. There is a natural rational map

$$D_{1,\ell} \dashrightarrow D_1, \qquad [\ell'] \mapsto \langle \ell, \ell' \rangle$$

which sends a point corresponding to a line $\ell' \subset X$ incident with ℓ to the plane spanned by ℓ and ℓ', viewed as a point of $\mathbf{P}W = \mathbf{P}(V/U)$; this factors through the discriminant divisor D_1 since such a plane intersects X at a q-bic curve containing at least 2 lines, so it has type at most $\mathbf{0} \oplus \mathbf{1}^{\oplus 2}$. In the case that X is a threefold and ℓ is a Hermitian, this extends to a morphism; see 4.7.21 and the comments that follow.

2.5.8. Unirationality via tangents to a line. — As in 2.5.2, let X be a smooth q-bic hypersurface and $\ell \subset X$ a $\mathbf{k}$-rational line. The following will produce another unirational parameterization of X which is in a sense dual to that of 2.5.4.

The projective bundle associated with the tangent bundle of X restricted to ℓ is canonically identified as the incidence correspondence

$$Y := \mathbf{P}(\mathcal{T}_X(-1)|_\ell) = \left\{ (x, [\ell']) \in \ell \times \mathbf{G}(2,V) \;\middle|\; x \in \ell' \text{ and } \mathrm{mult}_x(X \cap \ell') \geq 2 \right\}$$

between points $x \in \ell$ and lines $\ell' \subset \mathbf{P}V$ that are tangent to X at x. Writing $\pi \colon Y \to \ell$ for the structure morphism, this description endows Y with tautological bundles fitting into a short exact sequence

$$0 \to \pi^*\mathcal{O}_\ell(-1) \to \mathcal{S} \to \mathcal{O}_\pi(-1) \to 0,$$

where $\mathcal{S}$ is the rank 2 subbundle of V_Y whose fibre over a point $(x, [\ell'])$ is the 2-dimensional linear subspace of V underlying ℓ'. Then β induces a q-bic form

$$\beta_{\mathcal{S}} \colon \mathrm{Fr}^*(\mathcal{S}) \otimes \mathcal{S} \subset \mathrm{Fr}^*(V)_Y \otimes V_Y \to \mathcal{O}_Y$$

for which $\pi^*\mathcal{O}_\ell(-1)$ is an isotropic subbundle.

2.5.9. Lemma. — *There exists an exact commutative diagram of $\mathcal{O}_Y$-modules*

$$\begin{tikzcd} 0 \arrow[r] & \pi^*\mathcal{O}_\ell(-1) \arrow[r] \arrow[d, hook] & \mathcal{S} \arrow[r] \arrow[d, "\beta_{\mathcal{S}}"] & \mathcal{O}_\pi(-1) \arrow[r] \arrow[d, "0"] & 0 \\ 0 \arrow[r] & \mathcal{O}_\pi(q) \arrow[r] & \mathrm{Fr}^*(\mathcal{S})^\vee \arrow[r] & \pi^*\mathcal{O}_\ell(q) \arrow[r] & 0 \end{tikzcd}$$

where the left map is injective and the right map is zero. This induces an exact sequence

$$0 \to \mathrm{Fr}^*(\mathcal{S})^\perp \to \mathcal{O}_\pi(-1) \to \mathcal{O}_Z \to \operatorname{coker}(\beta_{\mathcal{S}}) \to \pi^*\mathcal{O}_\ell(q) \to 0$$

where Z is the effective Cartier divisor defined by the map $\pi^*\mathcal{O}_\ell(-1) \to \mathcal{O}_\pi(q)$.

Proof. — The diagram exists because $\pi^*\mathcal{O}_\ell(-1)$ is isotropic for $\beta_{\mathcal{S}}$. Let $y = (x, [\ell'])$ be a point of Y. Then $X \cap \ell'$ is the scheme of q-bic points in $\ell' = \mathbf{P}\mathcal{S}_y$ defined by $\beta_{\mathcal{S},y}$, see 2.1.4. Since x is a point of tangency between ℓ' and X by 2.5.8, it is the singular point of $X \cap \ell'$, so $\pi^*\mathcal{O}_\ell(-q) \subseteq \mathcal{S}^\perp$ by 2.2.7. This implies that the right vertical map vanishes.

For the left vertical map, observe that $X \cap \ell'$ is of type $\mathbf{N}_2$ for general y: The plane slice $X \cap \langle \ell, \ell' \rangle$ is of type $\mathbf{N}_3$ by 2.5.3 with singular point x and linear component ℓ; since x has multiplicity q, ℓ' will intersect the degree q component at one point besides x. Therefore $\mathrm{Fr}^*(\mathcal{S})^\perp$ is generically disjoint from $\pi^*\mathcal{O}_\ell(-1)$, giving the desired injectivity. The exact sequence now arises from the Snake Lemma. ■

By 2.5.9, $\mathrm{Fr}^*(\mathcal{S})^\perp$ is torsion-free of rank 1. The exact sequence

$$0 \to \mathrm{Fr}^*(\mathcal{S})^\perp \to \mathcal{S} \xrightarrow{\beta_{\mathcal{S}}} \mathrm{Fr}^*(\mathcal{S})^\vee \to \operatorname{coker}(\beta_{\mathcal{S}}) \to 0$$

means that it is a subbundle of $\mathcal{S}$ away from the vanishing locus of $\beta_{\mathcal{S}}$: this is the scheme $Y \times_{\mathbf{G}(2,V)} \mathbf{F}_1(X)$ whose points are those $(x, [\ell'])$ such that $\ell' \subset X$. Let Y° be the open complement in Y, so that $\mathrm{Fr}^*(\mathcal{S})^\perp$ induces a morphism $Y^\circ \to \mathbf{P}V$. Since $\mathrm{Fr}^*(\mathcal{S})^\perp$ is isotropic for β, this morphism factors through $X \subset \mathbf{P}V$.

2.5.10. Lemma. — *$Y^\circ \to X$ is dominant, purely inseparable, and finite of degree q.*

Proof. — By 2.5.3, the union of lines in X incident with ℓ forms a divisor. Let z be a point of the complement. Then the plane $\langle \ell, z\rangle$ intersects X at a q-bic curve of type $\mathbf{N}_3$ and z lies on the residual curve C of degree q. Let x be the point supporting $C \cap \ell$, and let $\ell' := \langle x, z\rangle$. Then $(x, [\ell'])$ is a preimage of z under $Y^\circ \to X$, showing that it is dominant. In fact, the scheme-theoretic fibre over z is the scheme $(C \cap \ell, [\ell'])$, showing that $Y^\circ \to X$ is finite and purely inseparable of degree q. ∎

The following summarizes the construction. Since a projective bundle over a line is rational, this provides another unirational parameterization of X.

2.5.11. Proposition. — *Let X be a smooth q-bic hypersurface of dimension at least 2. For every $\mathbf{k}$-rational line $\ell \subset X$, the rational map*

$$\mathbf{P}(\mathcal{T}_X(-1)|_\ell) \dashrightarrow X, \qquad (x, [\ell']) \mapsto X \cap \ell' - qx$$

is dominant, purely inseparable, and is defined and finite of degree q away from

$$\mathbf{P}(\mathcal{T}_X(-1)|_\ell) \times_{\mathbf{G}(2,V)} \mathbf{F}_1(X) = \left\{ (x, [\ell']) \in \mathbf{P}(\mathcal{T}_X(-1)|_\ell) \;\middle|\; \ell' \subset X \right\}. \qquad ∎$$

2.6. Cohomological properties

This Section collects some cohomological facts about q-bic hypersurfaces X in the projective n-space $\mathbf{P}V$.

2.6.1. Frobenius action on cohomology. — Let Y be a scheme over $\mathbf{k}$ and let Fr be the q-power absolute Frobenius morphism. Pullback induces q-linear maps on cohomology:

$$\mathrm{Fr}^* \colon \mathrm{H}^i(Y, \mathcal{O}_Y) \to \mathrm{H}^i(Y, \mathcal{O}_Y) \quad \text{for every } i \geq 0.$$

This can be made explicit in the case Y is a hypersurface in a projective space $\mathbf{P}V$: Let $g \in \mathrm{H}^0(\mathbf{P}V, \mathcal{O}_{\mathbf{P}V}(d))$ be an equation for Y. Then there is a commutative diagram of abelian sheaves on $\mathbf{P}V$ given by;

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\mathbf{P}V}(-d) \arrow[r, "g"'] \arrow[d, "g^{q-1}\mathrm{Fr}"] & \mathcal{O}_{\mathbf{P}V} \arrow[r] \arrow[d, "\mathrm{Fr}"] & \mathcal{O}_Y \arrow[r] \arrow[d, "\mathrm{Fr}"] & 0 \\
0 \arrow[r] & \mathrm{Fr}_*(\mathcal{O}_{\mathbf{P}V}(-d)) \arrow[r, "g^q"] & \mathrm{Fr}_*\mathcal{O}_{\mathbf{P}V} \arrow[r] & \mathrm{Fr}_*\mathcal{O}_Y \arrow[r] & 0
\end{tikzcd}$$

where the sheaf map Fr acts on local sections by $s \mapsto s^q$. Taking cohomology then yields a commutative diagram

$$\begin{tikzcd} \mathrm{H}^{n-1}(Y, \mathcal{O}_Y) \arrow[r, "\cong"'] \arrow[d, "\mathrm{Fr}"'] & \mathrm{H}^n(\mathbf{P}V, \mathcal{O}_{\mathbf{P}V}(-d)) \arrow[d, "g^{q-1}\,\mathrm{Fr}"] \\ \mathrm{H}^{n-1}(Y, \mathcal{O}_Y) \arrow[r, "\cong"] & \mathrm{H}^n(\mathbf{P}V, \mathcal{O}_{\mathbf{P}V}(-d)). \end{tikzcd}$$

The following shows that the q-power Frobenius always acts trivially on the coherent cohomology of a q-bic hypersurface:

2.6.2. Lemma. — *Let X be a q-bic hypersurface of dimension at least* 1. *Then the map* $\mathrm{Fr}\colon \mathrm{H}^{n-1}(X, \mathcal{O}_X) \to \mathrm{H}^{n-1}(X, \mathcal{O}_X)$ *induced by q-power Frobenius is zero.*

Proof. — Let f be an equation for X. Then the discussion of 2.6.1 shows that the Frobenius map on $\mathrm{H}^{n-1}(X, \mathcal{O}_X)$ is computed as $f^{q-1}\,\mathrm{Fr}$ applied to $\mathrm{H}^n(\mathbf{P}V, \mathcal{O}_{\mathbf{P}V}(-q-1))$. Upon choosing coordinates $(x_0 : \cdots : x_n)$ for $\mathbf{P}V = \mathbf{P}^n$, a basis of the latter space is given by elements

$$\xi := (x_0^{i_0} \cdots x_n^{i_n})^{-1} \quad \text{where } i_0 + \cdots + i_n = q+1 \text{ and each } i_j \geq 1.$$

Monomials appearing in f^{q-1} are of the form $a^q b$, where a and b are themselves monomials of degree $q-1$. Writing $a = x_0^{a_0} \cdots x_n^{a_n}$ with $a_0 + \cdots + a_n = q-1$, it follows that $f^{q-1}\,\mathrm{Fr}(\xi)$ is a sum of terms

$$a^q b \cdot (x_0^{i_0} \cdots x_n^{i_n})^{-q} = b \cdot (x_0^{i_0 - a_0} \cdots x_n^{i_n - a_n})^{-q}.$$

But now $(i_0 - a_0) + \cdots + (i_n - a_n) = 2 \leq n$. Therefore there is some index j such that $i_j - a_j \leq 0$ so the above term represents 0 in $\mathrm{H}^n(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(-q-1))$. Thus each potential contribution to $f^{q-1}\,\mathrm{Fr}(\xi)$ vanishes, and so $f^{q-1}\,\mathrm{Fr}(\xi) = 0$. ■

2.6.3. Étale cohomology. — Assume the base field $\mathbf{k}$ is separably closed, let X be a smooth q-bic hypersurface, and fix a prime $\ell \neq p$. For each $0 \leq i \leq 2n-2$, let

$$\mathrm{H}^i_{\text{ét}}(X, \mathbf{Z}_\ell)_{\text{prim}} := \ker\big(c_1(\mathcal{O}_X(1))^{n+1-i} \colon \mathrm{H}^i_{\text{ét}}(X, \mathbf{Z}_\ell) \to \mathrm{H}^{2n+2-i}_{\text{ét}}(X, \mathbf{Z}_\ell)\big)$$

be the primitive étale cohomology with respect to the action of the natural hyperplane class. Since X is a hypersurface, the Lefschetz hyperplane theorem shows that

$\mathrm{H}^i_{\text{ét}}(X, \mathbf{Z}_\ell) = 0$ for all $i \neq n-1$, and the universal coefficient theorem shows that the middle cohomology is torsion-free. Each of the primitive cohomology groups are representations for $\mathrm{U}_{n+1}(q) = \mathbf{Aut}(V, \beta)$ since the group acts on X through a linear action on $\mathbf{P}V$, see 2.3.1. The middle cohomology has the following properties:

2.6.4. Theorem. — *Let X be a smooth q-bic $(n-1)$-fold over a separably closed field. Then its middle étale cohomology group satisfies:*

(i) $\mathrm{rank}_{\mathbf{Z}_\ell} \mathrm{H}^{n-1}_{\text{ét}}(X, \mathbf{Z}_\ell)_{\mathrm{prim}} = q(q^n - (-1)^n)/(q+1)$;

(ii) $\mathrm{H}^{n-1}_{\text{ét}}(X, \mathbf{Q}_\ell)_{\mathrm{prim}}$ *is an irreducible representation of* $\mathrm{U}_{n+1}(q)$*; and*

(iii) *if $n-1$ is even, then $\mathrm{H}^{n-1}_{\text{ét}}(X, \mathbf{Q}_\ell)$ is spanned by algebraic cycles.*

Proof. — Item (i) is standard and may be obtained via an Euler characteristic computation, see [Mil80, p.246]; (ii) was first observed by [Tat65], but see also [HM78, Theorem 1]; this implies (iii) upon noting that the cycle classes of maximal linear spaces in X are not restricted from $\mathbf{P}V$, but see also [SK79, Theorem II]. ■

Since all smooth q-bic hypersurfaces are isomorphic to the Fermat hypersurface by 2.2.5, the rank of the middle cohomology may be determined by counting points:

2.6.5. Theorem. — *The zeta function of the Fermat q-bic $(n-1)$-fold over $\mathbf{F}_{q^2}$ is*

$$Z(X; t) = \frac{(1 - q^{n-1} t)^{(-1)^n b_{n-1,\mathrm{prim}}}}{(1-t)\cdots(1-q^{2n-2}t)}$$

where $b_{n-1,\mathrm{prim}} := q(q^n - (-1)^n)/(q+1)$.

Proof. — This is essentially due to Weil in [Wei49, Wei52]; see also [SK79, §3]. ■

In particular, this means that the Jacobian of a smooth q-bic curve is supersingular; see also [SK79, Proposition 3.10].

2.7. Linear spaces

Recognizing a q-bic hypersurface X the space of isotropic vectors for a q-bic form β, as in 2.1.3, endows the Fano schemes $\mathbf{F}_r(X)$ parameterizing r-planes contained in X with the alternative moduli interpretation as the space of $(r+1)$-dimensional isotropic subspaces of V. This likens the Fano schemes to orthogonal Grassmannian, bringing a perspective from which their special properties may be better understood. Throughout this Section, X is the q-bic hypersurface associated with a q-bic form (V,β) of dimension $n+1$ over a field $\mathbf{k}$.

2.7.1. Fano schemes. — For the generalities of the next two paragraphs, see, for example, [AK77] or [Kol96, Section V.4] for details. Given a projective scheme $Y \subseteq \mathbf{P}V$ and an integer $0 \leq r \leq n$, the *Fano scheme of r-planes in Y* is the closed subscheme $\mathbf{F}_r(Y) \subseteq \mathbf{G}(r+1,V)$ representing the functor $\mathrm{Sch}_{\mathbf{k}}^{\mathrm{opp}} \to \mathrm{Set}$ given by

$$T \mapsto \left\{ P \subseteq Y \times_{\mathbf{k}} T \;\middle|\; \begin{array}{c} P \text{ flat over } T \text{ such that, for all } t \in T, \\ P_t \subseteq \mathbf{P}V \otimes_{\mathbf{k}} \kappa(t) \text{ is an } r\text{-plane} \end{array} \right\}.$$

Since r-planes in a projective space are precisely projectivizations of linear subspaces of dimension $r+1$, there is a canonical identification $\mathbf{F}_r(\mathbf{P}V) = \mathbf{G}(r+1,V)$ and the tautological short exact sequence of the Grassmannian

$$0 \to \mathscr{S} \to V_{\mathbf{G}(r+1,V)} \to \mathscr{Q} \to 0$$

is such that the fibre of the universal subbundle $\mathscr{S}$ at a point $[P] \in \mathbf{F}_r(\mathbf{P}V)$ is the linear subspace of V underlying $P \subset \mathbf{P}V$.

2.7.2. — Let $Y \subset \mathbf{P}V$ be a hypersurface of degree d, say defined by a section

$$g \in \mathrm{H}^0(\mathbf{P}V, \mathscr{O}_{\mathbf{P}V}(d)) = \mathrm{Sym}^d(V^\vee).$$

Then $\mathbf{F}_r(Y)$ is the closed subscheme of $\mathbf{G}(r+1,V)$ consisting of r-planes on which the restriction of g vanishes. Thus $\mathbf{F}_r(Y)$ is the zero locus of the section

$$\mathscr{O}_{\mathbf{G}(r+1,V)} \xrightarrow{g} \mathscr{O}_{\mathbf{G}(r+1,V)} \otimes \mathrm{Sym}^d(V^\vee) \to \mathrm{Sym}^d(\mathscr{S}^\vee).$$

where the latter map comes from the d-th symmetric power of the dual tautological sequence above. In particular, if $\mathbf{F}_r(Y)$ is nonempty, then

$$\dim \mathbf{F}_r(Y) \geq \dim \mathbf{G}(r+1,V) - \mathrm{rank}_{\mathscr{O}_{\mathbf{G}(r+1,V)}} \mathrm{Sym}^d(\mathscr{S}^\vee) = (n-r)(r+1) - \binom{d+r}{r}.$$

In fact, equality holds for general Y, see [DM98, Théorème 2.1].

In contrast, it has long been observed that q-bic hypersurfaces contain many more linear subspaces and that this dimension bound is typically a gross underestimate: see [Col79, Example 1.27] and [Deb01, pp.51–52]. One way to resolve this conundrum is to recognize q-bic hypersurfaces as the moduli spaces of isotropic vectors for q-bic forms, as in 2.1.3. Then the associated Fano schemes take on the alternative interpretation as moduli spaces of isotropic subspaces in V:

2.7.3. Lemma. — *Let X be the q-bic hypersurface associated with a q-bic form (V,β). Then its Fano scheme $\mathbf{F}_r(X)$ represents the functor $\mathrm{Sch}_{\mathbf{k}}^{\mathrm{opp}} \to \mathrm{Set}$ given by*

$$T \mapsto \left\{ \mathcal{V}' \subset V_T \text{ a } \beta\text{-isotropic subbundle of rank } r+1 \right\}.$$

Thus $\mathbf{F}_r(X)$ is the vanishing locus in $\mathbf{G}(r+1,V)$ of the q-bic form

$$\beta_{\mathcal{S}} \colon \mathrm{Fr}^*(\mathcal{S}) \otimes \mathcal{S} \subset \mathrm{Fr}^*(V)_{\mathbf{G}(r+1,V)} \otimes V_{\mathbf{G}(r+1,V)} \xrightarrow{\beta} \mathcal{O}_{\mathbf{G}(r+1,V)}.$$

and $\dim \mathbf{F}_r(X) \geq (r+1)(n-2r-1)$ whenever it is nonempty.

Proof. — By its definition in 2.7.1, $\mathbf{F}_r(X)$ represents the presheaf on $\mathrm{Sch}_{\mathbf{k}}$ sending a $\mathbf{k}$-scheme T to the set of rank $r+1$ subbundles $\mathcal{V}' \subset V_T$ such that $\mathbf{P}\mathcal{V}' \subset X_T \subset \mathbf{P}V_T$. But 2.1.3 and 1.1.4 together imply that $\mathcal{V}'$ is a totally isotropic subbundle for β_T, yielding the first statement. That $\mathbf{F}_r(X)$ is the zero locus of $\beta_{\mathcal{S}}$ follows from universal property of the Grassmannian, see A.1.2. Finally,

$$\begin{aligned} \dim \mathbf{F}_r(X) &\geq \dim \mathbf{G}(r+1,V) - \mathrm{rank}_{\mathcal{O}_{\mathbf{G}(r+1,V)}}(\mathrm{Fr}^*(\mathcal{S}) \otimes \mathcal{S}) \\ &= (r+1)(n-r) - (r+1)^2 = (r+1)(n-2r-1). \end{aligned}$$ ■

The following verifies that the Fano schemes are nonempty in a certain range.

2.7.4. Lemma. — *The Fano scheme $\mathbf{F}_r(X)$ is nonempty for each $0 < r < \frac{n}{2}$.*

Proof. — This is a geometric question, so assume $\mathbf{k}$ is algebraically closed. Fix $0 < r < \frac{n}{2}$. Let $\mathbf{P}(\mathrm{Fr}^*(V)^\vee \otimes V^\vee)$ denote the parameter space of q-bic hypersurfaces in $\mathbf{P}V$, and consider the incidence correspondence

$$\Phi := \left\{ ([\mathbf{P}U],[X]) \in \mathbf{G}(r+1,V) \times \mathbf{P}(\mathrm{Fr}^*(V)^\vee \otimes V^\vee) \,\middle|\, \mathbf{P}U \subseteq X \right\}.$$

The fibre of the second projection $\mathrm{pr}_2 \colon \Phi \to \mathbf{P}(\mathrm{Fr}^*(V)^\vee \otimes V^\vee)$ over a point $[X]$ is $\mathbf{F}_r(X)$, so the goal is to show pr_2 is surjective. Note Φ is proper since $\mathrm{Gr}(r+1, V)$ is proper, and the fibre of the first projection $\mathrm{pr}_1 \colon \Phi \to \mathrm{Gr}(r+1,V)$ over a point $[\mathbf{P}U]$ is the projective space

$$\mathrm{pr}_1^{-1}([\mathbf{P}U]) = \mathbf{P}\big(\ker(\mathrm{Fr}^*(V)^\vee \otimes V^\vee \to \mathrm{Fr}^*(U)^\vee \otimes U^\vee)\big)$$

of q-bic equations vanishing on U. Thus it suffices to show that pr_2 is dominant, and this will follow if $\mathbf{F}_r(X)$ is nonempty for every smooth X. A smooth X is defined by a nonsingular q-bic form by **2.2.3**, so this follows from **1.2.15**, which implies that a smooth X contains a Hermitian linear subspace of dimension $\lfloor \frac{n-1}{2} \rfloor$. ■

Since $\mathbf{F}_r(X)$ is a Hilbert scheme, its tangent space to a point parameterizing a linear subspace $\mathbf{P}U \subset X$ is canonically identified as $\mathrm{H}^0(\mathbf{P}U, \mathcal{N}_{\mathbf{P}U/X})$: see, for example, [**FGI+05**, Proposition 6.5.2]. The next Proposition explicitly identifies the normal bundle of a linear subspace contained in the smooth locus of X. This will show, in particular, that such linear spaces give smooth points of $\mathbf{F}_r(X)$ around which the Fano scheme has the expected dimension. Toward this, the next, rather technical, Lemma reformulates the geometric property that a linear subspace $\mathbf{P}U \subset \mathbf{P}V$ is disjoint from the singular locus of X in terms of linear algebraic notions. For part of its statement, set $W := V/U$ and note that β induces bilinear pairings

$$\beta_W \colon U^\perp \otimes W \to \mathbf{k} \quad \text{and} \quad \beta_{\mathrm{Fr}^*(W)} \colon \mathrm{Fr}^*(W) \otimes \mathrm{Fr}^*(U)^\perp \to \mathbf{k}$$

computed by taking any lift along $V \to W$ and $\mathrm{Fr}^*(V) \to \mathrm{Fr}^*(W)$, respectively.

2.7.5. Lemma. — *Let $\mathbf{P}U \subset \mathbf{P}V$ be a linear subspace. The following are equivalent:*

(i) *$\mathbf{P}U$ is disjoint from the nonsmooth locus of X.*
(ii) *$\mathrm{Fr}^*(U)$ is linearly disjoint from $V^\perp$.*
(iii) *The map $\beta^\vee \colon \mathrm{Fr}^*(U) \to V^\vee$ is injective.*
(iv) *The map $(-)|_{\mathrm{Fr}^*(U)} \circ \beta \colon V \to \mathrm{Fr}^*(V)^\vee \to \mathrm{Fr}^*(U)^\vee$ is surjective.*
(v) *$\dim_{\mathbf{k}} \mathrm{Fr}^*(U)^\perp = \dim_{\mathbf{k}} V - \dim_{\mathbf{k}} U$.*

Suppose U is furthermore isotropic for β. Set $W := V/U$. Then these are equivalent to:

(vi) *The map* $(-)|_{\mathrm{Fr}^*(U)} \circ \beta_W \colon W \to U^{\perp,\vee} \to \mathrm{Fr}^*(U)^\vee$ *is surjective.*

(vii) $\dim_{\mathbf{k}}(\mathrm{Fr}^*(U)^\perp/U) = \dim_{\mathbf{k}} V - 2\dim_{\mathbf{k}} U$.

Proof. — That (i) $\Leftrightarrow$ (ii) follows from **2.2.7**; their equivalence with (iii) then follows from definitions. That (iii) $\Leftrightarrow$ (iv) follows by linear duality. Finally, (iv) $\Leftrightarrow$ (v) since $\mathrm{Fr}^*(U)^\perp$ is precisely the kernel of the map $\beta \colon V \to \mathrm{Fr}^*(U)^\vee$.

Now assume that U is isotropic for β. Then there is a commutative diagram

$$\begin{tikzcd}
V \arrow[r, "\beta"'] \arrow[d, two heads] & \mathrm{Fr}^*(V)^\vee \arrow[r, two heads] \arrow[d, two heads] & \mathrm{Fr}^*(U)^\vee \arrow[d, equal] \\
W \arrow[r, "\beta_W"] & U^{\perp,\vee} \arrow[r, two heads] & \mathrm{Fr}^*(U)^\vee
\end{tikzcd}$$

where the maps in the right square are the restriction maps. The top composite is the map of (iv), and the bottom composite that of (vi). Since $V \to W$ is surjective, this shows (iv) $\Leftrightarrow$ (vi). The kernel of the lower map is $\mathrm{Fr}^*(U)^\perp/U$, so (vi) $\Leftrightarrow$ (vii). ■

This gives the normal bundle of a linear subspace wholly contained in the smooth locus of a q-bic hypersurface.

2.7.6. Proposition. — *Let* $\mathbf{P}^r \cong \mathbf{P}U \subset \mathbf{P}V$ *be a linear subspace contained in the smooth locus of* X*. Then there is a canonical, split short exact sequence*

$$0 \to (\mathrm{Fr}^*(U)^\perp/U)_{\mathbf{P}U} \to \mathcal{N}_{\mathbf{P}U/X}(-1) \to \mathrm{Fr}^*(\Omega^1_{\mathbf{P}U}(1)) \to 0.$$

This yields a canonical identification

$$\mathcal{T}_{\mathbf{F}_r(X)} \otimes_{\mathcal{O}_{\mathbf{F}_r(X)}} \kappa([\mathbf{P}U]) \cong \mathrm{H}^0(\mathbf{P}U, \mathcal{N}_{\mathbf{P}U/X}) \cong U^\vee \otimes (\mathrm{Fr}^*(U)^\perp/U)$$

and this has dimension $(r+1)(n-2r-1)$.

Proof. — Since $\mathbf{P}U$ is contained in the smooth locus of X, there is a short exact sequence of normal bundles

$$0 \to \mathcal{N}_{\mathbf{P}U/X}(-1) \to \mathcal{N}_{\mathbf{P}U/\mathbf{P}V}(-1) \xrightarrow{\delta} \mathcal{N}_{X/\mathbf{P}V}(-1)|_{\mathbf{P}U} \to 0.$$

The rightmost sheaf is $\mathcal{O}_{\mathbf{P}U}(q)$; the Euler sequence together with 2.2.2 shows that the map δ fits into a commutative diagram

$$\begin{tikzcd}
V_{\mathbf{P}U} \arrow[r, "\beta"'] \arrow[d, two heads] & \mathrm{Fr}^*(V)^\vee_{\mathbf{P}U} \arrow[d, two heads, "\mathrm{eu}^{(q),\vee}"] \\
\mathcal{N}_{\mathbf{P}U/\mathbf{P}V}(-1) \arrow[r, "\delta"] & \mathcal{N}_{X/\mathbf{P}V}(-1)|_{\mathbf{P}U}\,.
\end{tikzcd}$$

Since $\mathcal{N}_{\mathbf{P}U/\mathbf{P}V}(-1) \cong W_{\mathbf{P}U}$ and the Euler section $\mathrm{eu}^{(q),\vee}$ factors through the quotient $\mathrm{Fr}^*(V)^\vee_{\mathbf{P}U} \twoheadrightarrow \mathrm{Fr}^*(U)^\vee_{\mathbf{P}U}$, the map δ may be identified as

$$\delta \colon W_{\mathbf{P}U} \xrightarrow{\beta_W} \mathrm{Fr}^*(U)^\vee_{\mathbf{P}U} \xrightarrow{\mathrm{eu}^{(q),\vee}} \mathcal{O}_{\mathbf{P}U}(q)$$

where β_W is as in 2.7.5(iv); since $\mathbf{P}U$ is contained in the smooth locus of X, the map β_W is surjective with kernel $\mathrm{Fr}^*(U)^\perp/U$. Thus the normal bundle sequence above fits into an exact commutative diagram

$$\begin{tikzcd}
 & (\mathrm{Fr}^*(U)^\perp/U)_{\mathbf{P}U} \arrow[r, equal] \arrow[d, hook] & (\mathrm{Fr}^*(U)^\perp/U)_{\mathbf{P}U} \arrow[d, hook] & & \\
0 \arrow[r] & \mathcal{N}_{\mathbf{P}U/X}(-1) \arrow[r] \arrow[d, two heads] & W_{\mathbf{P}U} \arrow[r, "\delta"] \arrow[d, two heads, "\beta_W"] & \mathcal{O}_{\mathbf{P}U}(q) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Fr}^*(\Omega^1_{\mathbf{P}U}(1)) \arrow[r] & \mathrm{Fr}^*(U)^\vee_{\mathbf{P}U} \arrow[r, "\mathrm{eu}^{(q),\vee}"] & \mathrm{Fr}^*(\mathcal{O}_{\mathbf{P}U}(1)) \arrow[r] & 0
\end{tikzcd}$$

where the lower row is identified as the Frobenius pullback of the Euler sequence on $\mathbf{P}U$. The left column is the sequenced promised by the Lemma. That the left column splits is because the middle column is splits, being a sequence of free $\mathcal{O}_{\mathbf{P}U}$-modules.

To compute global sections of $\mathcal{N}_{\mathbf{P}U/X}$, note that the Euler sequence implies that $\mathrm{H}^0(\mathbf{P}U, \mathrm{Fr}^*(\Omega^1_{\mathbf{P}U}(1))(1)) = 0$. Thus the sequence of the Lemma shows

$$\mathrm{H}^0(\mathbf{P}U, \mathcal{N}_{\mathbf{P}U/X}) \cong \mathrm{H}^0(\mathbf{P}U, \mathcal{O}_{\mathbf{P}U}(1) \otimes (\mathrm{Fr}^*(U)^\perp/U)) \cong U^\vee \otimes (\mathrm{Fr}^*(U)^\perp/U).$$

That this has dimension $(r+1)(n-2r-1)$ follows from 2.7.5(vii). ■

The identification of the tangent space to $\mathbf{F}_r(X)$ in 2.7.6 can also be done using the alternative functor underlying the Fano scheme given in 2.7.3. The computation given here, however, has the advantage of determining the normal bundle of a linear space in the smooth locus of X.

2.7.7. Corollary. — *Let $\mathbf{P}^r \cong \mathbf{P}U \subset \mathbf{P}V$ be a linear subspace contained in the smooth locus of X. Then $\mathbf{F}_r(X)$ is smooth at the point $[\mathbf{P}U]$ and has local dimension*

$$\dim_{[\mathbf{P}U]}(\mathbf{F}_r(X)) = (r+1)(n-2r-1).$$

In particular, $\operatorname{Sing}(\mathbf{F}_r(X)) \subseteq \{\, [\mathbf{P}U] \in \mathbf{F}_r(X) \mid \mathbf{P}U \cap \operatorname{Sing}(X) \neq \varnothing \,\}$.

Proof. — Let $\kappa := \kappa([\mathbf{P}U])$ be the residue field at $[\mathbf{P}U] \in \mathbf{F}_r(X)$. Then

$$\begin{aligned}(r+1)(n-2r-1) &\leq \dim_{[\mathbf{P}U]}(\mathbf{F}_r(X)) \\ &\leq \dim_\kappa(\mathcal{T}_{\mathbf{F}_r(X)} \otimes_{\mathcal{O}_{\mathbf{F}_r(X)}} \kappa) = (r+1)(n-2r-1)\end{aligned}$$

where the lower bound follows from 2.7.3 together with 2.7.4, and the right equality follows from 2.7.6. Thus equality holds throughout and $\mathbf{F}_r(X)$ is smooth at $[\mathbf{P}U]$. ■

2.7.8. Koszul resolution. — As explained in 2.7.3, the Fano scheme $\mathbf{F}_r(X)$ is cut out by the morphism of locally free $\mathcal{O}_{\mathbf{G}(r+1,V)}$-modules

$$\beta_{\mathcal{S}} \colon \operatorname{Fr}^*(\mathcal{S}) \otimes \mathcal{S} \to \mathcal{O}_{\mathbf{G}(r+1,V)}.$$

The associated Koszul complex

$$\operatorname{Kosz}_\bullet(\beta_{\mathcal{S}}) := \big[\, \wedge^{(r+1)^2} (\operatorname{Fr}^*(\mathcal{S}) \otimes \mathcal{S}) \to \cdots \to \operatorname{Fr}^*(\mathcal{S}) \otimes \mathcal{S} \xrightarrow{\beta_{\mathcal{S}}} \mathcal{O}_{\mathbf{G}(r+1,V)} \big]$$

is exact in positive homological degrees away from $\mathbf{F}_r(X)$. Since $\mathbf{G}(r+1,V)$ is regular, it is, in particular, Cohen–Macaulay and the complex will be furthermore form a resolution of $\mathcal{O}_{\mathbf{F}_r(X)}$ as an $\mathcal{O}_{\mathbf{G}(r+1,V)}$-module when $\mathbf{F}_r(X)$ is of expected dimension $(r+1)(n-2r-1)$. See, for example, [**Laz04**, p.320].

A convenient sufficient condition for when the Fano scheme is of expected dimension can be given in terms of the map

$$\beta^\vee \colon \operatorname{Fr}^*(\mathcal{S}_{\mathbf{F}_r(X)}) \to \mathcal{Q}^\vee_{\mathbf{F}_r(X)}$$

where $\beta^\vee$ factors through the subbundle $\mathcal{Q}^\vee_{\mathbf{F}_r(X)} \subset V_{\mathbf{F}_r(X)}$ given by the restriction of the tautological quotient bundle because $\mathcal{S}_{\mathbf{F}_r(X)}$ is isotropic for β. The following identifies the conormal map of $\mathbf{F}_r(X)$ in $\mathbf{G}(r+1,V)$, generalizing 2.2.2:

2.7.9. Proposition. — *There is a commutative diagram of $\mathscr{O}_{\mathbf{F}_r(X)}$-modules*

$$\begin{tikzcd}
\mathrm{Fr}^*(\mathscr{S}_{\mathbf{F}_r(X)}) \otimes \mathscr{S}_{\mathbf{F}_r(X)} \arrow[r, "\beta^\vee \otimes \mathrm{id}"'] \arrow[d, "\beta_{\mathscr{S},\mathbf{F}_r(X)}"'] & \mathscr{Q}^\vee_{\mathbf{F}_r(X)} \otimes \mathscr{S}_{\mathbf{F}_r(X)} \arrow[d, "\cong"] \\
\mathscr{C}_{\mathbf{F}_r(X)/\mathbf{G}(r+1,V)} \arrow[r, "\delta"] & \Omega^1_{\mathbf{G}(r+1,V)}|_{\mathbf{F}_r(X)}
\end{tikzcd}$$

in which δ is the conormal map of $\mathbf{F}_r(X)$ in $\mathbf{G}(r+1,V)$.

Proof. — The image of the morphism $\beta_{\mathscr{S}}$ from 2.7.8 defining $\mathbf{F}_r(X)$ is the ideal sheaf of the Fano scheme in the Grassmannian, whence its restriction $\beta_{\mathscr{S},\mathbf{F}_r(X)}$ surjects onto the conormal sheaf as on the left of the diagram. The right of the diagram arises from the usual identification of the cotangent bundle of the Grassmannian. Since δ acts by differentiating local equations, it is linear over q-powers, whence commutativity of the square. ■

Injectivity of the map $\beta^\vee$ then gives sufficient condition for, amongst other things, when $\mathbf{F}_r(X)$ is of expected dimension:

2.7.10. Corollary. — *If the map $\beta^\vee \colon \mathrm{Fr}^*(\mathscr{S}_{\mathbf{F}_r(X)}) \to \mathscr{Q}^\vee_{\mathbf{F}_r(X)}$ is injective, then*

(i) $\beta_{\mathscr{S},\mathbf{F}_r(X)} \colon \mathrm{Fr}^*(\mathscr{S}_{\mathbf{F}_r(X)}) \otimes \mathscr{S}_{\mathbf{F}_r(X)} \to \mathscr{C}_{\mathbf{F}_r(X)/\mathbf{G}(r+1,V)}$ *is an isomorphism,*

(ii) *there is a short exact sequence*

$$0 \to \mathscr{C}_{\mathbf{F}_r(X)/\mathbf{G}(r+1,V)} \xrightarrow{\delta} \Omega^1_{\mathbf{G}(r+1,V)}|_{\mathbf{F}_r(X)} \to \Omega^1_{\mathbf{F}_r(X)} \to 0,$$

(iii) *there is an isomorphism* $\Omega^1_{\mathbf{F}_r(X)} \cong (\mathscr{Q}^\vee_{\mathbf{F}_r(X)} / \mathrm{Fr}^*(\mathscr{S}_{\mathbf{F}_r(X)})) \otimes \mathscr{S}_{\mathbf{F}_r(X)}$, *and*

(iv) $\mathbf{F}_r(X)$ *is generically smooth, Cohen–Macaulay, and of dimension* $(r+1)(n-2r-1)$.

Proof. — If the map $\beta^\vee$ in question is injective, then the commutative diagram of 2.7.9 implies that $\beta_{\mathscr{S},\mathbf{F}_r(X)}$ is injective. Since it is always surjective, it is an isomorphism, giving (i). The diagram of 2.7.9 now implies that δ is injective, whence exactness of the sequence (ii) and the identification of (iii). Since the conormal sequence is split exact at generic points, so $\mathbf{F}_r(X)$ is generically smooth of the expected dimension, yielding (iv) in particular. ■

In this setting, $\mathbf{F}_r(X)$ is a local complete intersection closed subscheme of the Grassmannian and so its dualizing bundle may be computed by taking determinants of its sheaf of Kähler differentials; see [Har77, Theorem III.7.11]. Write $\mathcal{O}_{\mathbf{F}_r(X)}(1)$ for the restriction of the Plücker line bundle to the Fano scheme. The following additionally keeps track of a determinant twist:

2.7.11. Corollary. — *If $\beta^\vee \colon \mathrm{Fr}^*(\mathcal{S}_{\mathbf{F}_r(X)}) \to \mathcal{Q}^\vee_{\mathbf{F}_r(X)}$ is injective, then*

$$\omega_{\mathbf{F}_r(X)} \cong \mathcal{O}_{\mathbf{F}_r(X)}\big((q+1)(r+1)-(n+1)\big) \otimes \det(V^\vee)^{\otimes r+1}.$$

Proof. — Taking determinants of the sequence in 2.7.10(ii) yields

$$\omega_{\mathbf{F}_r(X)} \cong \det(\Omega^1_{\mathbf{F}_r(X)}) \cong \det(\Omega^1_{\mathbf{G}(r+1,V)}) \otimes \det(\mathrm{Fr}^*(\mathcal{S}) \otimes \mathcal{S})^\vee|_{\mathbf{F}_r(X)}.$$

The result follows upon using the identifications

$$\det(\mathcal{S}) \cong \mathcal{O}_{\mathbf{G}(r+1,V)}(-1), \quad \det(\mathcal{Q}) \cong \mathcal{O}_{\mathbf{G}(r+1,V)}(1) \otimes \det(V)$$

and that $\det(\Omega^1_{\mathbf{G}(r+1,V)}) \cong \mathcal{O}_{\mathbf{G}(r+1,V)}(-n-1) \otimes \det(V^\vee)^{\otimes r+1}$. ■

By 2.7.7, the singular locus of $\mathbf{F}_r(X)$ is contained in the locus parameterizing r-planes that intersect the singular locus of X. When $\beta^\vee$ is injective, all such r-planes give singular points of the Fano scheme:

2.7.12. Corollary. — *If $\beta^\vee \colon \mathrm{Fr}^*(\mathcal{S}_{\mathbf{F}_r(X)}) \to \mathcal{Q}^\vee_{\mathbf{F}_r(X)}$ is injective, then the singular locus of $\mathbf{F}_r(X)$ is the locus parameterizing $\mathbf{P}U \subset X$ that intersect the singular locus of X:*

$$\mathrm{Sing}(\mathbf{F}_r(X)) = \{\, [\mathbf{P}U] \in \mathbf{F}_r(X) \mid \mathbf{P}U \cap \mathrm{Sing}(X) \neq \varnothing \,\}.$$

Proof. — Since the conormal sequence is exact by 2.7.10(ii), the smoothness criterion from [Har77, Theorem II.8.17] means that $\mathbf{F}_r(X)$ is singular where $\Omega^1_{\mathbf{F}_r(X)}$ has torsion. The computation of δ in 2.7.9 shows that this occurs along the degeneracy locus of

$$\beta^\vee \colon \mathrm{Fr}^*(\mathcal{S}_{\mathbf{F}_r(X)}) \to \mathcal{Q}^\vee_{\mathbf{F}_r(X)} \subset V^\vee_{\mathbf{F}_r(X)}.$$

This drops rank over the points where the fibre of $\mathrm{Fr}^*(\mathcal{S}_{\mathbf{F}_r(X)})$ intersects $V^\perp \subset \mathrm{Fr}^*(V)$. By 2.2.7, these points are given by $\mathbf{P}U \subset X$ which intersects the singular locus. ■

It is clear from 2.7.10 that injectivity of $\beta^\vee$ should generally be a condition stronger than expected dimensionality of the Fano scheme. Indeed, the scheme of lines in a q-bic surface of type $\mathbf{N}_3 \oplus \mathbf{1}$ is of expected dimension 0, but $\beta^\vee$ is not injective: by 3.12.1, $\mathbf{F}_1(X)$ is a single nonreduced point. It may also be the case that $\beta^\vee$ is injective on some irreducible components of $\mathbf{F}_r(X)$ and not on others. This occurs on the scheme of lines on a q-bic of type $\mathbf{1}^{\oplus 2} \oplus \mathbf{N}_2$: see 3.10.2. In fact, the following gives a simple characterization of these two conditions:

2.7.13. Lemma. — *The morphism* $\beta^\vee \colon \mathrm{Fr}^*(\mathscr{S}_{\mathbf{F}_r(X)}) \to \mathscr{Q}^\vee_{\mathbf{F}_r(X)}$ *is injective if and only if*

$$\dim \{ [\mathbf{P}U] \in \mathbf{F}_r(X) \mid \mathbf{P}U \cap \mathrm{Sing}(X) \neq \varnothing \} < (r+1)(n-2r-1)$$

and $\mathbf{F}_r(X)$ *is of expected dimension* $(r+1)(n-2r-1)$ *if and only if*

$$\dim \{ [\mathbf{P}U] \in \mathbf{F}_r(X) \mid \mathbf{P}U \cap \mathrm{Sing}(X) \neq \varnothing \} \leq (r+1)(n-2r-1).$$

Proof. — Consider the first statement. If $\beta^\vee$ is injective, then by 2.7.10, $\mathbf{F}_r(X)$ has expected dimension $(r+1)(n-2r-1)$ and is generically smooth. Its singular locus is therefore a proper closed subscheme, and the conclusion follows from 2.7.12. For the converse, consider the diagram

$$\begin{tikzcd}
 & & \mathrm{Fr}^*(\mathscr{S}_{\mathbf{F}_r(X)}) \arrow[r, "\beta^\vee"'] \arrow[d, hook] & \mathscr{Q}^\vee_{\mathbf{F}_r(X)} \arrow[d, hook] & & \\
0 \arrow[r] & V^\perp_{\mathbf{F}_r(X)} \arrow[r] & \mathrm{Fr}^*(V)_{\mathbf{F}_r(X)} \arrow[r, "\beta^\vee"] & V^\vee_{\mathbf{F}_r(X)} \arrow[r] & \mathrm{Fr}^*(V)^{\perp,\vee}_{\mathbf{F}_r(X)} \arrow[r] & 0
\end{tikzcd}$$

defining $\beta^\vee \colon \mathrm{Fr}^*(\mathscr{S}_{\mathbf{F}_r(X)}) \to \mathscr{Q}^\vee_{\mathbf{F}_r(X)}$. Since

$$\dim \{ [\mathbf{P}U] \in \mathbf{F}_r(X) \mid \mathbf{P}U \cap \mathrm{Sing}(X) \neq \varnothing \} < (r+1)(n-2r-1) \leq \dim \mathbf{F}_r(X),$$

the fibres of $\mathrm{Fr}^*(\mathscr{S}_{\mathbf{F}_r(X)})$ and $V^\perp_{\mathbf{F}_r(X)}$ are disjoint at the generic point of $\mathbf{F}_r(X)$. Since $\mathrm{Fr}^*(\mathscr{S}_{\mathbf{F}_r(X)})$ is torsion-free, this shows that $\beta^\vee$ is injective.

For the second statement, by 2.7.7, the singular locus is contained in the set in question. Since the singular locus certainly has dimension less than the entirety of $\mathbf{F}_r(X)$, if the Fano scheme has expected dimension, then the stated inequality holds. Conversely, since the open complement of smooth points in $\mathbf{F}_r(X)$ has dimension

$(r+1)(n-2r-1)$ by **2.7.7**, so as long as the inequality holds, the entire Fano scheme has expected dimension. ■

As a simple application of this criterion, the following shows that when (V,β) has corank at most 1 and has no radical, then $\mathbf{F}_r(X)$ achieves its expected dimension whenever it is nonnegative, and $\beta^\vee \colon \mathrm{Fr}^*(\mathscr{S}_{\mathbf{F}_r(X)}) \to \mathscr{Q}^\vee_{\mathbf{F}_r(X)}$ is injective as soon as the expected dimension of $\mathbf{F}_r(X)$ is positive:

2.7.14. Lemma. — *Suppose (V,β) is a q-bic form of corank at most 1 with no radical.*

(i) *If $n \geq 2r+1$, then $\mathbf{F}_r(X)$ is of expected dimension $(r+1)(n-2r-1)$.*

(ii) *If $n \geq 2r+2$, then $\beta^\vee \colon \mathrm{Fr}^*(\mathscr{S}_{\mathbf{F}_r(X)}) \to \mathscr{Q}^\vee_{\mathbf{F}_r(X)}$ is injective.*

Proof. — Both statements are clear when β is nonsingular, so assume that β has corank 1 and no radical. Let $x = \mathbf{P}L$ be the unique singular point of X. Then by **2.7.13**, it suffices to show that the locus

$$Z := \{\, [\mathbf{P}U] \in \mathbf{F}_r(X) \mid x \in \mathbf{P}U \,\}$$

has dimension at most $(r+1)(n-2r-1)$ for (i), and strictly less for (ii). Proceed by induction on r. When $r=0$, $Z=\{x\}$ and the required inequalities are clear. Assume $r \geq 1$. The intersection $X \cap \mathbf{P}\,\mathrm{Fr}^{-1}(L^\perp)$ is a cone over a q-bic $(n-3)$-fold X' of corank 1 with no radical, and by **2.4.4**, any r-plane through x is the cone over an $(r-1)$-plane in X'. Applying induction to $\mathbf{F}_{r-1}(X')$ now shows that

$$\dim Z = \dim \mathbf{F}_{r-1}(X') = r(n-2r-1).$$

If $n \geq 2r+1$, this is at most $(r+1)(n-2r-1)$, proving (i); if $n \geq 2r+2$, this is strictly less than $(r+1)(n-2r-1)$, proving (ii). ■

Adapting the argument of [**BVdV79**, Theorem 6], see also [**DM98**, p.544], shows that the Fano schemes are connected whenever they are positive dimensional:

2.7.15. Proposition. — *The scheme $\mathbf{F}_r(X)$ is connected whenever $n \geq 2r+2$.*

Proof. — Let $\Phi \subset \mathbf{G}(r+1,V) \times \mathbf{P}(\mathrm{Fr}^*(V)^\vee \otimes V^\vee)$ be the incidence correspondence of pairs $([\mathbf{P}U],[X])$ of r-planes and q-bic hypersurfaces with $\mathbf{P}U \subseteq X$, as in the proof of 2.7.4. Let $Z \subset \Phi$ be the locus where $\mathrm{pr}_2 \colon \Phi \to \mathbf{P}(\mathrm{Fr}^*(V)^\vee \otimes V^\vee)$ is not smooth. Then the codimension of Z in Φ is at least $n-2r$. Indeed, it follows from 2.7.12 that pr_2 is smooth over the open subset of $\mathbf{P}(\mathrm{Fr}^*(V)^\vee \otimes V^\vee)$ parameterizing smooth q-bic hypersurfaces. A general point of the codimension 1 complement corresponds to a q-bic hypersurface X of corank 1 without a radical, see 1.4.5 and 1.4.8. By 2.7.14, $\mathbf{F}_r(X)$ is of expected dimension $(r+1)(n-2r-1)$, and the proof of 2.7.14 shows that $\mathrm{Sing}\,\mathbf{F}_r(X)$ has dimension $r(n-2r-1)$. Therefore

$$\mathrm{codim}(Z \subset \Phi) \geq 1 + \mathrm{codim}(\mathrm{Sing}\,\mathbf{F}_r(X) \subset \mathbf{F}_r(X)) = n-2r.$$

Now consider the Stein factorization

$$\mathrm{pr}_2 \colon \Phi \to \Phi' \to \mathbf{P}(\mathrm{Fr}^*(V)^\vee \otimes V^\vee)$$

of the second projection. Then Z contains the inverse image of the branch locus of $\Phi' \to \mathbf{P}(\mathrm{Fr}^*(V)^\vee \otimes V^\vee)$. So if $n \geq 2r+2$, the codimension estimate implies that the branch locus has codimension at least 2 in Φ'. By the Purity of the Branch Locus, [**Stacks**, 0BMB], $\Phi' \to \mathbf{P}(\mathrm{Fr}^*(V)^\vee \otimes V^\vee)$ is étale, and hence an isomorphism since projective space is simply connected. The properties of the Stein factorization means that pr_2 has connected fibres, and this means that each $\mathbf{F}_r(X)$ is connected. ■

The results so far are simplest when X itself is smooth:

2.7.16. Corollary. — *If $X \subset \mathbf{P}V$ is a smooth q-bic $(n-1)$-fold, then its Fano scheme $\mathbf{F}_r(X)$ of r-planes is smooth, of dimension*

$$\dim \mathbf{F}_r(X) = (r+1)(n-2r-1) \quad \textit{whenever } 0 < r < \tfrac{n}{2},$$

and empty otherwise, and irreducible when $n > 2r+1$. Moreover,

$$\Omega^1_{\mathbf{F}_r(X)} \cong (\mathrm{Fr}^*(\mathscr{S}_{\mathbf{F}_r(X)})^\perp/\mathscr{S}_{\mathbf{F}_r(X)})^\vee \otimes \mathscr{S}_{\mathbf{F}_r(X)}.$$

Proof. — The smoothness and dimension statement follow from 2.7.7. The statement about the sheaf of Kähler differentials follows from 2.7.10(iii) upon noting that there is a short exact sequence

$$0 \to \mathrm{Fr}^*(\mathscr{S}_{\mathbf{F}_r(X)})^{\perp}/\mathscr{S}_{\mathbf{F}_r(X)} \to \mathscr{Q}_{\mathbf{F}_r(X)} \xrightarrow{\beta} \mathrm{Fr}^*(\mathscr{S}_{\mathbf{F}_r(X)})^{\vee} \to 0$$

since β is nondegenerate by 2.2.3. Irreducibility when $n > 2r+1$ follows from its smoothness and the connectedness provided by 2.7.15. ■

2.7.17. Remark. — Combining 2.7.11 and 2.7.16 lead to an interesting collection of smooth projective varieties with a given canonical bundle. For instance, let $r > 0$ and let $n = (q+1)(r+1)-1$. If $X \subset \mathbf{P}V = \mathbf{P}^n$ is any a smooth q-bic hypersurface, then $\mathbf{F}_r(X)$ is a smooth projective variety of dimension $(r+1)^2(q-1)$ with trivial canonical bundle.

2.7.18. Numerical invariants. — Being the zero locus of a of $\mathrm{Fr}^*(\mathscr{S}^{\vee}) \otimes \mathscr{S}^{\vee}$ by 2.7.3, when $\mathbf{F}_r(X)$ has expected codimension $(r+1)^2$, its class in the Chow ring of $\mathbf{G}(r+1, V)$ is given by the top Chern class

$$[\mathbf{F}_r(X)] = c_{(r+1)^2}(\mathrm{Fr}^*(\mathscr{S}^{\vee}) \otimes \mathscr{S}^{\vee}) \in \mathrm{CH}^{(r+1)^2}(\mathbf{G}(r+1, V)).$$

When $r = 1$, this can be reasonably used to compute numerical invariants of the Fano schemes $\mathbf{F}_1(X)$ of lines. Notation as in [Ful98, p.271], write $\sigma_\lambda = \sigma_{\lambda_0,\ldots,\lambda_r}$ for the Schubert cycle associated with the partition $\lambda = (\lambda_0, \ldots, \lambda_r)$ with $n - r \geq \lambda_0 \geq \cdots \geq \lambda_r \geq 0$. Then the total Chern class of the dual of the tautological subbundle is

$$c(\mathscr{S}^{\vee}) = 1 + \sigma_1 + \sigma_{1^2} + \cdots + \sigma_{1^{r+1}},$$

where 1^{d+1} denotes the partition whose nonzero parts are $\lambda_0 = \cdots = \lambda_d = 1$.

2.7.19. Proposition. — *In the Chow ring of* $\mathbf{G}(2, V)$*,*

$$\begin{aligned} c_4(\mathrm{Fr}^*(\mathscr{S}^{\vee}) \otimes \mathscr{S}^{\vee}) &= (q+1)^2\sigma_{1,1}\big((q^2-2q+1)\sigma_{1,1} + q\sigma_1^2\big) \\ &= (q+1)(q^3+1)\sigma_{2,2} + q(q+1)^2\sigma_{3,1}. \end{aligned}$$

If $\mathbf{F}_1(X)$ *is of expected dimension* $2n-6$*, then*

$$\deg(\mathscr{O}_{\mathbf{F}_1(X)}(1)) = \frac{(2n-6)!}{(n-1)!(n-3)!}(q+1)^2\big((n-1)q^2 + (2n-8)q + (n-1)\big).$$

Proof. — Use the splitting principle to write $c(\mathscr{S}^\vee) = (1+a)(1+b)$ in terms of its Chern roots a and b, so that $c_1(\mathscr{S}^\vee) = \sigma_1 = a+b$ and $c_2(\mathscr{S}^\vee) = \sigma_{1,1} = ab$. The Chern roots of $\mathrm{Fr}^*(\mathscr{S}^\vee)$ are given by qa and qb, so $c_4(\mathrm{Fr}^*(\mathscr{S}^\vee) \otimes \mathscr{S}^\vee)$ is given by

$$\begin{aligned}(a+qa)(b+qb)(a+qb)(b+qa) &= (q+1)^2 ab\big((q^2+1)ab + q(a^2+b^2)\big) \\ &= (q+1)^2\sigma_{1,1}\big((q^2-2q+1)\sigma_{1,1} + q\sigma_1^2\big),\end{aligned}$$

giving the first formula; the second formula now comes upon applying Pieri's rule, see [**Ful98**, p.271], to compute the products $\sigma_{1,1}^2 = \sigma_{2,2}$ and $\sigma_{1,1}\sigma_1^2 = \sigma_{2,2} + \sigma_{3,1}$.

The degree of the Plücker line bundle on $\mathbf{F}_1(X)$ is now obtained as

$$\begin{aligned}\deg(\mathscr{O}_{\mathbf{F}_1(X)}(1)) &= \int_{\mathbf{G}(2,V)} \sigma_1^{2n-6} c_4(\mathrm{Fr}^*(\mathscr{S}^\vee) \otimes \mathscr{S}^\vee) \\ &= (q+1)^2\Big((q^2-2q+1)\int_{\mathbf{G}(2,V)} \sigma_1^{2n-6}\sigma_{1,1}^2 + q\int_{\mathbf{G}(2,V)} \sigma_1^{2n-4}\sigma_{1,1}\Big).\end{aligned}$$

Since $\sigma_{1,1} = c_2(\mathscr{S}^\vee)$, it represents the cycle of the Grassmannian of subspaces contained in a hyperplane of V. Thus

$$\deg(\mathscr{O}_{\mathbf{F}_1(X)}(1)) = (q+1)^2\big((q^2-2q+1)\deg(\mathscr{O}_{\mathbf{G}(2,n-1)}(1)) + q\deg(\mathscr{O}_{\mathbf{G}(2,n)}(1))\big).$$

As computed in [**Ful98**, Example 14.7.11], $\deg(\mathscr{O}_{\mathbf{G}(2,n)}(1)) = \frac{1}{n-2}\binom{2n-4}{n-1}$. Putting this into the above gives the formula in the statement. ■

2.8. Fano correspondence

The universal family $\mathbf{L} := \mathbf{L}_r(X) := \mathbf{P}\mathscr{S}$ of r-planes over $\mathbf{F} := \mathbf{F}_r(X)$, given by the projective bundle associated with the universal rank $r+1$ subbundle $\mathscr{S}$ on $\mathbf{F}$, defines an incidence correspondence

$$\begin{tikzcd} & \mathbf{L} \arrow[dl, "\mathrm{pr}_{\mathbf{F}}"'] \arrow[dr, "\mathrm{pr}_X"] & \\ \mathbf{F} & & X \end{tikzcd}$$

between the Fano scheme $\mathbf{F}$ and the q-bic hypersurface X. This is referred to as the *Fano correspondence*. Its basic geometric properties are collected in this Section, and for that purpose, assume throughout that the base field $\mathbf{k}$ is algebraically closed.

The first property says that X is swept out by r-planes whenever $\dim \mathbf{F}_r(X) > 0$:

2.8.1. Lemma. — *If $n \geq 2r+2$, then $\mathrm{pr}_X : \mathbf{L} \to X$ is surjective.*

Proof. — It suffices to show that for a general closed point $x = \mathbf{P}L$ of X, there exists an r-plane contained in X which passes through x. By 2.4.4, any such r-plane must be contained in the intersection

$$X \cap \mathbf{P}\,\mathrm{Fr}^*(L)^\perp \cap \mathbf{P}\,\mathrm{Fr}^{-1}(L^\perp).$$

For general x, this is a cone over a q-bic hypersurface X' of dimension $n-4$, and an r-plane through x as above is a cone over an $(r-1)$-plane in X'. Since

$$\dim \mathbf{F}_{r-1}(X') \geq r(n-2r-2) \geq 0$$

by 2.7.3, the result follows from 2.7.4. ■

The morphism $\mathrm{pr}_X : \mathbf{L} \to X$ never flat as the fibre dimension jumps over special points. When X is smooth, the locus over which the fibre dimension jumps is precisely the set of Hermitian points; compare with the comments in 2.4.5.

2.8.2. Lemma. — *Let X be a smooth q-bic $(n-1)$-fold. Then for a closed point $x \in X$,*

$$\dim \mathrm{pr}_X^{-1}(x) = \begin{cases} r(n-2r-1) & \text{if } n \geq 2r+1 \text{ and } x \text{ is Hermitian, and} \\ r(n-2r-2) & \text{if } n \geq 2r+2 \text{ and } x \text{ is not Hermitian.} \end{cases}$$

Proof. — As in the proof of 2.8.1, the reduced subscheme underlying $\mathrm{pr}_X^{-1}(x)$ may be identified with the space of $(r-1)$-planes in the base X' of the cone

$$X \cap \mathbf{P}\,\mathrm{Fr}^*(L)^\perp \cap \mathbf{P}\,\mathrm{Fr}^{-1}(L^\perp).$$

When x is a Hermitian point, then X' is a smooth q-bic $(n-3)$-fold, see 2.4.8, and so by 2.7.3, the fibre $\mathrm{pr}_X^{-1}(x)$ is nonempty of dimension $r(n-2r-1)$ whenever $n \geq 2r+1$. When x is not a Hermitian point, then X' is a q-bic $(n-4)$-fold of corank at most 1 which is not a cone, so by 2.7.14(i), $\mathbf{F}_{r-1}(X')$ is of expected dimension $r(n-2r-2)$. ■

2.8.3. — In particular, 2.8.2 shows that when $n \geq 2r+2$, the morphism $\mathrm{pr}_X \colon \mathbf{L} \to X$ is surjective and its generic fibre is of dimension $r(n-2r-2)$. To describe some further geometric properties of the fibres of pr_X, view $\mathbf{L}$ as a closed subscheme of the flag variety

$$\mathbf{G}(1,r+1,V)_X := \mathbf{G}(1,r+1,V) \times_{\mathbf{P}V} X = \{\, (x,[\mathbf{P}U]) \in X \times \mathbf{G}(r+1,V) \mid x \in \mathbf{P}U \,\}$$

restricted to X. Let $\pi \colon \mathbf{G}(1,r+1,V)_X \to X$ be the projection, and let

$$0 \to \pi^*\mathcal{O}_X(-1) \to \mathcal{S} \to \mathcal{S}' \to 0$$

be the tautological sequence, with $\mathcal{S}$ the rank $r+1$ tautological subbundle of $V_{\mathbf{G}(1,r+1,V)_X}$. The fibre of π over a point $x = \mathbf{P}L$ of X is canonically isomorphic to the Grassmannian $\mathbf{G}(r, V/L)$ in a way compatible with Plücker polarizations. The scheme $\mathbf{L}$ is defined by the q-bic form $\beta_{\mathcal{S}}$ on $\mathcal{S}$ obtained by restricting β. The following analyzes the basic structure of the equations of $\mathbf{L}$ relative to X, and in particular computes the degree of the general fibre with respect to the Plücker polarization $\mathcal{O}_{\mathrm{pr}_X}(1) := \mathrm{pr}_{\mathbf{F}}^*(\mathcal{O}_{\mathbf{F}}(1))$ over X:

2.8.4. Lemma. — *Let X be a smooth q-bic $(n-1)$-fold with $n \geq 2r+2$. Then each closed fibre of $\mathrm{pr}_X \colon \mathbf{L} \to X$ has multiplicity q^r, and*

$$\deg(\mathcal{O}_{\mathrm{pr}_X}(1)|_{\mathrm{pr}_X^{-1}(x)}) = q^r \deg(\mathbf{F}_{r-1}(X'))$$

for every non-Hermitian closed point $x \in X$, where X' is a smooth q-bic $(n-4)$-fold.

Proof. — Since $\pi^*\mathcal{O}_X(-1)$ is an isotropic subbundle of $\mathcal{S}$, $\beta_{\mathcal{S}}$ induces two morphisms

$$\mathrm{Fr}^*(\pi^*\mathcal{O}_X(-1)) \otimes \mathcal{S}' \to \mathcal{O}_{\mathbf{G}(1,r+1,V)_X} \quad \text{and} \quad \mathrm{Fr}^*(\mathcal{S}') \otimes \pi^*\mathcal{O}_X(-1) \to \mathcal{O}_{\mathbf{G}(1,r+1,V)_X}$$

and their vanishing locus $\mathbf{L}'$ is a closed subscheme of $\mathbf{G}(1,r+1,V)_X$ containing $\mathbf{L}$. Taken together, the equations mean that the reduced fibre of $\mathbf{L}'$ over a non-Hermitian point $x = \mathbf{P}L$ of X is the Grassmannian of r-spaces in $V' := (\mathrm{Fr}^*(L)^{\perp} \cap \mathrm{Fr}^{-1}(L^{\perp}))/L$ viewed as a subscheme of $\mathbf{G}(r, V/L)$. Since the first set of equations are linear in the fibre coordinates over X, whereas the second set are q-powers, the closed fibres of

$\mathbf{L}' \to X$ have multiplicity q^r. Finally, the scheme $\mathbf{L}$ is the closed subscheme of $\mathbf{L}'$ cut out by a morphism

$$\mathrm{Fr}^*(\mathscr{S}') \otimes \mathscr{S}' \to \mathscr{O}_{\mathbf{L}'}$$

induced by $\beta_{\mathscr{S}}$. Restricted to the reduced fibre $\mathbf{G}(r, V')$ over x, this gives the equations to the Fano scheme $\mathbf{F}_{r-1}(X')$ of $(r-1)$-planes to a q-bic hypersurface X' of corank at most 1 and which is not a cone in $\mathbf{P}V'$. The argument of 2.8.2 shows that this Fano scheme is of expected dimension, and so its Plücker degree coincides with the corresponding Fano scheme of a smooth q-bic. ■

2.8.5. Action of the correspondence. — The incidence correspondence $\mathbf{L}$ is a closed subscheme of $\mathbf{F} \times X$, and may be viewed as a correspondence of degree $-r$ from $\mathbf{F}$ to X; see [Ful98, Chapter 16] for generalities on correspondences. This induces morphisms

$$\mathbf{L}_* \colon \mathrm{CH}_*(\mathbf{F}) \to \mathrm{CH}_{*+r}(X) \quad \text{and} \quad \mathbf{L}^* \colon \mathrm{CH}^*(X) \to \mathrm{CH}^{*-r}(\mathbf{F})$$

given by $\mathbf{L}_*(\alpha) := \mathrm{pr}_{X,*}(\mathrm{pr}_{\mathbf{F}}^*(\alpha) \cdot \mathbf{L})$ and $\mathbf{L}^*(\beta) := \mathrm{pr}_{\mathbf{F},*}(\mathrm{pr}_X^*(\beta) \cdot \mathbf{L})$, where the dot denotes the intersection product of cycles on $\mathbf{F} \times X$.

The next statement shows that the correspondence $\mathbf{L}$ relates the two polarizations

$$h := c_1(\mathscr{O}_X(1)) \in \mathrm{CH}^1(X) \quad \text{and} \quad g := c_1(\mathscr{O}_{\mathbf{F}}(1)) \in \mathrm{CH}^1(\mathbf{F})$$

given by the hyperplane class h of X, and the Plücker polarization g of $\mathbf{F}$.

2.8.6. Lemma. — *If $n \geq 2r+2$, then for every $r+1 \leq k \leq n-1$,*

$$0 \neq \mathbf{L}^*(h^k) = c_{k-r}(\mathscr{Q}) \in \mathrm{CH}^{k-r}(\mathbf{F}).$$

In particular, $\mathbf{L}^*(h^{r+1}) = g$ *in* $\mathrm{CH}^1(\mathbf{F})$.

Proof. — View h^k as the restriction to X of the class of a general $(n-k)$-plane in $\mathbf{P}^n$. Then $\mathbf{L}^*(h^k)$ represents the class of the subscheme of r-planes in X that are incident with h^k; in other words, this is represented by the intersection of $\mathbf{F}_r(X)$ with the Schubert variety Σ_k corresponding to incidence with a general $(n-k)$-plane. Since there is an r-plane through every point of X by 2.8.1, it follows that $\Sigma_k \cap \mathbf{F}_r(X)$ is always nonempty. Therefore

$$\mathbf{L}^*(h^k) = [\Sigma_k] = c_{k-r}(\mathscr{Q}) \in \mathrm{CH}^{k-r}(\mathbf{F})$$

and each class is nonzero; see [Ful98, Example 14.7.3] for the second equality. ■

Using the cycle class map and the Poincaré duality pairing, $\mathbf{L}$ acts on any Weil cohomology theory. In particular, it yields a map on ℓ-adic cohomology

$$\mathbf{L}^* \colon \mathrm{H}^*_{\text{ét}}(X, \mathbf{Q}_\ell) \to \mathrm{H}^{*-2r}_{\text{ét}}(\mathbf{F}, \mathbf{Q}_\ell).$$

The following shows that, in the case $r = 1$, the action is injective on the middle cohomology of X. The injectivity statement is analogous to one involving schemes of lines to smooth Fano hypersurfaces over the complex number, see [Shi90]. The second statement is analogous to one for cubic hypersurfaces, see [Huy22].

2.8.7. Lemma. — *The Fano correspondence defines an injective map*

$$\mathbf{L}^* \colon \mathrm{H}^{n-1}_{\text{ét}}(X, \mathbf{Z}_\ell) \to \mathrm{H}^{n-3}_{\text{ét}}(\mathbf{F}_1(X), \mathbf{Z}_\ell)$$

and satisfies

$$(\alpha \cdot \beta) = -\frac{1}{q(q+1)} \int_{\mathbf{F}_1(X)} \mathbf{L}^*(\alpha) \cdot \mathbf{L}^*(\beta) \cdot g^{n-2}$$

for all primitive classes $\alpha, \beta \in \mathrm{H}^{n-1}_{\text{ét}}(X, \mathbf{Q}_\ell)_{\mathrm{prim}}$.

Proof. — Since $\mathrm{H}^{n-1}_{\text{ét}}(X, \mathbf{Z}_\ell)$ is torsion-free, see 2.6.4, it suffices to prove that $\mathbf{L}^*$ is injective upon inverting ℓ. When $n-1$ is even, 2.8.6 shows that the class of $h^{(n-1)/2}$ is mapped to a nonzero class in $\mathrm{H}^{n-3}_{\text{ét}}(\mathbf{F}_1(X), \mathbf{Q}_\ell)$. Since $\mathbf{L}^*$ is equivariant for the action of $\mathrm{U}_{n+1}(q)$ and the hyperplane class and the primitive subspace span distinct irreducible representations by 2.6.4(ii), it remains to show that $\mathbf{L}^*$ is injective on $\mathrm{H}^{n-1}_{\text{ét}}(X, \mathbf{Q}_\ell)_{\mathrm{prim}}$; by Poincaré duality, this will be implied by the second statement.

To relate the pairings, since $\mathbf{L}$ is a $\mathbf{P}^1$-bundle over $\mathbf{F}$, there is an isomorphism

$$\mathrm{H}^{n-1}_{\text{ét}}(\mathbf{L}, \mathbf{Q}_\ell) \cong \mathrm{pr}_{\mathbf{F}}^* \mathrm{H}^{n-1}_{\text{ét}}(\mathbf{F}, \mathbf{Q}_\ell) \oplus \mathrm{pr}_{\mathbf{F}}^* \mathrm{H}^{n-3}_{\text{ét}}(\mathbf{F}, \mathbf{Q}_\ell) \cdot \xi$$

where $\xi := c_1(\mathscr{O}_{\mathrm{pr}_{\mathbf{F}}}(1))$. Then for any class $\alpha \in \mathrm{H}^{n-1}_{\text{ét}}(X, \mathbf{Q}_\ell)_{\text{prim}}$,

$$\mathrm{pr}_X^*(\alpha) = \mathrm{pr}_{\mathbf{F}}^*(\alpha') + \mathrm{pr}_{\mathbf{F}}^*(\mathbf{L}^*(\alpha)) \cdot \xi \quad \text{for some } \alpha' \in \mathrm{H}^{n-1}_{\text{ét}}(\mathbf{F}, \mathbf{Q}_\ell).$$

Using the relation $\xi^2 + \mathrm{pr}_{\mathbf{F}}^*(c_1(\mathscr{S})) \cdot \xi + \mathrm{pr}_{\mathbf{F}}^*(c_2(\mathscr{S})) = 0$ in $\mathrm{CH}^*(\mathbf{L})$ given by Grothendieck's construction of the Chern classes, see [Gro58, p.144] or [Ful98, Remark 3.2.4], and the fact that α is primitive, it follows that

$$0 = \mathrm{pr}_X^*(\alpha \cdot h) = \mathrm{pr}_X^*(\alpha) \cdot \xi = -\mathrm{pr}_{\mathbf{F}}^*(\mathbf{L}^*(\alpha) \cdot c_2(\mathscr{S})) + \mathrm{pr}_X^*(\mathbf{L}^*(\alpha) \cdot c_1(\mathscr{S}) - \alpha') \cdot \xi$$

since $\mathrm{pr}_X^*(h) = \xi$ by the discussion of 2.8.3. Therefore $\mathbf{L}^*(\alpha) \cdot c_2(\mathscr{S}) = 0$ and

$$\mathrm{pr}_X^*(\alpha) = \mathrm{pr}_{\mathbf{F}}^*(\mathbf{L}^*(\alpha)) \cdot (\mathrm{pr}_{\mathbf{F}}^*(c_1(\mathscr{S})) + \xi) \quad \text{for all } \alpha \in \mathrm{H}^{n-1}_{\text{ét}}(X, \mathbf{Q}_\ell)_{\text{prim}}.$$

Therefore, for any pair α and β of primitive classes, since $\mathrm{pr}_{\mathbf{F},*}$ extracts the coefficient of ξ, the Chern class relation above implies that

$$\mathrm{pr}_{\mathbf{F},*}\,\mathrm{pr}_X^*(\alpha \cdot \beta) = \mathrm{pr}_{\mathbf{F},*}\,\mathrm{pr}_{\mathbf{F}}^*(\mathbf{L}^*(\alpha) \cdot \mathbf{L}^*(\beta)) \cdot (\mathrm{pr}_{\mathbf{F}}^*(c_1(\mathscr{S})) + \xi)^2 = -\mathbf{L}^*(\alpha) \cdot \mathbf{L}^*(\beta) \cdot g$$

in $\mathrm{H}^{2n-4}_{\text{ét}}(\mathbf{F}, \mathbf{Q}_\ell)$. Write $\alpha \cdot \beta = \deg(\alpha \cdot \beta)[x]$ in terms of the class of a point in $\mathrm{H}^{2n-2}_{\text{ét}}(X, \mathrm{Q}_\ell)$ so that $\mathrm{pr}_X^*(\alpha \cdot \beta) = (\alpha \cdot \beta)[\mathbf{L}_x]$ is a multiple of the class of a fibre of $\mathrm{pr}_X \colon \mathbf{L} \to X$. Multiplying by g^{n-4} on both sides and taking degrees now shows that

$$(\alpha \cdot \beta) \int_{\mathbf{L}} \mathbf{L}_x \cdot g^{n-4} = -\int_{\mathbf{F}} \mathbf{L}^*(\alpha) \cdot \mathbf{L}^*(\beta) \cdot g^{n-3}$$

and the integral on the left is the Plücker degree of a general fibre of pr_X. By 2.8.4, this is $q(q+1)$, yielding the result. ■

2.8.8. Incidence schemes. — Let X be a smooth q-bic hypersurface. Suppose that $n \geq 2r+2$ so that $\mathbf{F}_k(X)$ is of positive dimension and $\mathrm{pr}_X \colon \mathbf{L}_k(X) \to X$ is surjective for every $0 \leq k \leq r$, see 2.8.1. For every r-plane $P \subset X$, let $D_{k,P}$ be the closed subscheme of $\mathbf{F}_k(X)$ obtained by taking the closure of the locus

$$\left\{ P' \in \mathbf{F}_k(X) \,\middle|\, P' \neq P \text{ and } P' \cap P \neq \varnothing \right\}$$

of k-planes in X which are incident with P. This is equivalently the support of the $(k(n-2k-2)+r)$-cycle obtained by applying the correspondence

$$\mathbf{L}_k(X)^* \colon \mathrm{CH}^{n-r-1}(X) \to \mathrm{CH}^{n-r-k-1}(\mathbf{F}_k(X))$$

to the cycle $[P]$. The following gives a basic property of the cycles $[D_{k,P}]$:

2.8.9. Lemma. — *The $D_{k,P}$ are algebraically equivalent for varying r-planes P, and*

$$(q+1)[D_{k,P}] \sim_{\mathrm{alg}} c_{n-r-k-1}(\mathcal{Q}).$$

Proof. — The classes $[D_{k,P}]$ are images of the classes $[P]$ under the correspondence $\mathbf{L}_k(X)^*$. All r-planes in X are parameterized by the Fano scheme $\mathbf{F}_r(X)$ which, since $n \geq 2r+2$, is connected by 2.7.15. So the classes $[P]$ are algebraically equivalent for varying P, whence by [Ful98, Proposition 10.3], the $[D_{k,P}]$ are also algebraically equivalent for varying P.

For the second statement, by the first part and 2.8.6, it suffices to show that there exists a $(r+1)$-plane section of X which is a union of r-planes. By taking general hyperplane sections, it suffices to consider the case $n = 2r+2$, in which case, such a $(r+1)$-plane is obtained by taking successively taking the embedded tangent space at a Hermitian point and applying induction using 2.4.8. ■

2.9. Hermitian structures

Given a nonsingular q-bic form (V, β), the constructions of 1.2.16 give a canonical q^2-linear map $\phi \colon V \to V$. This may be viewed as a $\mathbf{F}_{q^2}$-rational structure of V with which β is related to a Hermitian form. This section will describe some of the geometric content of this structure. Paragraphs 2.9.1–2.9.7 describe the structure provided by ϕ on all of $\mathbf{P}V$.

The structure induced on the associated q-bic hypersurface X is described starting from 2.9.8. In particular, this restricts to a morphism $\phi_X \colon X \to X$ of schemes over $\mathbf{k}$, which then induces a canonical filtration $X^\bullet$ of X by complete intersections: see 2.9.9 and 2.9.10. This filtration is related to the Fano schemes in 2.9.15.

The endomorphism ϕ is preserved by the action of finite subgroups of $\mathbf{GL}(V)$; namely, $\mathrm{GL}_{n+1}(q^2)$ and, upon taking β into consideration, $\mathrm{U}_{n+1}(q)$. Thus the varieties appearing in this section are intimately related to Deligne–Lusztig varieties, as introduced in [DL76]. The objects in the first half are related to Drinfel's upper half plane over finite fields: compare with [Lan19] and [Eke04]. The objects in the second half are related to Deligne–Lusztig varieties of type A_n^2, and this relation will be used in 4.7 to determine the ℓ-adic cohomology of the surface of lines associated with a smooth q-bic threefold.

2.9.1. Endomorphism. — The canonical q^2-linear map $\phi \colon V \to V$ defined in 1.2.16 induces, for each $0 \leq r \leq n$, endomorphisms of schemes over $\mathbf{k}$

$$\phi_{\mathbf{G}(r+1,V)} \coloneqq (\beta^{-1} \circ \beta^{(q),\vee}) \circ \mathrm{Fr}^2_{\mathbf{G}(r+1,V)/\mathbf{k}} \colon \mathbf{G}(r+1,V) \to \mathbf{G}(r+1,V)^{(q^2)} \to \mathbf{G}(r+1,V).$$

This is defined by the subbundle $\mathrm{Fr}^{2,*}(\mathcal{S}_{\mathbf{G}(r+1,V)}) \hookrightarrow V_{\mathbf{G}(r+1,V)}$ obtained by composing the q^2-Frobenius pullback of the tautological subbundle with the linearization of ϕ, as in 1.1.1. Therefore $\phi_{\mathbf{G}(r+1,V)}$ is the composition of the $\mathbf{k}$-linear q^2-power Frobenius with the isomorphism induced by the linear isomorphism $\beta^{-1} \circ \beta^{(q),\vee} \colon \mathrm{Fr}^{2,*}(V) \to V$. In particular, this shows that $\phi_{\mathbf{G}(r+1,V)}$ is purely inseparable of degree $q^{2(r+1)(n-r)}$.

2.9.2. Cyclic subspaces. — The endomorphisms $\phi_{\mathbf{G}(r+1,V)}$ identify various structures on the Grassmannians. For instance, it follows from 1.2.19 that the fixed points of $\phi_{\mathbf{G}(r+1,V)}$ correspond to the Hermitian subspaces in V of dimension $r+1$. As another simple construction, iterating $\phi_{\mathbf{P}V}$ yields a rational map from $\mathbf{P}V$ to the Grassmannians:

2.9.3. Proposition. — *For each $0 \leq r \leq n$, the injection of $\mathcal{O}_{\mathbf{P}V}$-modules*

$$\bigoplus\nolimits_{i=0}^{r} \phi_{\mathbf{P}V}^{i,*}(\mathrm{eu}) \colon \bigoplus\nolimits_{i=0}^{r} \mathcal{O}_{\mathbf{P}V}(-q^{2i}) \to V_{\mathbf{P}V}$$

is locally split away from the Hermitian $(r-1)$-planes in $\mathbf{P}V$. The yields a rational map

$$\mathrm{cyc}^r_\phi \colon \mathbf{P}V \dashrightarrow \mathbf{G}(r+1,V), \quad x \mapsto \langle x, \phi_{\mathbf{P}V}(x), \ldots, \phi^r_{\mathbf{P}V}(x) \rangle$$

defined away from the Hermitian $(r-1)$-planes of $\mathbf{P}V$, and injective away from the Hermitian r-planes of $\mathbf{P}V$.

Proof. — The morphism $\phi_{\mathbf{P}V}$ is constructed in 2.9.1 as the linearization of $\phi\colon V \to V$. Therefore the sheaf map in the statement degenerates along points corresponding to vectors $v \in V$ such that the cyclic subspace $U := \langle v, \phi(v), \ldots, \phi^r(v)\rangle$ has dimension at most r. By 1.2.20, this means that v is contained in a Hermitian subspace of dimension at most r, yielding the first two statements. If U does not contain a Hermitian subspace of dimension $r+1$, then $U \cap \phi^r(U) = \langle \phi^r(v)\rangle$. Since ϕ is injective, this determines v. This gives the injectivity statement. ■

The rational map cyc^r_ϕ factors through the locally closed subscheme of the Grassmannian parameterizing cyclic subspaces for ϕ. Let $\mathbf{G}(r+1,V)_{\mathrm{cyc}}$ denote the closure of this locus, so that cyc^r_ϕ may be viewed as a dominant rational map $\mathbf{P}V \dashrightarrow \mathbf{G}(r+1,V)_{\mathrm{cyc}}$. There is a map in the other direction:

2.9.4. Proposition. — *For each $0 \leq r \leq n$, there is a dominant rational map*

$$[\mathrm{id}\cap\phi^r]\colon \mathbf{G}(r+1,V)_{\mathrm{cyc}} \dashrightarrow \mathbf{P}V$$

defined away from the locus of subspaces containing a 2-dimensional Hermitian subspace of V, and injective away from the locus of subspaces containing a Hermitian vector of V.

Proof. — The scheme $\mathbf{G}(r+1,V)_{\mathrm{cyc}}$ is an irreducible component of

$$\mathrm{Degen}_{r+2}\big(\mathscr{S} \oplus \phi^*_{\mathbf{G}(r+1,V)}(\mathscr{S}) \to V_{\mathbf{G}(r+1,V)}\big)$$

the subscheme of the Grassmannian where the fibre of the tautological subbundle and its pullback by ϕ intersect in a space of dimension at least r. Therefore, for each $0 \leq i \leq r$, the intersection $\mathscr{S} \cap \phi^{i,*}_{\mathbf{G}(r+1,V)}(\mathscr{S})$ is torsion-free subsheaf of $V_{\mathbf{G}(r+1,V)_{\mathrm{cyc}}}$ of rank $r+1-i$, which by 1.2.20, is a subbundle away from the locus of subspaces containing a $(r+2-i)$-dimensional Hermitian subspace of V. Taking $i = r$ gives the first statement, and injectivity follows by the same argument as in 2.9.3. ■

It follows from the functorial description given in 2.9.4 that $\mathbf{G}(r+1,V)_{\mathrm{cyc}}$ is stable under the endomorphism $\phi_{\mathbf{G}(r+1,V)}$ of 2.9.1. Write $\phi_{\mathbf{G}(r+1,V)_{\mathrm{cyc}}}$ for the induced endomorphism.

2.9.5. Corollary. — *For each* $0 \leq r \leq n$*,*

$$[\mathrm{id} \cap \phi^r] \circ \mathrm{cyc}^r_\phi = \phi^r_{\mathbf{P}V} \quad \textit{and} \quad \mathrm{cyc}^r_\phi \circ [\mathrm{id} \cap \phi^r] = \phi^r_{\mathbf{G}(r+1,V)_{\mathrm{cyc}}}.$$

Therefore cyc^r_ϕ *and* $[\mathrm{id} \cap \phi^r]$ *are dominant, generically finite, and purely inseparable.*

Proof. — This follows from the functorial descriptions given in 2.9.3 and 2.9.4. ■

2.9.6. Resolution. — The graph closures of the rational maps cyc^r_ϕ and $[\mathrm{id} \cap \phi^r]$ constructed in 2.9.3 and 2.9.4 admit moduli descriptions which then provide resolutions of the two rational maps. Indeed, these are the closed subschemes of the incidence correspondence between $\mathbf{P}V$ and $\mathbf{G}(r+1,V)_{\mathrm{cyc}}$ with points

$$\begin{aligned} \Gamma_{\mathrm{cyc}^r_\phi} &= \{\, (L \subset U) \mid \langle L, \phi(L), \ldots, \phi^r(L) \rangle \subseteq U \,\}, \\ \Gamma_{[\mathrm{id} \cap \phi^r]} &= \{\, (L \subset U) \mid L \subseteq U \cap \phi^r(U) \,\}, \end{aligned}$$

where the points of the incidence correspondence are written as flags $(L \subset U)$ with $[L] \in \mathbf{P}V$ and $[U] \in \mathbf{G}(r+1,V)_{\mathrm{cyc}}$. The map ϕ induces morphisms:

$$\begin{aligned} \phi^r \times \mathrm{id} \colon \Gamma_{\mathrm{cyc}^r_\phi} &\to \Gamma_{[\mathrm{id} \cap \phi^r]}, & (L \subset U) &\mapsto (\phi^r(L) \subset U), \\ \mathrm{id} \times \phi^r \colon \Gamma_{[\mathrm{id} \cap \phi^r]} &\to \Gamma_{\mathrm{cyc}^r_\phi}, & (L \subset U) &\mapsto (L \subset \phi^r(U)). \end{aligned}$$

These morphisms resolve the rational maps in the following sense:

2.9.7. Proposition. — *There is a commutative diagrams of schemes over* $\mathbf{k}$ *given by*

$$\begin{tikzcd} \Gamma_{\mathrm{cyc}^r_\phi} \arrow[r, "\phi^r \times \mathrm{id}"'] \arrow[d, "\mathrm{pr}_1"'] & \Gamma_{[\mathrm{id} \cap \phi^r]} \arrow[r, "\mathrm{id} \times \phi^r"'] \arrow[d, "\mathrm{pr}_2"'] & \Gamma_{\mathrm{cyc}^r_\phi} \arrow[d, "\mathrm{pr}_1"] \\ \mathbf{P}V \arrow[r, dashed, "\mathrm{cyc}^r_\phi"] & \mathbf{G}(r+1,V)_{\mathrm{cyc}} \arrow[r, dashed, "{[\mathrm{id} \cap \phi^r]}"] & \mathbf{P}V \end{tikzcd}$$

such that

(i) $\mathrm{pr}_1 \colon \Gamma_{\mathrm{cyc}^r_\phi} \to \mathbf{P}V$ *is an isomorphism away from the union of the Hermitian $(r-1)$-planes of* $\mathbf{P}V$*;*

(ii) $\mathrm{pr}_2 \colon \Gamma_{[\mathrm{id} \cap \phi^r]} \to \mathbf{G}(r+1, V)_{\mathrm{cyc}}$ *is an isomorphism away from the locus of subspaces containing a 2-dimensional Hermitian subspace of V; and*

(iii) $\phi^r \times \mathrm{id}$ *and* $\mathrm{id} \times \phi^r$ *are finite purely inseparable of degrees $q^{r(r+1)}$ and $q^{r(2n-r-1)}$.*

Proof. — Since $\Gamma_{\mathrm{cyc}^r_\phi}$ and $\Gamma_{[\mathrm{id} \cap \phi^r]}$ are the graphs of the rational maps from **2.9.3** and **2.9.4**, items (i) and (ii) follow from the identification of the indetereminacy loci. For (iii), note that the morphisms in question are induced by the finite purely inseparable morphisms $\phi_{\mathbf{P}V} \times \mathrm{id}$ and $\mathrm{id} \times \phi_{\mathbf{G}(r+1,V)}$ on the product $\mathbf{P}V \times \mathbf{G}(r+1, V)$, so they too share these properties.

Compute the degree of $\phi^r \times \mathrm{id}$ by considering the fibre of a general geometric point of $\Gamma_{[\mathrm{id} \cap \phi^r]}$. By **1.2.20**, a general point may be taken to be $(L_r \subset U)$ with

$$L_i = \langle \phi^i(v) \rangle, \quad U = \langle v, \phi(v), \ldots, \phi^r(v) \rangle, \quad \text{and} \quad V = \langle v, \phi(v), \ldots, \phi^n(v) \rangle.$$

The fibre of $\phi^r \times \mathrm{id}$ over $(L_r \subset U)$ is supported on the point $(L_0 \subset U)$, and represents the deformation functor on the category of Artinian local $\mathbf{k}$-algebras given by

$$A \mapsto \left\{ (\tilde{L} \subset U \otimes_{\mathbf{k}} A) \in \Gamma_{\mathrm{cyc}^r_\phi}(A) \;\middle|\; \tilde{L} \otimes_A \mathbf{k} = L_0 \text{ and } \phi^r(\tilde{L}) = L_r \otimes_{\mathbf{k}} A \right\}.$$

Since A is local, $\tilde{L}$ is free. Since $\tilde{L}$ is a deformation of L_0, a basis is of the form

$$\tilde{v} = v + a_1 \phi(v) + \cdots + a_r \phi^r(v) \quad \text{for some } a_1, \ldots, a_r \in \mathfrak{m}_A.$$

The condition that $(\tilde{L} \subset U \otimes_{\mathbf{k}} A)$ is an A-point of $\Gamma_{\mathrm{cyc}^r_\phi}$ means that

$$\phi^i(\tilde{v}) = \phi^i(v) + a_1^{q^{2i}} \phi^{i+1}(v) + \cdots + a_r^{q^{2i}} \phi^{i+r}(v) \in U \otimes_{\mathbf{k}} A$$

for each $0 \leq i \leq r$. Since $\phi^j(v) \notin U$ for $j > r$, this implies that

$$a_{r-i}^{q^{2(i+1)}} = 0 \quad \text{for } 0 \leq i \leq r-1.$$

This also implies the condition that $\phi^r(\tilde{L}) = L_r \otimes_{\mathbf{k}} A$. Therefore

$$(\phi^r \times \mathrm{id})^{-1}(L_r \subset U) \cong \mathrm{Spec}\big(\mathbf{k}[\epsilon_1, \ldots, \epsilon_r]/(\epsilon_1^{q^{2r}}, \ldots, \epsilon_r^{q^2})\big)$$

and so $\deg(\phi^r \times \mathrm{id}) = q^{r(r+1)}$. Since the composite $(\phi^r \times \mathrm{id}) \circ (\mathrm{id} \times \phi^r)$ has degree q^{2rn} by 2.9.5, this implies that $\deg(\mathrm{id} \times \phi^r) = q^{r(2n-r-1)}$. ■

2.9.8. Restriction to $\mathbf{F}_r(X)$. — The constructions made so far on all of $\mathbf{G}(r+1, V)$ restrict well to the Fano schemes of the smooth q-bic hypersurface X. Indeed, since the $\mathbf{F}_r(X)$ are moduli of isotropic subspaces by 2.7.3 and since ϕ preserves isotropic vectors by 1.2.18, the maps $\phi_{\mathbf{G}(r+1,V)}$ restrict to endomorphisms

$$\phi_{\mathbf{F}_r(X)} \colon \mathbf{F}_r(X) \to \mathbf{F}_r(X)$$

of schemes over $\mathbf{k}$, for each $0 \leq r < n/2$.

The $r = 0$ case, corresponding to $\mathbf{F}_0(X) = X$, has a particularly simple geometric interpretation. Recall from 2.2.15 that the *residual point of tangency* to a point $x \in X$ is the other special point of the q-bic hypersurface $X \cap \mathbf{T}_{X,x}$ of corank 1; for a smooth q-bic curve, this is the residual point of intersection with the tangent line, see 3.5.2.

2.9.9. Lemma. — *$\phi_X \colon X \to X$ sends a point to its residual point of tangency.*

Proof. — The discussion of 2.2.2 gives a commutative diagram of $\mathcal{O}_X$-modules

$$\begin{tikzcd}
\mathcal{O}_X(-q^2) \arrow[r, "\mathrm{eu}^{(q^2)}"'] \arrow[d, "f_\beta^q"'] & \mathrm{Fr}^{2,*}(V)_X \arrow[r, "\beta^{(q),\vee}"'] & \mathrm{Fr}^*(V)_X^\vee \arrow[d, "\beta^{-1}"] \\
\mathrm{Fr}^*(\mathcal{N}_{X/\mathbf{P}V}(-1))^\vee \arrow[r, "\phi_X"] & \mathcal{T}_X^{\mathrm{e}} \arrow[r, hook] & V_X
\end{tikzcd}$$

where the $\mathcal{O}_X$-module morphism ϕ_X is as constructed in 2.2.14 and which gives the residual point of tangency, and the top composition is the restriction of the map defining $\phi_{\mathbf{P}V}$ from 2.9.1. This gives the result. ■

2.9.10. Geometric filtration. — The endomorphism ϕ_X determines a filtration $X^\bullet$ of X by closed subschemes as follows: Viewing X as the moduli space of isotropic vectors for the q-bic form (V, β) as in 2.1.3, consider the closed subfunctors

$$X^\bullet \colon X =: X^0 \supseteq X^1 \supseteq X^2 \supseteq \cdots \supseteq X^{\lfloor n/2 \rfloor}$$

where the value of $X^k\colon \mathrm{Sch}_{\mathbf{k}}^{\mathrm{opp}} \to \mathrm{Set}$ on a $\mathbf{k}$-scheme T is given by

$$T \mapsto \Big\{ \iota\colon \mathscr{V}' \hookrightarrow V_T \;\Big|\; \bigoplus\nolimits_{i=0}^{k} \phi_T^i \circ \mathrm{Fr}^{2i,*}(\iota)\colon \bigoplus\nolimits_{i=0}^{k} \mathrm{Fr}^{2i,*}(\mathscr{V}') \to V_T \text{ is isotropic for } \beta \Big\}$$

where $\iota\colon \mathscr{V}' \hookrightarrow V_T$ is an isotropic subbundle of rank 1, $\phi_T^i\colon \mathrm{Fr}^{2i,*}(V)_T \to V_T$ is the linearization of the q^{2i}-linear map induced by $\phi^i\colon V \to V$, and a morphism $\varphi\colon \mathscr{E} \to V_T$ is said to be *isotropic* if the composition

$$\mathrm{Fr}^*(\varphi)^\vee \circ \beta \circ \varphi\colon \mathscr{E} \to V_T \to \mathrm{Fr}^*(V)_T^\vee \to \mathrm{Fr}^*(\mathscr{E})^\vee$$

vanishes. The pieces of the filtration are representable by complete intersections:

2.9.11. Proposition. — *Let $0 \leq k \leq n/2$. Then X^k is represented by the complete intersection in $\mathbf{P}V$ with equations*

$$\beta(\phi_{\mathbf{P}V}^{*,i}(\mathrm{eu})^{(q)}, \mathrm{eu})\colon \mathscr{O}_{\mathbf{P}V} \to \mathscr{O}_{\mathbf{P}V}(q^{2i+1}+1) \quad \textit{for } 0 \leq i \leq k.$$

The singular locus of X^k is supported on the union of the Hermitian $(k-1)$-planes in X.

Proof. — The case $k=0$ is 2.1.3. So consider $0 < k \leq n/2$. View X as the moduli space of isotropic vectors for (V,β) and view X^k as a closed subfunctor of X. Then to recognize X^k as the stated complete intersection, it suffices to show that for every isotropic $v \in V$,

$$\langle v, \phi(v), \ldots, \phi^k(v)\rangle \text{ is isotropic} \quad\Longleftrightarrow\quad \beta(\phi^i(v)^{(q)}, v) = 0 \text{ for } 0 \leq i \leq k.$$

The linear space $\langle v, \phi(v), \ldots, \phi^k(v)\rangle$ is isotropic if and only if

$$\beta(\phi^i(v)^{(q)}, \phi^j(v)) = 0 \quad \text{for each } 0 \leq i,j \leq k.$$

By successively applying the identities 1.2.18 and 1.2.17,

$$\beta(\phi^i(v)^{(q)}, \phi^j(v)) = \begin{cases} \beta(\phi^{i-j}(v)^{(q)}, v)^{q^{2j}} & \text{if } i \geq j, \text{ and} \\ \beta(\phi^{j-i-1}(v)^{(q)}, v)^{q^{2i+1}} & \text{if } i < j. \end{cases}$$

This shows that

$$\beta(\phi^i(v)^{(q)}, \phi^j(v)) = 0 \text{ for } 0 \leq i,j \leq r \quad\Longleftrightarrow\quad \beta(\phi^i(v)^{(q)}, v) = 0 \text{ for } 0 \leq i \leq r$$

which shows that X^k is the claimed complete intersection.

Observe that, as in 2.2.2, the conormal map of X^k fits into a commutative diagram

$$\begin{tikzcd}
\bigoplus_{i=0}^k \mathcal{O}_{X^k}(-q^{2i+1}) \arrow[rr, "\bigoplus_{i=0}^k \phi_{\mathbf{P}V}^{i,*}(\mathrm{eu})^{(q)}"'] \arrow[d, "\cong"'] & & \mathrm{Fr}^*(V)_{X^k} \arrow[d, "\beta^\vee"] \\
\mathscr{C}_{X^k/\mathbf{P}V}(1) \arrow[r, "\delta"] & \Omega^1_{\mathbf{P}V}(1)|_{X^k} \arrow[r, hook] & V^\vee_{X^k}.
\end{tikzcd}$$

The singular locus of X^k is supported on the degeneracy locus of $\mathscr{C}_{X^k/\mathbf{P}V}(1) \to V^\vee$. Since $\beta^\vee$ is an isomorphism, this is the degeneracy locus of the top map. But this is a Frobenius pullback of the sheaf map from 2.9.3 defining $\mathrm{cyc}_\phi^r : \mathbf{P}V \dashrightarrow \mathbf{G}(r+1,V)$. Since Frobenius pullbacks preserve the support of degeneracy loci, the discussion there implies that the singular locus of X is supported on the union of the Hermitian $(k-1)$-planes in X. ■

2.9.12. Filtration in Hermitian coordinates. — The equations of the X^k are particularly simple upon choosing a basis $V = \langle v_0, \ldots, v_n \rangle$ consisting of Hermitian vectors, see 1.2.13. Fix such a basis and let $(x_0 : \cdots : x_n)$ be the corresponding coordinates on $\mathbf{P}V = \mathbf{P}^n$. Then the Gram matrix $A := (a_{ij})_{i,j=0}^n := \mathrm{Gram}(\beta; v_0, \ldots, v_n)$ of β with respect to such a basis is Hermitian by 1.2.3, meaning $A^\vee = A^{(q)}$. This shows that the endomorphism $\phi_{\mathbf{P}V}$ from 2.9.1 is given in these coordinates simply by

$$\phi_{\mathbf{P}^n} : \mathbf{P}^n \to \mathbf{P}^n \qquad (x_0 : \cdots : x_n) \mapsto (x_0^{q^2} : \cdots : x_n^{q^2}).$$

In particular, the Hermitian subspaces of X coincide with its $\mathbf{F}_{q^2}$-rational ones. The equations of X^r from 2.9.11 are now given by

$$X^r = \bigcap\nolimits_{k=0}^r \mathrm{V}\Big(\sum\nolimits_{i,j=0}^n a_{ij} x_i x_j^{q^{2k+1}+1}\Big) \subset \mathbf{P}^n.$$

When X is given by the Fermat equation, this filtration is seen to coincide with that defined by Lusztig in [Lus76, Definition 2], see also [Rod00, §6.2].

The final piece $X^{\lfloor n/2 \rfloor}$ of the filtration from 2.9.10 essentially consists of the maximal Hermitian isotropic subspaces of (V, β):

2.9.13. Lemma. — *The scheme $X^{\lfloor n/2 \rfloor}$ is supported on the union of the maximal Hermitian subspaces contained in X. Furthermore, the sections*

$$\beta(\phi_{\mathbf{P}V}^{*,k}(\mathrm{eu})^{(q)}, \mathrm{eu}) \colon \mathcal{O}_{\mathbf{P}V} \to \mathcal{O}_{\mathbf{P}V}(q^{2k+1}+1)$$

vanish on $X^{\lfloor n/2 \rfloor}$ for all integers $k \geq 0$.

Proof. — Write $\dim X = n-1$ as $2m$ or $2m+1$ so that a maximal isotropic contained in X is an m-plane. It follows directly from the functorial description of $X^{\lfloor n/2 \rfloor}$ in 2.9.10 and 1.2.20 that it contains the Hermitian m-planes of X. Thus to prove the first statement, it remains to see that a point $x \in X^{\lfloor n/2 \rfloor}$ is contained in a Hermitian m-plane of X. Consider the even- and odd-dimensional cases separately:

- When $n-1 = 2m$, the definition of $X^{\lfloor n/2 \rfloor} = X^m$ together with 1.2.20 imply that x is contained in an m-plane in X, any of which is Hermitian by 2.4.16.
- When $n-1 = 2m+1$, the definition of $X^{\lfloor n/2 \rfloor} = X^{m+1}$ together with the fact that any linear subvariety of X has dimension at most m imply that

  $$\langle x, \phi_X(x), \ldots, \phi_X^m(x), \phi_X^{m+1}(x) \rangle = \langle x, \phi_X(x), \ldots, \phi_X^m(x) \rangle.$$

  Thus, by 1.2.20, x lies in a Hermitian m-plane of X.

For the second statement, the sections $\beta(\phi_{\mathbf{P}V}^{*,k}(\mathrm{eu})^{(q)}, \mathrm{eu})$ vanish on $X^{\lfloor n/2 \rfloor}$ for $0 \leq k \leq n/2$ by 2.9.11, so it remains to show vanishing when $k > n/2$. So consider any $v \in V$ contained in a maximal isotropic Hermitian subspace. Then by 1.2.20,

$$\phi^k(v) = a_0 v + a_1 \phi(v) + \cdots + a_m \phi^m(v) \quad \text{for some } a_0, a_1, \ldots, a_m \in \mathbf{k}.$$

Thus $\beta(\phi^k(v)^{(q)}, v) = \sum_{i=0}^m a_i^q \beta(\phi^i(v), v) = 0$ by 2.9.11. The vanishing of the sections $\beta(\phi_{\mathbf{P}V}^k(\mathrm{eu})^{(q)}, \mathrm{eu})$ now follow from the first part together with the functorial description of $X^{\lfloor m/2 \rfloor}$ from 2.9.10. ■

A finer analysis of the structure of $X^{\lfloor n/2 \rfloor}$ gives a geometric method to count the number of maximal isotropic Hermitian subspaces contained in a smooth q-bic hypersurface X. The following count is classical: see [Seg65, n.32] and [BC66, Theorem 9.2]; see also [Shi01, Corollary 2.22].

2.9.14. Corollary. — *The number of maximal isotropic Hermitian subspaces in a smooth q-bic $(n-1)$-fold X is*

$$\#\mathbf{F}_m(X)_{\mathrm{Herm}} = \begin{cases} \prod_{i=0}^{m}(q^{2i+1}+1) & \text{if } n-1=2m \text{ is even, and} \\ \prod_{i=0}^{m}(q^{2i+3}+1) & \text{if } n-1=2m+1 \text{ is odd.} \end{cases}$$

Proof. — In the even case, 2.9.13 implies that X^m is supported on the union of the maximal Hermitian subspaces in X. By 2.9.11, the singular locus of X^m is supported on the union of the Hermitian $(m-1)$-planes in X; in particular, X^m is reduced. Therefore

$$\#\mathbf{F}_m(X)_{\mathrm{Herm}} = \deg(X_m) = \prod_{i=0}^{m}(q^{2i+1}+1).$$

In the odd case, it follows from 2.9.13 and 2.9.11 that X^{m+1} is the union of the isotropic Hermitian m-planes and, by symmetry, each m-plane appears with the same multiplicity $a > 1$. Proceed by induction on m to show that $a = q+1$, at which point the result follows by using 2.9.11 and counting degrees. When $m = 0$, so X is a smooth q-bic curve and X^1 is supported on its Hermitian points, it follows from 2.4.13 that

$$a = \deg(X^1)/\#X_{\mathrm{Herm}} = q+1.$$

Suppose $m > 1$. For each $i \geq 0$, set

$$X_i := \mathrm{V}(\beta(\phi_{\mathbf{P}V}^{*,i}(\mathrm{eu})^{(q)}, \mathrm{eu})) \subseteq \mathbf{P}V$$

so that each X_i is a q^{2i+1}-bic hypersurface in $\mathbf{P}V$ and $X^k = \bigcap_{i=0}^{k} X_i$ for $0 \leq k \leq m+1$. The computations of 2.9.12 imply that Hermitian points x of X are also Hermitian points of X_i for all $i \geq 0$ and, by 2.2.9, $\mathbf{T}_{X,x} = \mathbf{T}_{X_i,x}$ as hyperplanes in $\mathbf{P}V$. This, together with 2.4.8, implies that $X \cap \mathbf{T}_{X,x}$ is a cone over a q-bic $(n-3)$-fold X', and $X^{m+1} \cap \mathbf{T}_{X,x}$ is a cone over

$$X'^m \cap X'_{m+1} = X'^m$$

where the notation is as above, and where the second equality follows from 2.9.13. Induction now gives that any irreducible component $P \subseteq X^{m+1}$ through x has multiplicity at least $q+1$.

To conclude $a = q+1$, it remains to show that P is scheme-theoretically contained in $\mathbf{T}_{X,x}$. For this, let $y \in P$ be a closed point not lying on any Hermitian $(m-1)$-plane of X. Then y is a smooth point of X^m by 2.9.11. Therefore the nonreduced structure of P at y is scheme-theoretically contained in $\mathbf{T}_{X^m,y} \cap X_{m+1}$. Thus it suffices to show that $\mathbf{T}_{X^m,y} \subset \mathbf{T}_{X,x}$ for all such y. By the computation of 2.2.9, it follows that

$$\mathbf{T}_{X^m,y} = \bigcap\nolimits_{i=0}^{m} \left\{ z \in \mathbf{P}V \;\middle|\; \beta(\phi^i(y)^{(q)}, z) = 0 \right\}.$$

Since $P = \langle y, \phi(y), \ldots, \phi^m(y) \rangle$ by 1.2.20 and since $x \in P$, the equation $z \mapsto \beta(x^{(q)}, z)$ defining $\mathbf{T}_{X,x}$ vanishes on $\mathbf{T}_{X^m,y}$, as desired. ■

For each $0 \leq r < n/2$, set $\mathbf{F}_r(X)_{\mathrm{cyc}} := \mathbf{F}_r(X) \cap \mathbf{G}(r+1, V)_{\mathrm{cyc}}$. The functorial description of X^r from 2.9.10 shows that the rational map from 2.9.3 fits into a commutative diagram

$$\begin{tikzcd} X^r \arrow[r, dashed, "\mathrm{cyc}^r_\phi"'] \arrow[d, hook] & \mathbf{F}_r(X)_{\mathrm{cyc}} \arrow[d, hook] \\ \mathbf{P}V \arrow[r, dashed, "\mathrm{cyc}^r_\phi"] & \mathbf{G}(r+1, V)_{\mathrm{cyc}} \end{tikzcd}$$

and that the diagram is Cartesian upon restricting to the complement of the Hermitian $(r-1)$-planes on the left. Together with 2.9.5, this shows the first two statements of:

2.9.15. Corollary. — *For each $0 \leq r < n/2$, the map $\mathrm{cyc}^r_\phi : X^r \dashrightarrow \mathbf{F}_r(X)_{\mathrm{cyc}}$ is dominant and injective away from the Hermitian r-planes in X^r, and*

$$\dim \mathbf{F}_r(X)_{\mathrm{cyc}} = \begin{cases} n-r-1 & \text{if } r < (n-1)/2, \text{ and} \\ 0 & \text{if } n-1 = 2m \text{ and } r = m. \end{cases}$$

In particular, if $n-1 = 2m$ or $n-1 = 2m+1$, then $\mathbf{F}_m(X)_{\mathrm{cyc}} = \mathbf{F}_m(X)$.

Proof. — Consider the dimension of $\mathbf{F}_r(X)_{\mathrm{cyc}}$. When $r < (n-1)/2$, then the Hermitian r-planes contained in X^r is a proper closed subscheme. Therefore $X^r \dashrightarrow \mathbf{F}_r(X)_{\mathrm{cyc}}$ is dominant and generically finite by the first statement, so

$$\dim \mathbf{F}_r(X)_{\mathrm{cyc}} = \dim X^r = n-r-1$$

by 2.9.11. The remaining case is when $n-1=2m$ and $r=m$. Then, as in 2.9.13, X^m is the union of the isotropic Hermitian m-planes of $\mathbf{P}V$ and the map $X^m \dashrightarrow \mathbf{F}_m(X)_{\mathrm{cyc}}$ collapses each m-plane to a point. In this case, $\dim \mathbf{F}_m(X)_{\mathrm{cyc}} = 0$. Finally, when $r=m$ with $n-1=2m+1$, a comparison with 2.7.16 shows that

$$\dim \mathbf{F}_m(X) = \dim \mathbf{F}_m(X)_{\mathrm{cyc}}.$$

Therefore $\mathbf{F}_m(X)_{\mathrm{cyc}}$ is an irreducible component of $\mathbf{F}_m(X)$. Since the Fano scheme is irreducible, the two schemes coincide. ■

The proof of 2.9.15 also shows that when $0 \leq r < (n-1)/2$, the rational map from 2.9.4 fits into a commutative diagram

$$\begin{tikzcd} \mathbf{F}_r(X)_{\mathrm{cyc}} \arrow[r, dashed, "{[\mathrm{id}\cap\phi^r]}"'] \arrow[d, hook] & X^r \arrow[d, hook] \\ \mathbf{G}(r+1,V)_{\mathrm{cyc}} \arrow[r, dashed, "{[\mathrm{id}\cap\phi^r]}"] & \mathbf{P}V \end{tikzcd}$$

such that the horizontal maps are defined away from the locus parameterizing subspaces which contain a 2-dimensional Hermitian subspace of V. Furthermore, as in 2.9.5, its composite with cyc_ϕ^r from above are the endomorphisms

$$[\mathrm{id}\cap\phi^r]\circ \mathrm{cyc}_\phi^r = \phi_{X^r}^r \quad \text{and} \quad \mathrm{cyc}_\phi^r \circ [\mathrm{id}\cap\phi^r] = \phi^r_{\mathbf{F}_r(X)_{\mathrm{cyc}}}$$

of X^r and $\mathbf{F}_r(X)_{\mathrm{cyc}}$ induced by ϕ, as in 2.9.8. To describe a resolution of these rational maps, with the notation as from 2.9.6, set

$$\tilde{X}^r := X^r \times_{\mathbf{P}V} \Gamma_{\mathrm{cyc}_\phi^r} \times_{\mathbf{G}(r+1,V)} \mathbf{F}_r(X)_{\mathrm{cyc}} = \{\, (x\in P) \mid \langle x, \phi_X(x), \ldots, \phi_X^r(x)\rangle \subset P\,\},$$

$$\tilde{\mathbf{F}}_r(X) := X^r \times_{\mathbf{P}V} \Gamma_{[\mathrm{id}\cap\phi^r]} \times_{\mathbf{G}(r+1,V)} \mathbf{F}_r(X)_{\mathrm{cyc}} = \{\, (x\in P) \mid x \in P\cap\phi_X^r(P)\,\}.$$

Then ϕ_X and $\phi_{\mathbf{F}_r(X)_{\mathrm{cyc}}}$ induce morphisms

$$\phi^r\times\mathrm{id}\colon \tilde{X}^r \to \tilde{\mathbf{F}}_r(X), \qquad (x\in P)\mapsto (\phi^r(x)\in P),$$

$$\mathrm{id}\times\phi^r\colon \tilde{\mathbf{F}}_r(X)\to\tilde{X}^r, \qquad (x\in P)\mapsto (x\in\phi^r(P)).$$

As before, these give resolutions of the rational maps in the following sense:

2.9.16. Proposition. — *There is a commutative diagram of schemes over* $\mathbf{k}$ *given by*

$$\begin{tikzcd} \tilde{X}^r \arrow[r, "\phi^r \times \mathrm{id}"'] \arrow[d, "\mathrm{pr}_1"'] & \tilde{\mathbf{F}}_r(X)_{\mathrm{cyc}} \arrow[r, "\mathrm{id} \times \phi^r"'] \arrow[d, "\mathrm{pr}_2"'] & \tilde{X}^r \arrow[d, "\mathrm{pr}_1"] \\ X^r \arrow[r, dashed, "\mathrm{cyc}^r_\phi"] & \mathbf{F}_r(X)_{\mathrm{cyc}} \arrow[r, dashed, "{[\mathrm{id} \cap \phi^r]}"] & X^r \end{tikzcd}$$

such that

(i) $\mathrm{pr}_1 \colon \tilde{X}^r \to X^r$ *is an isomorphism away from the union of the Hermitian* $(r-1)$*-planes contained in* X*;*

(ii) $\mathrm{pr}_2 \colon \tilde{\mathbf{F}}_r(X) \to \mathbf{F}_r(X)$ *is an isomorphism away from the locus parameterizing isotropic subspaces of* V *which contain a* 2*-dimensional Hermitian subspace;*

(iii) $\phi^r \times \mathrm{id}$ *and* $\mathrm{id} \times \phi^r$ *are finite purely inseparable of degree* $q^{r(r+1)}$ *and* $q^{r(2n-3r-3)}$.

Proof. — Items (i) and (ii) follow from their counterparts in **2.9.7**. For (iii), that they are finite purely inseparable is because they are restrictions of finite purely inseparable morphisms, see **2.9.7**(iii). Since the preimage of $(x \in P)$ under $\phi^r \times \mathrm{id}$ parameterizes deformations of the point x in P; since P is totally isotropic, the computation given in **2.9.7**(iii) still applies to show that $\deg(\phi^r \times \mathrm{id}) = q^{r(r+1)}$. Since the degree of $(\mathrm{id} \times \phi^r) \circ (\phi^r \times \mathrm{id})$ is $q^{2r(n-1-r)}$, this implies $\deg(\mathrm{id} \times \phi^r) = q^{r(2n-3r-3)}$. ∎

Chapter 3

q-bic Points, Curves, and Surfaces

Points, curves, and surfaces amongst q-bic hypersurfaces offer a variety of simple examples with which to illustrate the general theory. The Sections that follow step through the projective equivalence classes of low-dimensional q-bics which are not cones, and discuss particular features of their geometry. In most cases, automorphism group schemes are explicitly presented; schemes of cone points are often discussed, and are used to illustrate how various types deform into one another; and, for q-bic surfaces, schemes of lines are described.

Throughout this Chapter, $\mathbf{k}$ is an algebraically closed field of characteristic $p > 0$.

3.1. q-bic points

Much of the special projective geometry of q-bic hypersurfaces is related to the fact that q-bic hypersurfaces of dimension 0, or *q-bic points*, come in only three shapes. Indeed, the classification of q-bic forms 1.4.1 shows that:

3.1.1. Proposition. — *Let (V, β) be a nonzero q-bic form of dimension 2 and let $X \subset \mathbf{P}V$ be the associated q-bic hypersurface of dimension 0. Then either*

$$\operatorname{type}(\beta) = \begin{cases} \mathbf{1}^{\oplus 2} \\ \mathbf{N}_2 \\ \mathbf{1} \oplus \mathbf{0} \end{cases} \quad \textit{and} \quad X \cong \begin{cases} \mathrm{V}(x_0^{q+1} + x_1^{q+1}) \\ \mathrm{V}(x_0^q x_1) \\ \mathrm{V}(x_0^{q+1}) \end{cases}$$

for some choice of coordinates $(x_0 : x_1)$ on $\mathbf{P}V = \mathbf{P}^1$. ∎

Notably, this implies that any multiple point in a scheme of q-bic points must appear with multiplicity at least q. The significance of this observation arises in conjunction with the fact 2.1.4 that linear sections of q-bic hypersurfaces are q-bics: it implies that tangent lines to q-bic hypersurfaces always have contact order at least q at the point of tangency. Consequently, singular points of q-bics are points of multiplicity at least q.

The shapes of the q-bic points make it clear how they may specialize to one another in families. To make this precise, consider the parameter space

$$q\text{-}\mathbf{bics}(V) := \mathbf{A}(\mathrm{Fr}^*(V)^\vee \otimes V^\vee) \cong \mathbf{A}^4$$

of q-bic forms on the 2-dimensional vector space V, as in 1.4.4. The closure relations and strata dimensions of the type stratification from 1.4.6 are as follows:

3.1.2. Proposition. — *The Hasse diagram for the type stratification of q-$\mathbf{bics}(V)$ is*

$$\mathbf{1}^{\oplus 2} \rightsquigarrow \mathbf{N}_2 \rightsquigarrow \mathbf{0} \oplus \mathbf{1} \rightsquigarrow \mathbf{0}^{\oplus 2}$$

and the strata dimension are given by

λ	$\mathbf{1}^{\oplus 2}$	$\mathbf{N}_2$	$\mathbf{0} \oplus \mathbf{1}$	$\mathbf{0}^{\oplus 2}$
$\dim q\text{-}\mathbf{bics}(V)_\lambda$	4	3	2	0

Proof. — The first two closure relations are because $\mathbf{N}_2$ is the generic type in corank 1, see 1.4.8. Since everything specializes to the zero form, the remaining closure relation is clear. The strata dimensions are computed using 1.4.7; see 3.2.2 and 3.3.1 for concrete computations of the automorphism group schemes in the first two types, and apply 1.3.8 for type $\mathbf{0} \oplus \mathbf{1}$. ∎

3.1.3. Special families of q-bic points. — Flat families of q-bic points for which the general member has type $\mathbf{1}^{\oplus 2}$ and special members have type $\mathbf{N}_2$ are thought of as degenerations. It is clear from 3.1.2 that there are many such families, even over 1-dimensional bases. A geometrically simple class of such degenerations involves fixing one point and letting the remaining q come together. For example, some degenerations of this form over $\mathbf{A}^1 = \mathrm{Spec}(\mathbf{k}[t])$ can be given by a q-bic form over $\mathbf{k}[t]$ with Gram matrix

$$\begin{pmatrix} 0 & 1 \\ t & f \end{pmatrix} \colon \mathrm{Fr}^*(V[t]) \otimes_{\mathbf{k}[t]} V[t] \to \mathbf{k}[t]$$

where $V[t] := V \otimes_{\mathbf{k}} \mathbf{k}[t]$, $f \in \mathbf{k}[t]$, and the fixed point corresponds to the subspace spanned by $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$.

The family in which $f = 0$ is particularly special, and arises geometrically by fixing an additional point amongst the original set of q-bic points. Algebraically, this is the following:

3.1.4. Proposition. — *Let (V, β) be a q-bic form of type $\mathbf{1}^{\oplus 2}$. For every decomposition $V = L_- \oplus L_+$ into two isotropic subspaces of dimension 1, there exists a unique q-bic form*

$$\beta^{L_\pm} \colon \mathrm{Fr}^*(V[t]) \otimes_{\mathbf{k}[t]} V[t] \to \mathbf{k}[t]$$

over $\mathbf{k}[t]$ such that

(i) *$L_-[t]$ and $L_+[t]$ are isotropic for $\beta^{L_\pm}$;*
(ii) *the induced pairing $\mathrm{Fr}^*(L_-[t]) \otimes_{\mathbf{k}[t]} L_+[t] \to \mathbf{k}[t]$ is perfect;*
(iii) *the induced pairing $\mathrm{Fr}^*(L_+[t]) \otimes_{\mathbf{k}[t]} L_-[t] \to \mathbf{k}[t]$ has image the ideal (t); and*
(iv) *$\beta^{L_\pm}|_{t=0}$ is of type $\mathbf{N}_2$ and $\beta^{L_\pm}|_{t=1} = \beta$.*

Proof. — Choose a basis $L_\pm = \langle v_\pm \rangle$ so that β has Gram matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$: see 3.2.1 below. Then the q-bic form on $V[t]$ with Gram matrix

$$\mathrm{Gram}(\beta^{L_\pm}; v_- \otimes 1, v_+ \otimes 1) = \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}$$

is the unique form satisfying the stated properties. ■

3.1.5. $\mathbf{G}_m$-action. — The q-bic $(V[t], \beta^{L_\pm})$ over $\mathbf{k}[t]$ in 3.1.4 defines a scheme

$$\mathcal{X} \subset \mathbf{P}^1 \times \mathbf{A}^1 := \mathbf{P}V \times \mathrm{Spec}(\mathbf{k}[t])$$

which is viewed as a family of q-bic points over $\mathbf{A}^1$. In the coordinates $(x_- : x_+)$ of $\mathbf{P}V = \mathbf{P}^1$ dual to the basis chosen above, $\mathcal{X} = \mathrm{V}(x_-^q x_+ + t x_- x_+^q)$. This degeneration is special because the total space $\mathscr{X}$ admits a $\mathbf{G}_m$-action that is compatible with a $\mathbf{G}_m$-action on the base $\mathbf{A}^1$. Namely, let $\mathbf{G}_m$ act linearly on $\mathbf{P}^1 \times \mathbf{A}^1$ with

$$\mathrm{wt}(L_-) = -1, \quad \mathrm{wt}(L_+) = q, \quad \mathrm{wt}(t) = q^2 - 1.$$

In terms of the coordinates $(x_- : x_+)$, $\lambda \in \mathbf{G}_m$ acts by

$$\lambda \cdot \big((x_- : x_+), t\big) = \big((\lambda x_- : \lambda^{-q} x_+), \lambda^{q^2-1} t\big).$$

Then $\mathscr{X}$ is stable for the $\mathbf{G}_m$-action. In fact, slightly more is true:

3.1.6. Lemma. — *The q-bic form $\beta^{L_\pm}$ is invariant for the $\mathbf{G}_m$-action of 3.1.5.*

Proof. — In terms of the coordinates chosen in 3.1.4, the action of $\lambda \in \mathbf{G}_m$ on $\beta^{L_\pm}$ is

$$\lambda \cdot \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix} = \begin{pmatrix} \lambda^q & 0 \\ 0 & \lambda^{-q^2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \lambda^{q^2-1} t & 0 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-q} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}. \qquad \blacksquare$$

3.1.7. — The automorphism group scheme $\mathbf{Aut}(V[t], \beta^{L_\pm})$, see 1.3.1, of the q-bic form over $\mathbf{k}[t]$ from 3.1.4 has two 1-dimensional irreducible components. Let

$$\mathscr{G} \cong \left\{ \begin{pmatrix} \lambda & \epsilon \\ 0 & \lambda^{-q} \end{pmatrix} \in \mathbf{GL}_{2,\mathbf{A}_1} \;\middle|\; \lambda \in \boldsymbol{\mu}_{q^2-1}, \epsilon^q + t\lambda^{q-1}\epsilon = 0 \right\}$$

be the component that dominates $\mathbf{A}^1$; compare with 3.2.4, 3.3.1, and 1.3.2. The linear action of $\mathbf{G}_m$ on $\mathbf{P}^1 \times \mathbf{A}^1$ induces one on $\mathscr{G}$ given by $\lambda \cdot \left(\begin{smallmatrix} a & b \\ 0 & a^{-q} \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & \lambda^{q+1} b \\ 0 & a^{-q} \end{smallmatrix}\right)$. The action of $\mathscr{G}$ is compatible with this $\mathbf{G}_m$-action:

3.1.8. Lemma. — *The action map $\mathscr{G} \times_{\mathbf{A}^1} \mathscr{X} \to \mathscr{X}$ is $\mathbf{G}_m$-equivariant over $\mathbf{A}^1$.*

Proof. — Since $\mathbf{G}_m$ acts linearly on $\mathbf{P}^1 \times \mathbf{A}^1$ the action map

$$\mathbf{GL}_{2,\mathbf{A}^1} \times_{\mathbf{A}^1} \mathbf{P}^1_{\mathbf{A}^1} \to \mathbf{P}^1_{\mathbf{A}^1}$$

is $\mathbf{G}_m$-equivariant. The claim now follows as the action map $\mathscr{G} \times_{\mathbf{A}^1} \mathscr{X} \to \mathscr{X}$ is simply the restriction of linear action map from the ambient projective space. $\blacksquare$

3.2. Type $\mathbf{1}^{\oplus 2}$

Let X be q-bic points associated with a q-bic form (V,β) of type $\mathbf{1}^{\oplus 2}$. Such X is as simple as possible: it is a set of $q+1$ reduced points on the projective line. Of course, not any set of $q+1$ points on the line determines a scheme of q-bic points; the points must be arranged in a particularly symmetric way. One way to make sense of this is the following, which dictates how, upon choosing two points to serve as 0 and ∞ for $\mathbf{P}V = \mathbf{P}^1$, the remaining $q-1$ points must be distributed:

3.2.1. Lemma. — *For any decomposition $V = L_- \oplus L_+$ into a pair of isotropic 1-dimensional subspaces and any $a, b \in \mathbf{k}^\times$, there exists a basis $L_\pm = \langle v_\pm \rangle$ such that*

$$\mathrm{Gram}(\beta; v_-, v_+) = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}.$$

Proof. — Begin with any basis $L_\pm = \langle v'_\pm \rangle$. Then the associated Gram matrix is $\left(\begin{smallmatrix} 0 & a' \\ b' & 0 \end{smallmatrix}\right)$ for some $a', b' \in \mathbf{k}^\times$. Scaling by $\lambda_\pm \in \mathbf{k}^\times$, affects the Gram matrix as

$$\mathrm{Gram}(\beta; \lambda_- v'_-, \lambda_+ v'_+) = \begin{pmatrix} 0 & \lambda_+^q \lambda_- a' \\ \lambda_+ \lambda_-^q b' & 0 \end{pmatrix}.$$

Take any solution to $\lambda_+^{q^2-1} = ab'/a'b$, let $\lambda_- := a/a'\lambda_+^q$, and set $v_\pm := \lambda_\pm v'_\pm$. ■

Taking $a = -b = 1$ gives coordinates $(x_0 : x_1)$ so that $X = \mathrm{V}(x_0^q x_1 - x_0 x_1^q)$ is the set of $\mathbf{F}_q$ points of $\mathbf{P}V = \mathbf{P}^1$. This gives a pleasant presentation for its group of linear automorphisms of X:

3.2.2. Proposition. — *Let (V,β) be a q-bic form of type $\mathbf{1}^{\oplus 2}$. Then its automorphism group scheme admits the presentation:*

$$\mathbf{Aut}(V,\beta) \cong \left\{ \begin{pmatrix} \lambda a & \lambda^{-q} b \\ \lambda c & \lambda^{-q} d \end{pmatrix} \,\middle|\, \lambda \in \boldsymbol{\mu}_{q^2-1}, a, b, c, d \in \mathbf{F}_q, ad - bc = 1 \right\}.$$

Proof. — Choose a basis of V as in 3.2.1 so that $a = -b = 1$. Then $\mathbf{Aut}(V,\beta)$ is isomorphic to the closed subgroup scheme of $\mathbf{GL}_2$ consisting of matrices satisfying

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} a_0^q & c_0^q \\ b_0^q & d_0^q \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} = \begin{pmatrix} a_0^q c_0 - a_0 c_0^q & a_0^q d_0 - b_0 c_0^q \\ b_0^q c_0 - a_0 d_0^q & b_0^q d_0 - b_0 d_0^q \end{pmatrix}.$$

Since the columns of the matrices are linearly independent, $a_0^q c_0 - a_0 c_0^q = 0$ implies $(a_0 : c_0) \in \mathbf{P}^1(\mathbf{F}_q)$. Similarly, $(b_0 : d_0) \in \mathbf{P}^1(\mathbf{F}_q)$. Therefore

$$\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} = \begin{pmatrix} \lambda a & \mu b' \\ \lambda c & \mu d' \end{pmatrix} \quad \text{for some } a, b', c, d' \in \mathbf{F}_q \text{ and } \lambda, \mu \in \mathbf{G}_m.$$

Since elements of $\mathbf{F}_q$ are fixed under q-powers, the remaining equations may be written $\lambda^q \mu \Delta = \lambda \mu^q \Delta = 1$ where $\Delta := ad' - b'c$. Thus $\mu = 1/\lambda^q \Delta$ and, since $\Delta^q = \Delta$, $\lambda^{q^2-1} = 1$. Setting $b := b'/\Delta$ and $d := d'/\Delta$ gives the presentation. ■

The presentation can be phrased more invariantly as follows:

3.2.3. Corollary. — *There is a short exact sequence of groups*

$$1 \to \boldsymbol{\mu}_{q-1} \to \boldsymbol{\mu}_{q^2-1} \ltimes \mathbf{SL}_2(\mathbf{F}_q) \to \mathbf{Aut}(V,\beta) \to 1$$

where the action of $\boldsymbol{\mu}_{q^2-1}$ on $\mathbf{SL}_2(\mathbf{F}_q)$ in the product and the inclusion of $\boldsymbol{\mu}_{q-1}$ are

$$\lambda \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} a & \lambda^{q+1} b \\ \lambda^{-q-1} c & d \end{pmatrix} \quad \textit{and} \quad \zeta \mapsto \left(\zeta, \begin{pmatrix} \zeta^{-1} & 0 \\ 0 & \zeta \end{pmatrix}\right).$$

Proof. — The presentation of 3.2.2 means that the map of sets

$$\varphi \colon \boldsymbol{\mu}_{q^2-1} \ltimes \mathbf{SL}_2(\mathbf{F}_q) \to \mathbf{Aut}(V,\beta), \quad \left(\lambda, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) \mapsto \left(\begin{smallmatrix} \lambda a & \lambda^{-q} b \\ \lambda c & \lambda^{-q} d \end{smallmatrix}\right)$$

is a surjection. It is a homomorphism: write

$$\left(\begin{smallmatrix} \lambda_1 a_1 & \lambda_1^{-q} b_1 \\ \lambda_1 c_1 & \lambda_1^{-q} d_1 \end{smallmatrix}\right)\left(\begin{smallmatrix} \lambda_2 a_2 & \lambda_2^{-q} b_2 \\ \lambda_2 c_2 & \lambda_2^{-q} d_2 \end{smallmatrix}\right) = \varphi\left(\lambda_1 \lambda_2, \left(\begin{smallmatrix} a_1 a_2 + \lambda_1^{-q-1} b_1 c_2 & \lambda_1^{q+1} a_1 b_2 + b_1 d_2 \\ c_1 a_2 + \lambda_1^{-q-1} d_1 c_2 & \lambda_1^{q+1} c_1 a_2 + d_1 d_2 \end{smallmatrix}\right)\right)$$

and observe that the matrix on the right is the product

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} a_2 & \lambda_1^{q+1} b_2 \\ \lambda_1^{-q-1} c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\left(\lambda_1 \cdot \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right).$$

The kernel of φ consists of pairs $\left(\lambda, \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)\right)$ such that $\lambda a = \lambda^{-q} d = 1$. Thus $\lambda = a^{-1}$, so $\lambda \in \mathbf{F}_q^\times$ and thus $d = \lambda = a^{-1}$. Identifying $\mathbf{F}_q^\times$ with $\boldsymbol{\mu}_{q-1}$ gives the statement. ∎

An interesting subgroup scheme of $\mathbf{Aut}(V,\beta)$ is the stabilizer of a given isotropic line; in other words, these are the linear automorphisms fixing a chosen point of X.

3.2.4. Lemma. — *Let $L \subset V$ be a 1-dimensional isotropic subspace. Then*

$$\mathbf{Aut}(L \subset V, \beta) := \{\, g \in \mathbf{Aut}(V,\beta) \mid g \cdot L = L \,\} \cong \boldsymbol{\mu}_{q^2-1} \ltimes \mathbf{F}_q.$$

Proof. — Choose a basis as in the computation of 3.2.2 such that, in addition, L is the span of $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. Then by inspection,

$$\mathbf{Aut}(L \subset V, \beta) = \left\{ \left(\begin{smallmatrix} \lambda a & \lambda^{-q} b \\ 0 & \lambda^{-q} a^{-1} \end{smallmatrix}\right) \,\middle|\, \lambda \in \boldsymbol{\mu}_{q^2-1}, a \in \mathbf{F}_q^\times, b \in \mathbf{F}_q \right\}.$$

Since $a^q = a$, setting $\lambda \mapsto a\lambda$ gives the isomorphism with $\boldsymbol{\mu}_{q^2-1} \ltimes \mathbf{F}_q$. ∎

3.3. Type $\mathbf{N}_2$

When q of the points of a reduced scheme of q-bic points collide, the result is one of type $\mathbf{N}_2$. Such a scheme X is supported on two points, corresponding to the kernels of the underlying q-bic form (V,β):

$$L_- := \mathrm{Fr}^*(V)^\perp \quad \text{and} \quad L_+ := \mathrm{Fr}^{-1}(V^\perp).$$

Since $\mathbf{k}$ is perfect, these spaces span V: see 1.3.6 and also 1.4.3. Then by 2.2.7,

- L_- underlies the reduced point of X, and
- L_+ underlies the q-fold point of X.

Concretely, choose basis vectors $L_\pm = \langle v_\pm \rangle$ such that $\beta(v_-^{(q)}, v_+) = 1$ and let $x_\pm \in L_\pm^\vee$ be the dual coordinate. This choice of coordinates on $(x_- : x_+)$ on $\mathbf{P}V = \mathbf{P}^1$ gives the equation $X = \mathrm{V}(x_-^q x_+)$ and identifies the points above directly. In these coordinates, the automorphism group scheme of (V,β) admits the following simple presentation:

3.3.1. Proposition. — *Let (V,β) be a q-bic form of type $\mathbf{N}_2$. Then its automorphism scheme is isomorphic to the 1-dimensional closed subscheme of $\mathbf{GL}_2$ given by*

$$\mathbf{Aut}(V,\beta) \cong \left\{ \begin{pmatrix} \lambda & \epsilon \\ 0 & \lambda^{-q} \end{pmatrix} \;\middle|\; \lambda \in \mathbf{G}_m, \epsilon \in \boldsymbol{\alpha}_q \right\}.$$

Proof. — This is a special case of 1.3.7. For a direct computation, use the coordinates as above, and note that any automorphism preserves L_- and $\mathrm{Fr}^*(L_+)$, so $\mathbf{Aut}(V,\beta)$ is isomorphic to the closed subgroup scheme of $\mathbf{GL}_2$ consisting of matrices satisfying

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \lambda^q & 0 \\ 0 & \mu^q \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda & \epsilon \\ 0 & \mu \end{pmatrix} = \begin{pmatrix} 0 & \lambda^q \mu \\ 0 & 0 \end{pmatrix}$$

with $\epsilon^q = 0$. Therefore $\mu = \lambda^{-q}$ as required. ■

3.4. q-bic curves

Plane curves that are q-bic hypersurfaces are *q-bic curves*. The classification of q-bic forms in 1.4.1 shows that, there are three isomorphism classes of q-bic curves which are not cones; see also [HH16, p.213] for a direct computation.

3.4.1. Proposition. — *Let X be the q-bic curve associated with a nonzero q-bic form (V,β) of dimension 3. Then either X is a cone over q-bic points or*

$$\mathrm{type}(\beta) = \begin{cases} \mathbf{1}^{\oplus 3} \\ \mathbf{N}_2 \oplus \mathbf{1} \\ \mathbf{N}_3 \end{cases} \quad \textit{and} \quad X \cong \begin{cases} \mathrm{V}(x_0^{q+1} + x_1^{q+1} + x_2^{q+1}), \textit{ or} \\ \mathrm{V}(x_0^q x_1 + x_2^{q+1}), \textit{ or} \\ \mathrm{V}(x_0^q x_1 + x_1^q x_2), \end{cases}$$

for some choice of coordinates $(x_0 : x_1 : x_2)$ on $\mathbf{P}V = \mathbf{P}^2$. ■

Either by direct inspection, or else by considering linear projection in view of the shapes of q-bic points from 3.1.1, it follows that:

3.4.2. Lemma. — *Reduced irreducible components of singular q-bic curves are rational.*

Proof. — Looking at the classification given in 3.4.1, it suffices to observe that

$$\mathbf{P}^1 \to \mathrm{V}(x_0^d + x_1^{d-1}x_2), \qquad (t_0 : t_1) \mapsto (t_0^{d-1}t_1 : t_0^d : -t_1^d)$$

is the normalization onto the irreducible degree $d = q+1$ curve of type $\mathbf{N}_2 \oplus \mathbf{1}$, and that onto the degree $d = q$ irreducible component of the curve of type $\mathbf{N}_3$. ∎

As with q-bic points in 3.1.2, the type stratification on the 9-dimensional affine parameter space q-$\mathbf{bics}(V)$, see 1.4.4 and 1.4.6, has a totally ordered Hasse diagram:

3.4.3. Proposition. — *The Hasse diagram for the type stratification of q-$\mathbf{bics}(V)$ is*

$$\mathbf{1}^{\oplus 3} \rightsquigarrow \mathbf{N}_2 \oplus \mathbf{1} \rightsquigarrow \mathbf{N}_3 \rightsquigarrow \mathbf{0} \oplus \mathbf{1}^{\oplus 2} \rightsquigarrow \mathbf{0} \oplus \mathbf{N}_2 \rightsquigarrow \mathbf{0}^{\oplus 2} \oplus \mathbf{1} \rightsquigarrow \mathbf{0}^{\oplus 3}$$

and the strata dimension are given by

λ	$\mathbf{1}^{\oplus 3}$	$\mathbf{N}_2 \oplus \mathbf{1}$	$\mathbf{N}_3$	$\mathbf{0} \oplus \mathbf{1}^{\oplus 2}$	$\mathbf{0} \oplus \mathbf{N}_2$	$\mathbf{0}^{\oplus 2} \oplus \mathbf{1}$	$\mathbf{0}^{\oplus 3}$
$\dim q\text{-}\mathbf{bics}(V)_\lambda$	9	8	7	6	5	3	0

Proof. — The relations $\mathbf{1}^{\oplus 3} \rightsquigarrow \mathbf{N}_2 \oplus \mathbf{1} \rightsquigarrow \mathbf{N}_3$ follow because $\mathbf{N}_2 \oplus \mathbf{1}$ is the general corank 1 form, see 1.4.8. The relation $\mathbf{N}_3 \rightsquigarrow \mathbf{0} \oplus \mathbf{1}^{\oplus 2}$ can be seen geometrically via plane sections of a smooth q-bic surface X: Fix one of the finitely many lines $\ell \subset X$, see 2.7.16. Then a general plane section of X containing ℓ must be a q-bic curve of type $\mathbf{N}_3$. But by 2.4.8, the tangent plane section of X at a Hermitian point contained in ℓ is a q-bic curve of type $\mathbf{0} \oplus \mathbf{1}^{\oplus 2}$, as required. The remaining specialization relations are now deduced from those of q-bic points, see 3.1.2.

The strata dimension are computed via 1.4.7; see 3.6.1 and 3.7.1 for the automorphisms in the second two types, and apply 1.3.8 and 3.1.2 for the cone types. ∎

3.5. Type $\mathbf{1}^{\oplus 3}$

Smooth q-bic curves X are well-studied for many reasons: see the bibliographic comments in the Introduction. Some familiar equations for such a curve include:

$$X \cong \begin{cases} \mathrm{V}(x_0^{q+1} + x_1^{q+1} + x_2^{q+1}) & \text{the Fermat curve,} \\ \mathrm{V}(x_0^q x_1 + x_0 x_1^q - x_2^{q+1}) & \text{the Hermitian curve, and} \\ \mathrm{V}(x_0^q x_1 - x_0 x_1^q - x_2^{q+1}) & \text{the Drinfeld–Deligne–Lusztig curve.} \end{cases}$$

Potential source of linguistic confusion: *Hermitian curve* refers to the specific equation given here, whereas *Hermitian q-bic curve* refers to any q-bic curve given with a choice of Hermitian coordinates as in 1.2.11 and 2.9.12. Thankfully, the Hermitian curve is a Hermitian q-bic curve; so is the Fermat curve, but *not* the Drinfeld–Deligne–Lusztig curve.

3.5.1. Tangents. — Classical algebraic geometers noticed smooth q-bic curves because the Gauss map of such a curve is purely inseparable of degree q. This means that tangent lines to X are exactly q-fold tangent at the general point of X. From the perspective here, this is a feature arising from the shapes of q-bic points, see 3.1.1 and the comments that follow.

One question that remains, however, is where the order of tangency jumps to $q+1$. The following shows that these are precisely the Hermitian points of X, as defined in 2.1.5. Moreover, it gives a geometric description of the canonical endomorphism $\phi_X : X \to X$ constructed in 2.9.8.

3.5.2. Lemma. — *Let $x \in X$ be a closed point. Then*

$$\mathrm{mult}_x(X \cap \mathbf{T}_{X,x}) = \begin{cases} q & \textit{if } x \textit{ is not a Hermitian point, and} \\ q+1 & \textit{if } x \textit{ is a Hermitian point.} \end{cases}$$

The endomorphism $\phi_X : X \to X$ sends a point x to its residual intersection point $X \cap \mathbf{T}_{X,x} - qx$ of X and the tangent line at x.

Proof. — By 2.2.13, $X \cap \mathbf{T}_{X,x}$ is the scheme of q-bic points defined by the fibre at x of the tangent form $\beta_{\tan} : \mathrm{Fr}^*(\mathcal{T}_X^{\mathrm{e}}) \otimes \mathcal{T}_X^{\mathrm{e}} \to \mathcal{O}_X$. This form is everywhere of corank 1 by 2.2.14, so the classification of q-bic points 3.1.1 implies $\mathrm{mult}_x(X \cap \mathbf{T}_{X,x})$ is either q or $q+1$, and the latter occurs when $\beta_{\tan,x}$ is of type $\mathbf{0} \oplus \mathbf{1}$. By the exact sequence

$$0 \to \mathrm{Fr}^*(\mathscr{N}_{X/\mathbf{P}V}(-1))^\vee \xrightarrow{\phi_X} \mathscr{T}_X^{\mathrm{e}} \xrightarrow{\beta_{\tan}} \mathrm{Fr}^*(\mathscr{T}_X^{\mathrm{e}})^\vee \xrightarrow{\mathrm{eu}^{(q),\vee}} \mathrm{Fr}^*(\mathscr{O}_X(-1))^\vee \to 0$$

from 2.2.14, $\beta_{\tan}$ has type $\mathbf{0} \oplus \mathbf{1}$ along the locus where the image of the sheaf map ϕ_X coincides with $\mathrm{eu}\colon \mathscr{O}_X(-1) \to \mathscr{T}_X^e$. Exactness means this is the vanishing locus of

$$\beta \circ \mathrm{eu}\colon \mathscr{O}_X(-1) \to \mathrm{Fr}^*(\mathscr{T}_X(-1))^\vee = \ker\big(\mathrm{eu}^{(q),\vee}\colon \mathrm{Fr}^*(\mathscr{T}_X^{\mathrm{e}})^\vee \to \mathrm{Fr}^*(\mathscr{O}_X(-1))^\vee\big).$$

This is dual to the section defining the scheme of Hermitian points, see 2.4.10 and 2.4.8, thereby yielding the first statement. The second statement follows from the discussion of 2.2.15 and 2.9.9. ■

3.5.3. Hermitian points. — By 2.4.13, X has q^3+1 Hermitian points, the divisor of which is defined by the vanishing of the morphism

$$\mathrm{eu}^\vee \circ \beta^\vee\colon \mathrm{Fr}^*(\mathscr{T}_X(-1)) \to \mathscr{O}_X(1)$$

see 2.4.10. Since $\mathscr{T}_X(-1) \cong \mathscr{O}_X(-q+1)$, the divisor of Hermitian points lies in the linear system $|\mathscr{O}_X(q^2-q+1)|$; in particular, the scheme of Hermitian points of X is a complete intersection in $\mathbf{P}V$. Compare this to the fact that the set of $\mathbf{F}_q$-rational points of $\mathbf{P}V$ is a complete intersection. The following gives an explicit lift

$$\tilde{\phi}_X \in \mathrm{H}^0(\mathbf{P}V, \mathscr{O}_{\mathbf{P}V}(q^2-q+1))$$

of ϕ_X, providing explicit equations for the scheme of Hermitian points of X in $\mathbf{P}V$.

3.5.4. Lemma. — *Let $(x_0 : x_1 : x_2)$ be coordinates so that $X = \mathrm{V}(x_0^q x_1 + x_0 x_1^q - x_2^{q+1})$. Then a lift of ϕ_X is given by*

$$\tilde{\phi}_X := \frac{x_0^{q^2} x_1 - x_0 x_1^{q^2}}{x_0^q x_1 + x_0 x_1^q} x_2 = \big(x_0^{q(q-1)} - x_0^{(q-1)(q-1)} x_1^{q-1} + \cdots - x_1^{q(q-1)}\big) x_2.$$

Proof. — As explained in 2.9.12, in these coordinates, the Hermitian points of X are its $\mathbf{F}_{q^2}$-rational points. Thus it suffices to show that each point of intersection between X with the vanishing locus $D := \mathrm{V}(\tilde{\phi}_X)$ of the putative lift is a $\mathbf{F}_{q^2}$-point of X. The irreducible components of D are lines of the form $\mathrm{V}(x_2)$ or else $\mathrm{V}(x_0 - \alpha x_1)$

where $\alpha \in \mathbf{F}_{q^2}$ is such that $\alpha^q + \alpha \neq 0$. The first line intersects X at the $q+1$ points given by $(x_0 : x_1 : 0)$ where $x_0, x_1 \in \mathbf{F}_q$. A line in the second set intersect X at

$$X \cap \mathrm{V}(x_0 - \alpha x_1) = \{ (\alpha x_1 : x_1 : x_2) \mid (x_2/x_1)^{q+1} = \alpha^q + \alpha \}.$$

Setting $z := x_2/x_1$ for such an intersection point, it follows that

$$z^{q^2} = z \cdot z^{(q+1)(q-1)} = z \cdot \frac{(\alpha^q + \alpha)^q}{\alpha^q + \alpha} = z \cdot \frac{\alpha + \alpha^q}{\alpha^q + \alpha} = z$$

so that $z \in \mathbf{F}_{q^2}$. ■

3.6. Type $\mathbf{N}_2 \oplus \mathbf{1}$

A general singular q-bic curve X is one of type $\mathbf{N}_2 \oplus \mathbf{1}$. It is irreducible, has a unique unibranch singularity, and, as shown in 3.4.2, has normalization given by the projective line. Its group of linear automorphisms is computed as a special case of 1.3.7, and can be made explicit as follows:

3.6.1. Proposition. — *Let (V, β) be a q-bic form of type $\mathbf{N}_2 \oplus \mathbf{1}$. Then $\mathbf{Aut}(V, \beta)$ is isomorphic to the 1-dimensional closed subgroup scheme of $\mathbf{GL}_3$ given by*

$$\begin{pmatrix} \lambda & \epsilon_1 & \epsilon_2 \\ 0 & \lambda^{-q} & 0 \\ 0 & -\zeta\epsilon_2^q/\lambda^q & \zeta \end{pmatrix}$$

where $\lambda \in \mathbf{G}_m$, $\epsilon_1 \in \boldsymbol{\alpha}_q$, $\epsilon_2 \in \boldsymbol{\alpha}_{q^2}$, and $\zeta \in \boldsymbol{\mu}_{q+1}$.

Proof. — A direct computation here is short. Choose a basis $V = \langle e_0, e_1, e_2 \rangle$ so that $\mathrm{Gram}(\beta; e_0, e_1, e_2) = \mathbf{N}_2 \oplus \mathbf{1}$. The $\perp$-filtration of 1.1.16 is $\langle e_0 \rangle \subset \langle e_0, e_2 \rangle$, and the first piece of the $\mathrm{Fr}^*(\perp)$-filtration of 1.1.20 $\langle e_1^{(q)} \rangle$. Since the automorphism group scheme preserves these filtrations, see 1.3.2, it is isomorphic to the closed subgroup scheme of $\mathbf{GL}_3$ consisting of matrices which satisfy

$$\begin{pmatrix} a_{00}^q & 0 & 0 \\ 0 & a_{11}^q & 0 \\ a_{02}^q & 0 & a_{22}^q \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ 0 & a_{11} & 0 \\ 0 & a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $a_{01}^q = a_{21}^q = 0$. Expanding gives the three equations

$$a_{00}^q a_{11} = a_{22}^{q+1} = 1 \quad \text{and} \quad a_{02}^q a_{11} + a_{22}^q a_{21} = 0$$

The first two equations determine the diagonal entries. Setting $\lambda := a_{00}$ and $\zeta := a_{22}$, the third equation implies $a_{21} = -\zeta a_{02}^q/\lambda^q$. Combined with the fact that $a_{21}^q = 0$, this implies $a_{02}^{q^2} = 0$. Setting $\epsilon_1 := a_{01}$ and $\epsilon_2 := a_{02}$ finishes the computation. ■

3.6.2. Remark. — The morphism $\mathbf{Aut}(V,\beta) \to \mathbf{Aut}(X)$ from 2.3.1 giving the linear automorphisms of X is not surjective here: since X is normalized by $\mathbf{P}^1$, any automorphism of $\mathbf{P}^1$ infinitesimally preserving the preimage ∞ of the cusp will descend to an automorphism of X. In particular, $\mathbf{Aut}(X)$ contains the 2-dimensional group of automorphisms of $\mathbf{A}^1 = \mathbf{P}^1 \setminus \infty$, whereas $\mathbf{Aut}(V,\beta)$ is only 1-dimensional by 3.6.1. An explicit computation of $\mathbf{Aut}(X)$ in the case $q = 2$ can be found [BM76, Proposition 6]; the method may be adapted to the case of general q.

3.6.3. Cone points. — Consider the cone points of X as defined in 2.4.5. By 1.2.5 and 1.2.10(iii), the underlying q-bic form does not have any nontrivial Hermitian points. Therefore, by 2.4.7, the cone points of X are precisely its singular point $x_+ := \mathbf{P}\,\mathrm{Fr}^{-1}(V^\perp)$ and the special point $x_- := \mathbf{P}\,\mathrm{Fr}^*(V)^\perp$.

The scheme X_{cone} of cone points from 2.4.9 is more interesting. Either by direct computation or else by analyzing the argument of 2.4.12, it follows that x_- is a reduced point of X_{cone}. Then by the degree computation of 2.4.11, x_+ appears with multiplicity q^3. This gives some indication as to how a smooth q-bic curve degenerates to one of type $\mathbf{N}_2 \oplus \mathbf{1}$: fix one of the q^3+1 Hermitian points, and let the remaining q^3 come together. Compare with the analysis of 3.1.3 for q-bic points.

3.7. Type $\mathbf{N}_3$

A q-bic curve of type $\mathbf{N}_3$, though not yet a cone, is reducible with a linear component and a component of degree q. The latter is a rational curve with a unibranch singularity which is analogous to the standard cusp in characteristic 3.

The two components intersect at the unique singular point $x_+ = \mathbf{P}\,\mathrm{Fr}^{-1}(V^\perp)$. The special point $x_- := \mathbf{P}\,\mathrm{Fr}^*(V)^\perp$ lies on the linear component, and together with x_+, spans the line. The linear subspace underlying this component is therefore the second step of the $\perp$-filtration of (V,β), see 1.1.16. This linear component also comprise of the Hermitian points and cone points of X: see 1.2.10(iii) and 2.4.7.

Finally, the scheme of linear automorphisms of X is determined as follows:

3.7.1. Proposition. — *Let (V,β) be a q-bic form of type $\mathbf{N}_3$. Then $\mathbf{Aut}(V,\beta)$ is isomorphic to the 2-dimensional closed subgroup scheme of $\mathbf{GL}_3$ consisting of*

$$\begin{pmatrix} \lambda & t & \epsilon \\ 0 & \lambda^{-q} & 0 \\ 0 & -\lambda^{q(q-1)}t^q & \lambda^{q^2} \end{pmatrix}$$

where $\lambda \in \mathbf{G}_m$, $t \in \mathbf{G}_a$, and $\epsilon \in \boldsymbol{\alpha}_q$.

Proof. — Choose a basis $V = \langle e_0, e_1, e_2 \rangle$ so that $\mathrm{Gram}(\beta; e_0, e_1, e_2) = \mathbf{N}_3$. Then the $\perp$-filtration is $\langle e_0 \rangle \subset \langle e_0, e_2 \rangle$, and the first piece of the $\mathrm{Fr}^*(\perp)$-filtration is $\langle e_2^{(q)} \rangle$, see 1.1.16 and 1.1.20. Since automorphisms preserve these filtrations by 1.3.2, $\mathbf{Aut}(V,\beta)$ is isomorphic to the closed subscheme of $\mathbf{GL}_3$ consisting of matrices which satisfy

$$\begin{pmatrix} a_{00}^q & 0 & 0 \\ a_{01}^q & a_{11}^q & a_{21}^q \\ 0 & 0 & a_{22}^q \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ 0 & a_{11} & 0 \\ 0 & a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and $a_{02}^q = 0$. Expanding gives $a_{00}^q a_{11} = a_{11}^q a_{22} = 1$ and $a_{01}^q a_{11} + a_{11}^q a_{21} = 0$. The first two equations give the diagonal entries; set $\lambda := a_{00}$. The third equation implies $a_{21} = -a_{01}^q \lambda^{q(q-1)}$. Setting $t := a_{01}$ and $\epsilon := a_{02}$ completes the computation. ■

3.8. q-bic surfaces

The remainder of this Chapter is devoted to the study of q-bic surfaces: the q-bic hypersurfaces in projective 3-space. The classification of q-bic forms from 1.4.1 gives:

3.8.1. Proposition. — *Let X be the q-bic surface associated with a nonzero q-bic form (V, β) of dimension 4. Then either X is a cone over q-bic curve or*

$$\operatorname{type}(\beta) = \begin{cases} \mathbf{1}^{\oplus 4} \\ \mathbf{N}_2 \oplus \mathbf{1}^{\oplus 2} \\ \mathbf{N}_4 \\ \mathbf{N}_3 \oplus \mathbf{1} \\ \mathbf{N}_2^{\oplus 2} \end{cases} \quad \textit{and} \quad X \cong \begin{cases} \mathrm{V}(x_0^{q+1} + x_1^{q+1} + x_2^{q+1} + x_3^{q+1}), \textit{ or} \\ \mathrm{V}(x_0^q x_1 + x_2^{q+1} + x_3^{q+1}), \textit{ or} \\ \mathrm{V}(x_0^q x_1 + x_1^q x_2 + x_2^q x_3), \textit{ or} \\ \mathrm{V}(x_0^q x_1 + x_1^q x_2 + x_3^{q+1}), \textit{ or} \\ \mathrm{V}(x_0^q x_1 + x_2^q x_3), \end{cases}$$

for some choice of coordinates $(x_0 : x_1 : x_2 : x_3)$ on $\mathbf{P}V \cong \mathbf{P}^3$. ∎

The type stratification of the affine space q-$\mathbf{bics}(V)$, see 1.4.4 and 1.4.6, is much more interesting than in the case of points and curves. In the following, temporarily drop the "$\oplus$" symbol for notation on types.

3.8.2. Proposition. — *The Hasse diagram for the type stratification of q-$\mathbf{bics}(V)$ is*

$$\begin{tikzcd} \mathbf{1}^4 \arrow[r, rightsquigarrow] & \mathbf{N}_2\mathbf{1}^2 \arrow[r, rightsquigarrow] & \mathbf{N}_4 \arrow[r, rightsquigarrow] \arrow[dr, rightsquigarrow] & \mathbf{N}_2^2 \arrow[rr, rightsquigarrow] & & \mathbf{0N}_2\mathbf{1} \arrow[r, rightsquigarrow] & \mathbf{0N}_3 \arrow[r, rightsquigarrow] & \mathbf{0}^2\mathbf{1}^2 \arrow[r, rightsquigarrow] & \mathbf{0}^2\mathbf{N}_2 \arrow[r, rightsquigarrow] & \mathbf{0}^3\mathbf{1} \\ & & & \mathbf{N}_3\mathbf{1} \arrow[r, rightsquigarrow] & \mathbf{01}^3 \arrow[ur, rightsquigarrow] & & & & & \end{tikzcd}$$

and the strata dimension are given by

λ	$\mathbf{1}^4$	$\mathbf{N}_2\mathbf{1}^2$	$\mathbf{N}_4$	$\mathbf{N}_3\mathbf{1}$	$\mathbf{N}_2^2$	$\mathbf{01}^3$	$\mathbf{0N}_2\mathbf{1}$	$\mathbf{0N}_3$	$\mathbf{0}^2\mathbf{1}^2$	$\mathbf{0}^2\mathbf{N}_2$	$\mathbf{0}^3\mathbf{1}$
$\dim q\text{-}\mathbf{bics}(V)_\lambda$	16	15	14	13	12	12	11	10	8	7	4

Proof. — The relations $\mathbf{1}^{\oplus 4} \rightsquigarrow \mathbf{N}_2 \oplus \mathbf{1}^{\oplus 2} \rightsquigarrow \mathbf{N}_4$ are because the general corank 1 form is of type $\mathbf{N}_2 \oplus \mathbf{1}^{\oplus 2}$, see 1.4.8. Analogously, $\mathbf{N}_2^{\oplus 2} \rightsquigarrow \mathbf{0} \oplus \mathbf{N}_2 \oplus \mathbf{1}$ because the former is the general corank 2 form. That $\mathbf{N}_3 \oplus \mathbf{1} \rightsquigarrow \mathbf{0} \oplus \mathbf{1}^{\oplus 3}$ can be seen by specializing the subform $\mathbf{N}_3 \rightsquigarrow \mathbf{0} \oplus \mathbf{1}^{\oplus 2}$, possible by 3.4.3. The specialization relations to the right of $\mathbf{N}_2^{\oplus 2}$ and $\mathbf{0} \oplus \mathbf{1}^{\oplus 3}$ now follow from specializing 2- or 3-dimensional subforms and using the relations from 3.1.2 and 3.4.3.

To see that $\mathbf{N}_4 \rightsquigarrow \mathbf{N}_3 \oplus \mathbf{1}$, consider the family β_t of q-bic forms on $V = \langle e_0, e_1, e_2, e_3 \rangle$ over $\mathbf{A}^1 = \mathrm{Spec}(\mathbf{k}[t])$ given by

$$\mathrm{Gram}(\beta_t; e_0, e_1, e_2, e_3) = \left(\begin{smallmatrix} 0 & 1 & t & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right).$$

Over a $\mathbf{k}$-point of $\mathbf{A}^1$, the $\perp$-filtration and $\mathrm{Fr}^*(\perp)$-filtration of this form are

$$\langle e_0 \rangle \subset \langle e_0, t^q e_3 + (1 - t^{q+1})(te_1 + e_2) \rangle, \text{ and}$$
$$\langle e_3 \rangle \subset \langle e_3, t^{1/q} e_1 + (1 - t^{(q+1)/q})(e_1 - te_2) \rangle.$$

For general t, the second steps differ, whence $\mathrm{type}(\beta_t) = \mathbf{N}_4$; for $t = 1$, the second step coincides, so $\mathrm{type}(\beta_t) = \mathbf{N}_3 \oplus \mathbf{1}$. See 1.4.1 and the comments of 1.4.2.

It remains to show that $\mathbf{N}_3 \oplus \mathbf{1}$ does *not* specialize to $\mathbf{N}_2^{\oplus 2}$. Suppose there were such a specialization. Then there would exist a discrete valuation ring R over $\mathbf{k}$ and a flat family $\mathscr{X} \subset \mathbf{P}V \otimes_{\mathbf{k}} R$ of q-bic surfaces with generic fibre of type $\mathbf{N}_3 \oplus \mathbf{1}$ and special fibre of type $\mathbf{N}_2^{\oplus 2}$. Consider the relative Fano scheme of lines $\mathbf{F}_1(\mathscr{X}/R)$ of this family. As in 2.7.3, this is defined in $\mathbf{G}(2, V) \otimes_{\mathbf{k}} R$ by a section of a rank 4 bundle. In particular, every component of $\mathbf{F}_1(\mathscr{X}/R)$ has codimension at most 4 in $\mathbf{G}(2, V)$. On the other hand, by 3.12.1 the generic fibre of $\mathbf{F}_1(\mathscr{X}/R)$ is a single point, whereas by 3.14.3, the special fibre has a one-dimensional component and two isolated points. In any case, this implies that $\mathbf{F}_1(\mathscr{X}/R)$ must have an isolated 0-dimensional component in its special fibre, which would be codimension 5 in $\mathbf{G}(2, V) \otimes_{\mathbf{k}} R$—a contradiction! Thus such a $\mathscr{X}$ cannot exist.

The dimensions of the strata are now determined by 1.4.7; for explicit computations of the automorphism group schemes, see 3.10.5, 3.11.3, 3.12.2, and 3.14.4 for the singular surfaces which are not cones, and for the cones, see 1.3.8 together with the computations 3.3.1, 3.6.1, and 3.7.1. ■

3.8.3. Lines. — Lines in X correspond to maximal isotropic subspaces for (V, β). So by 2.4.15, every line in X contains a cone point. The general results of 2.7 show that the scheme $\mathbf{F}_1(X)$ of lines in X has expected dimension 0, in which case it has degree $(q+1)(q^3+1)$: see 2.7.3 and 2.7.19. By 2.7.14(i), $\mathbf{F}_1(X)$ is 0-dimensional whenever β has corank at most 1 and no radical; in fact, the converse holds:

3.8.4. Lemma. — *The Fano scheme* $\mathbf{F}_1(X)$ *has expected dimension* 0 *if and only if* (V,β) *has corank at most* 1 *and no radical.*

Proof. — If β has a radical, then X is a cone by 2.4.1, and $\mathbf{F}_1(X)$ contains a curve. If β does not have a radical, then $\mathrm{type}(\beta) = \mathbf{N}_2^{\oplus 2}$ by 2.4.2. Then X contains the 1-dimensional family of lines given by the $\mathbf{P}\,\mathrm{Fr}^{-1}(L^\perp)$, where $\mathbf{P}L$ is a singular point of X, and again, $\mathbf{F}_1(X)$ is at least 1-dimensional. ■

The remainder of the Section is devoted to a general analysis of surfaces of corank 1 and 2. Write $L_- := \mathrm{Fr}^*(V)^\perp$ and $L_+ := \mathrm{Fr}^{-1}(V^\perp)$ for the two kernels of β, so that by 1.1.8, they fit into an exact sequence

$$0 \to L_- \to V \xrightarrow{\beta} \mathrm{Fr}^*(V) \to \mathrm{Fr}^*(L_+) \to 0.$$

By 2.2.7, the singular locus of X is supported on the linear space $\mathbf{P}L_+$.

3.8.5. Corank 1. — Assume that X has corank 1, so that its type is amongst

$$\mathbf{N}_2 \oplus \mathbf{1}^{\oplus 2},\ \ \mathbf{N}_4,\ \ \mathbf{N}_3 \oplus \mathbf{1},\ \ \mathbf{0} \oplus \mathbf{1}^{\oplus 3}.$$

Write $x_\pm := \mathbf{P}L_\pm$ for the points corresponding to the kernels of β, so x_+ is the unique singular point of X. By 3.8.4, if X is not the cone, then $\mathbf{F}_1(X)$ is 0-dimensional and by 2.7.7, points corresponding to lines not passing through x_+ are reduced.

3.8.6. Projection from x_+. — Let $\tilde{X} \to X$ be the blowup at the singular point x_+, let $W := V/L_+$, and let $\pi\colon \tilde{X} \to \mathbf{P}W$ be the morphism resolving linear projection centred at x_+. The structure of π splits into two cases:

- if X is not a cone, then π is an isomorphism away from the finitely many points of $\mathbf{P}W$ corresponding to the lines in X through x_+; and
- if X is a cone, then π factors through the smooth q-bic curve $C \subset \mathbf{P}W$ induced by β on the quotient, and there is an isomorphism $\tilde{X} \cong \mathbf{P}(\mathcal{O}_C(-1) \oplus L_{+,C})$.

The first case follows from the structure of q-bic points from 3.1.1: any line in $\mathbf{P}V$ through the singular point x_+ must contain it to multiplicity at least q, and when

x_+ is not a vertex point, the general line contains it to multiplicity at most q. The second case follows by general projective geometry; see also A.2.4.

The blowup $\tilde{X}$ is not always smooth. A direct computation shows that it is smooth for X of type $\mathbf{N}_2 \oplus \mathbf{1}^{\oplus 2}$ and $\mathbf{0} \oplus \mathbf{1}^{\oplus 3}$, but not in the remaining cases $\mathbf{N}_4$ and $\mathbf{N}_3 \oplus \mathbf{1}$.

3.8.7. Corank 2. — Assume that X has corank 2, so that its type is amongst

$$\mathbf{N}_2^{\oplus 2},\;\; \mathbf{0} \oplus \mathbf{N}_3,\;\; \mathbf{0} \oplus \mathbf{N}_2 \oplus \mathbf{1},\;\; \mathbf{0}^{\oplus 2} \oplus \mathbf{1}^{\oplus 2}.$$

Write $\ell_\pm := \mathbf{P}L_\pm$ for the lines corresponding to the kernels of β, so that ℓ_+ is the singular locus of X. The analysis of corank 2 surfaces relies on the following simple consequence of the scheme structure of the nonsmooth locus:

3.8.8. Lemma. — *Any $\mathbf{P}^2$-section of X containing ℓ_+ is a q-bic curve containing ℓ_+ to multiplicity at least q.*

Proof. — Let $\mathbf{P}U \cong \mathbf{P}^2$ be a plane containing ℓ_+. Then $L_+ \subseteq U$ so the restriction of β to U satisfies $\mathrm{Fr}^*(L_+) \subseteq \ker(\beta_U \colon \mathrm{Fr}^*(U) \to U^\vee)$. Since the q-bic curve $X \cap \mathbf{P}U$ is that given by (U, β_U) by 2.1.4, the result follows from 2.2.7. ■

3.8.9. Projection from ℓ_+. — Let $W := V/L_+$ and let $\psi \colon \mathbf{P}V \dashrightarrow \mathbf{P}W$ be the linear projection from $\ell_+ = \mathbf{P}L_+$. As in A.2.1, let

$$\mathbf{P}\psi := \left\{ ([V'],[W']) \in \mathbf{P}V \times \mathbf{P}W \;\middle|\; \psi(V') \subseteq W' \right\}$$

be the incidence correspondence for 1-dimensional subspaces under the quotient map $V \to W$. Then there is a commutative diagram

$$\begin{tikzcd} \tilde{X} \arrow[r, hook] \arrow[d] & \mathbf{P}\psi \arrow[r] \arrow[d] & \mathbf{P}W \\ X \arrow[r, hook] & \mathbf{P}V \arrow[ur, dashed, "\psi"'] & \end{tikzcd}$$

where, by A.2.4, the vertical maps are blowups along ℓ_+, and $\mathbf{P}\psi \to \mathbf{P}W$ is the projective bundle associated with the bundle $\mathcal{V}$ defined by the pullback diagram

$$\begin{tikzcd}
0 \arrow[r] & L_{+,\mathbf{P}W} \arrow[r] \arrow[d, equal] & \mathcal{V} \arrow[r] \arrow[d, hook] & \mathcal{O}_{\mathbf{P}W}(-1) \arrow[r] \arrow[d, hook, "\mathrm{eu}_{\mathbf{P}W}"] & 0 \\
0 \arrow[r] & L_{+,\mathbf{P}W} \arrow[r] & V_{\mathbf{P}W} \arrow[r] & W_{\mathbf{P}W} \arrow[r] & 0\,.
\end{tikzcd}$$

The inverse image $X_{\mathbf{P}\psi} := X \times_{\mathbf{P}V} \mathbf{P}\psi$ of X along the blowup is the bundle of q-bic curves over $\mathbf{P}W$ defined by the q-bic form

$$\beta_{\mathcal{V}} \colon \mathrm{Fr}^*(\mathcal{V}) \otimes \mathcal{V} \subset \mathrm{Fr}^*(V)_{\mathbf{P}W} \otimes V_{\mathbf{P}W} \xrightarrow{\beta} \mathcal{O}_{\mathbf{P}W}.$$

The general $\mathbf{P}^2$-section of X containing ℓ_+ is not completely contained in X. Thus $\mathcal{V}^\perp = \mathrm{Fr}^*(L_{+,\mathbf{P}W})$ and so the equation of $\tilde{X}$ is obtained by pulling the morphism

$$\beta_{\mathcal{V}} \colon \mathcal{V} \to \mathrm{Fr}^*(\mathcal{V}/L_{+,\mathbf{P}W})^\vee \cong \mathcal{O}_{\mathbf{P}W}(q)$$

up to $\mathbf{P}\psi$, pre-composing and post-composing by the $\mathrm{eu}_{\mathbf{P}\psi/\mathbf{P}W}$ and $\mathrm{eu}_{\mathbf{P}\psi/\mathbf{P}W}^{(q),\vee}$, respectively. From this, it follows that $X_{\mathbf{P}\psi}$ contains the exceptional subbundle $\mathbf{P}L_{+,\mathbf{P}W}$ to multiplicity at least q, globalizing 3.8.8.

3.8.10. Strict transform. — The blowup $\tilde{X} \to X$ along ℓ_+, being the strict transform of X along $\mathbf{P}\psi \to \mathbf{P}V$, is obtained by factoring out the equation the exceptional subbundle $\mathbf{P}L_{+,\mathbf{P}W}$. This can be expressed in terms of the morphism $\beta_{\mathcal{V}} \colon \mathcal{V} \to \mathcal{O}_{\mathbf{P}W}(q)$ constructed at the end of 3.8.9; the construction breaks up into two cases:

If X is a cone over ℓ_+, then $X_{\mathbf{P}\psi}$ contains $\mathbf{P}L_{+,\mathbf{P}W}$ to multiplicity $q+1$, and the equation of $\tilde{X}$ is the pullback of $\beta_{\mathcal{V}}$ to $\mathbf{P}\psi$. Thus $\tilde{X}$ consists of the fibres of $\mathbf{P}\psi \to \mathbf{P}W$ over the smooth q-bic points determined by the q-bic form induced by β on W.

If X is not a cone over ℓ_+, then $X_{\mathbf{P}\psi}$ contains the $\mathbf{P}L_{+,\mathbf{P}W}$ to multiplicity q, and the equation of $\tilde{X}$ is given by $\mathrm{eu}_{\mathbf{P}\psi/\mathbf{P}W} \circ \beta_{\mathcal{V}}$. Thus, if $\beta_{\mathcal{V}} \colon \mathcal{V} \to \mathcal{O}_{\mathbf{P}W}(q)$ is furthermore surjective or, equivalently, if X does not contain a $\mathbf{P}^2$ as an irreducible component, then $\tilde{X}$ is the projective subbundle of $\mathbf{P}\psi$ associated with

$$\mathrm{Fr}^*(\mathcal{V})^\perp = \ker(\beta_{\mathcal{V}} \colon \mathcal{V} \to \mathcal{O}_{\mathbf{P}W}(q)).$$

Even if $\beta_{\mathscr{V}}$ is not surjective, it is a nonzero map; since any subsheaf of $\mathscr{O}_{\mathbf{P}W}(q) = \mathscr{O}_{\mathbf{P}^1}(q)$ is a line bundle, $\mathrm{Fr}^*(\mathscr{V})^\perp$ is a rank 2 isotropic subbundle of $\mathscr{V}$, and hence yields a family over $\mathbf{P}W$ of lines in X. In summary, this shows that:

3.8.11. Proposition. — *A q-bic surface X of corank 2 is irreducible if and only if it does not contain a plane. If X is irreducible, then the blowup $\tilde{X} \to X$ along its singular line ℓ_+ is the ruled surface $\mathbf{P}\,\mathrm{Fr}^*(\mathscr{V})^\perp \to \mathbf{P}W$.*

Proof. — If X contains a plane, then it must be an irreducible component as X is also 2-dimensional. If X does not contain a plane, then 3.8.10 shows that the blowup $\tilde{X} \to X$ along the singular line ℓ_+ is isomorphic to the projective bundle $\mathbf{P}\,\mathrm{Fr}^*(\mathscr{V})^\perp \to \mathbf{P}W$. In particular, $\tilde{X}$ is irreducible, and thus so is X. ■

The q-bic surfaces of corank that contain a plane are classified as follows:

3.8.12. Lemma. — *Let X be a q-bic surface of corank 2.*

- *If X is of type either $\mathbf{N}_2^{\oplus 2}$ or $\mathbf{0} \oplus \mathbf{N}_2 \oplus \mathbf{1}$, then X contains no planes.*
- *If X is of type $\mathbf{0} \oplus \mathbf{N}_3$, then X contains the unique plane $\mathbf{P}(L_- + L_+)$.*
- *If X is of type $\mathbf{0}^{\oplus 2} \oplus \mathbf{1}^{\oplus 2}$, then X consists of $q+1$ planes intersecting at $\mathbf{P}L_+$.*

Proof. — The cases in which X is a cone reduce, upon by passing to the quotient by the radical of (V, β), to the classification of q-bic curves which contain lines: see 3.4.1. It remains to show that that a q-bic surface X of type $\mathbf{N}_2^{\oplus 2}$ does not contain a plane. Since X is connected, any plane in such X must contain the singular line ℓ_+ in its intersection with other irreducible components of X. But, as in 1.3.6, $V = L_- \oplus L_+$ and β induces an isomorphism $L_+ \to \mathrm{Fr}^*(L_-)^\vee$. Thus if $U \subset V$ is any 3-dimensional subspace containing L_+, $U \subset W$ containing L_+, the composite $L_+ \subset U \to \mathrm{Fr}^*(U)^\vee$ has rank 1 and thus U cannot be isotropic. In other words, a q-bic surface of type $\mathbf{N}_2^{\oplus 2}$ cannot contain a plane. ■

The family $\tilde{X} \to \mathbf{P}W$ of lines in X at the end of 3.8.10 sweep out X:

3.8.13. Lemma. — *Let X be a q-bic surface of corank 2, not of type $\mathbf{0}^{\oplus 2} \oplus \mathbf{1}^{\oplus 2}$. Then the morphism $\mathbf{P}W \to \mathbf{F}_1(X)$ classifying the family of lines given by $\tilde{X} \to \mathbf{P}W$ is injective.*

Proof. — By construction, the fibre of $\tilde{X} \to \mathbf{P}W$ over a point $y \in \mathbf{P}W$ is the line ℓ residual to $q\ell_+$ in the intersection $X \cap \mathbf{P}\mathscr{V}_y$. If $\mathscr{V}_y \neq \mathrm{Fr}^{-1}(L_+^\perp) \subset V$, which can only occur when X has a vertex, then $\ell = \ell_-$; otherwise, $\ell \cap \ell_-$ is a single point $x = \mathbf{P}L$ and, by 2.4.4, $\mathscr{V}_y = \mathrm{Fr}^{-1}(L^\perp)$. Thus all but at most one fibre of $\tilde{X} \to \mathbf{P}W$ are lines which intersect ℓ_- at a distinct point, with the remaining fibre being ℓ_-. It follows that the classifying map $\mathbf{P}W \to \mathbf{F}_1(X)$ is injective. ∎

This implies that, in the cases given by 3.8.13, there is a component of $\mathbf{F}_1(X)$ whose reduction is a geometrically rational curve. The scheme structure of the component, however, is generally quite complicated: see 3.14.3 and 3.15.4.

3.9. Type $\mathbf{1}^{\oplus 4}$

Let X be the smooth q-bic surface associated with a q-bic form (V, β) of type $\mathbf{1}^{\oplus 4}$. Convenient equations for X include;

$$X = \begin{cases} \mathrm{V}(x_0^{q+1} + x_1^{q+1} + x_2^{q+1} + x_3^{q+1}) & \text{the Fermat surface, and} \\ \mathrm{V}(x_0^q x_1 + x_0 x_1^q + x_2^q x_3 + x_2 x_3^q) & \text{the Hermitian surface.} \end{cases}$$

3.9.1. Hermitian points and lines. — The surface X contains $(q^2+1)(q^3+1)$ Hermitian points, and through each such point passes $q+1$ Hermitian lines: see 2.4.13 and 2.4.8. Since each line in X is Hermitian by 2.4.16, each line in X contains q^2+1 Hermitian points, giving another way to verify that there are exactly $(q+1)(q^3+1)$ lines in X. The union of the lines in X is the complete intersection $X \cap X^1$, see 2.9.10 and 2.9.13. For example, for the Fermat equation, 2.9.12 gives

$$X \cap X^1 \cong \mathrm{V}(x_0^{q+1} + x_1^{q+1} + x_2^{q+1} + x_3^{q+1}) \cap \mathrm{V}(x_0^{q^3+1} + x_1^{q^3+1} + x_2^{q^3+1} + x_3^{q^3+1}).$$

3.9.2. Unirational parameterization. — By 2.5.1, the surface X admits a purely inseparable unirational parameterization of degree q. Shioda first constructed an explicit coordinate parameterization in [Shi74]; see also [Kat17, Corollary 5.3] for a coordinate computation akin to the method presented in 2.5.2. The construction of 2.5.8 identifies an explicit blowup of a particular Hirzebruch surface for this parameterization, and will be described in the following. That the Fermat q-bic surface is purely inseparably covered by the specific Hirzebruch surface below was described by Hirokado in [Hiroo, Proposition 3.6] in the case $q = p$ via explicit computations using the theory of 1-foliations.

3.9.3. — Fix a line $\ell \subset X$ and let

$$Y := \mathbf{P}(\mathcal{T}_X(-1)|_\ell) \cong \mathbf{P}(\mathcal{O}_\ell(1) \oplus \mathcal{O}_\ell(-q))$$

be projective bundle over ℓ associated with the restricted tangent bundle of X; see 2.7.6 for the second identification. Let $\tilde{Y} \to Y$ be the blowup along the points

$$\left\{ (x, [\ell']) \in Y \;\middle|\; \ell' \neq \ell \text{ and } \ell' \subset X \right\}$$

with notation as in 2.5.8; this consists of $q(q^2+1)$ points with q points lying over each of the q^2+1 Hermitian points of X contained in ℓ. Let $\tilde{X} \to X$ be the blowup along the Hermitian points contained in ℓ. Then there exists a commutative diagram

$$\begin{tikzcd} \tilde{Y} \arrow[r, "\varphi"'] \arrow[d] & \tilde{X} \arrow[d] \\ Y \arrow[r, dashed] & X \end{tikzcd}$$

where $Y \dashrightarrow X$ is the rational map constructed in 2.5.11 and $\varphi \colon \tilde{Y} \to \tilde{X}$ is a finite purely inseparable morphism of degree q.

3.9.4. Divisors. — The Picard rank of the surfaces $\tilde{Y}$ and $\tilde{X}$ are

$$\operatorname{rank}_{\mathbf{Z}} \mathrm{NS}(\tilde{Y}) = \operatorname{rank}_{\mathbf{Z}} \mathrm{NS}(\tilde{X}) = q(q^2+1)+2.$$

This matches the étale Betti number of $\tilde{X}$, which can be computed from 2.6.4(i). The pullback map $\varphi^* \colon \mathrm{NS}(\tilde{X}) \to \mathrm{NS}(\tilde{Y})$ acts on certain special classes as follows:

$$\varphi^* E_x = q\tilde{Y}_x, \quad \varphi^*\ell' = qE_{(x,\ell')} + \tilde{Y}_x, \quad \varphi^*\tilde{\ell} = \sigma.$$

Here, $x \in \ell$ is a Hermitian point and ℓ' is a line in X intersecting ℓ exactly at x. The divisors on $\tilde{X}$ are: E_x is the exceptional divisor of $\tilde{X} \to X$ over x; $\tilde{\ell}$ is the strict transform of ℓ; and σ is the pullback of the class of the negative section $\mathbf{P}\mathscr{O}_\ell(1)$. The divisors on $\tilde{Y}$ are: $E_{(x,\ell')}$ is the exceptional divisor of $\tilde{Y} \to Y$ above $(x,[\ell'])$; and $\tilde{Y}_x$ is the strict transform of the fibre of Y over x.

3.10. Type $\mathbf{N}_2 \oplus \mathbf{1}^{\oplus 2}$

General q-bic surfaces of corank 1 are of type $\mathbf{N}_2 \oplus \mathbf{1}^{\oplus 2}$, see **3.8.2**. By **1.3.6**, the underlying q-bic form admits a canonical orthogonal decomposition

$$(V,\beta) = (U,\beta_U) \perp (W,\beta_W)$$

where, notation as in **3.8.5**, the restriction of β to $U := L_- \oplus L_+$ is of type $\mathbf{N}_2$, and the complement W is of type $\mathbf{1}^{\oplus 2}$.

3.10.1. Cone points. — Applying **2.4.7** shows that X has $q+3$ cone points:

(i) the singular point x_+,

(ii) the special point x_-, and

(iii) the $q+1$ Hermitian points of the type $\mathbf{1}^{\oplus 2}$ subform (W,β_W).

The cone points of the third type are determined with the help of **1.2.5** and the computation of **1.2.10**(iii). In each case, there is a unique plane witnessing the cone point property: this is $\mathbf{P}\,\mathrm{Fr}^{-1}(L_+^{\perp})$ in (i), and $\mathbf{T}_{X,x}$ in (ii) and (iii).

Consider the cones obtained in (i) and (ii). Since

$$\mathrm{Fr}^{-1}(L_+^{\perp}) = \langle L_+, W\rangle \quad \text{and} \quad \mathrm{Fr}^*(L_-)^{\perp} = \langle L_-, W\rangle$$

the associated cones have base given by the q-bic points defined by (W,β_W). Consider a cone point $x = \mathbf{P}L$ as in (iii). Its embedded tangent space has underlying linear space $\mathrm{Fr}^*(L)^{\perp} = \langle U, L\rangle$ and so the associated cone is

$$X \cap \mathbf{P}\,\mathrm{Fr}^*(L)^{\perp} = \langle x, x_-\rangle \cup q\langle x, x_+\rangle.$$

Since every line contains a cone point by **2.4.15**, this gives the first statement of:

3.10.2. Proposition. — *Every line in X passes through exactly one of x_- or x_+. The subscheme of $\mathbf{F}_1(X)$ corresponding to lines through*

- *x_- consists of $q+1$ reduced points, and*
- *x_+ consists of $q+1$ points of multiplicity q^3.*

Proof. — This follows from the facts collected in 3.8.3 and 3.8.5: the lines through x_- are contained in the smooth locus and hence give reduced points in $\mathbf{F}_1(X)$; since the Fano scheme has degree $(q+1)(q^3+1)$, the lines through x_+ must appear with multiplicity q^3 each, by symmetry. ■

The scheme of cone points X_{cone}, as defined in 2.4.9, can now be described by examining how cone points of a smooth q-bic surface must come together in a degeneration to X. The analysis relies on the observation that each cone point in a smooth q-bic surface is the intersection of two distinct lines contained in the surface.

3.10.3. Proposition. — *The multiplicity of the scheme of cone points at a point x is*

$$\mathrm{mult}_x(X_{\mathrm{cone}}) = \begin{cases} q^5 & \text{if } x \text{ is the singular point } x_+ \text{ as in 3.10.1(ii),} \\ 1 & \text{if } x \text{ is the special point } x_- \text{ as in 3.10.1(i), and} \\ q^2 & \text{if } x \text{ is a Hermitian point as in 3.10.1(iii).} \end{cases}$$

Proof. — Consider a flat family of q-bic surfaces containing X whose general fibre is smooth and such that the special point x_- extends to a section. A specific such family may be obtained via the construction of 3.1.4: consider, say,

$$\mathscr{X} = \mathrm{V}(x_0^q x_1 + t x_0 x_1^q + x_2^{q+1} + x_3^{q+1}) \subset \mathbf{P}^4 \times \mathbf{A}^1$$

and abusively conflate the point $x_- = (1:0:0:0)$ with the corresponding constant section over $\mathbf{A}^1$. Observe that the relative schemes $\mathscr{X}_{\mathrm{cone}}$ of cone points and $\mathbf{F}_1(\mathscr{X}/\mathbf{A}^1)$ of lines are flat over $\mathbf{A}^1$.

Consider how lines in a smooth general fibre X_t of $\mathscr{X}$ limit to lines in X. Since the tangent space to x_- remains constant along the family $\mathscr{X}$, a line through x_- in X_t remains a line through x_- in X; let $\ell^0, \ell^1, \ldots \ell^q$ denote the $q+1$ such lines, viewed interchangeably as lying in X_t or as the corresponding limit in X. Since $\ell^0 + \ell^1 + \cdots + \ell^q$ is a hyperplane section, any other line $\ell_t \subset X_t$ intersects exactly one of the ℓ^i. Its limit $\ell_0 \subset X$ must also intersect ℓ^i. Since each of the points $[\ell^i]$ is reduced in $\mathbf{F}_1(X)$

by 3.10.2, ℓ_0 cannot coincide with ℓ^i and thus their point y_i of intersection one of the Hermitian points as in 3.10.1(iii). Thus the q^2 cone points of $\ell^i \subset X_t$ distinct from x_- limit to y_i, implying it has multiplicity q^2 in X_{cone}. Since $\deg(X_{\mathrm{cone}}) = (q^2+1)(q^3+1)$ by 2.4.11, it follows that x_+ has multiplicity q^5. ■

3.10.4. Remark. — In fact, a slight modification of the arguments of 3.10.1 and 3.10.3 shows that for a q-bic hypersurface of type $\mathbf{N}_2 \oplus \mathbf{1}^{\oplus n-1}$, its cone points consist of its singular point, special point, and the Hermitian points coming from the subform $\mathbf{1}^{\oplus n-1}$, and that the multiplicities are q^{2n-1}, 1, and q^2, respectively.

Linear automorphisms of X can be obtained by specializing 1.3.7:

3.10.5. Proposition. — *Let (V, β) be a q-bic form of type $\mathbf{N}_2 \oplus \mathbf{1}^{\oplus 2}$. Then $\mathbf{Aut}(V, \beta)$ is isomorphic to the 1-dimensional subgroup scheme of $\mathbf{GL}_4$ consisting of*

$$\left(\begin{array}{cc|cc} \lambda & \epsilon & y_1 & y_2 \\ 0 & \lambda^{-q} & 0 & 0 \\ \hline 0 & x_1 & \multicolumn{2}{c}{\multirow{2}{*}{A}} \\ 0 & x_2 & & \end{array}\right)$$

with $A \in \mathrm{U}_2(q)$, $\lambda \in \mathbf{G}_m$, $\epsilon, x_1, x_2 \in \boldsymbol{\alpha}_q$, and $y_1, y_2 \in \boldsymbol{\alpha}_{q^2}$, subject to the equation

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda^{-q}(A^{\vee,(q)}\beta_W)^{-1}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$ ■

3.11. Type $\mathbf{N}_4$

The next q-bic surfaces of corank 1 are those of type $\mathbf{N}_4$. By 2.4.7, the cone points such a surface X are its singular point x_+ and its special point x_-: the computation of 1.2.10(iii) implies X does not have any Hermitian points. The associated cones are of type $\mathbf{0} \oplus \mathbf{N}_2$, and their components are

$$X \cap \mathbf{P}\,\mathrm{Fr}^{-1}(L_+^{\perp}) = \ell_0 \cup q\ell_+ \quad \text{and} \quad X \cap \mathbf{P}\,\mathrm{Fr}^*(L_-)^{\perp} = q\ell_0 \cup \ell_-$$

where $\ell_0 := \langle x_-, x_+ \rangle$, $\ell_+ := \mathbf{P}\,\mathrm{Fr}^{-1}(\mathrm{Fr}^{-1}(L_+^\perp)^\perp)$, and $\ell_- := \mathbf{P}\,\mathrm{Fr}^*(\mathrm{Fr}^*(L_-)^\perp)^\perp$. That is, ℓ_0 is the line between $x_\pm$, and $\ell_\pm$ are the first pieces of the $\perp$- and $\mathrm{Fr}^*(\perp)$-filtrations of (V, β), see 1.1.16 and 1.1.20.

The scheme of cone points is now easily determined:

3.11.1. Proposition. — $\mathrm{mult}_{x_+}(X_{\mathrm{cone}}) = q^3 + 1$ *and* $\mathrm{mult}_{x_-}(X_{\mathrm{cone}}) = q^2(q^3+1)$.

Proof. — Consider a flat family of q-bic surfaces with general fibre of type $\mathbf{N}_2 \oplus \mathbf{1}^{\oplus 2}$ and special fibre X. This is possible by 3.8.2. Then the scheme of cone points also fit into a flat family over the base. By the classification of cone points in the two types, each of the Hermitian cone points 3.10.1(iii) in the general fibre specializes to either x_+ or x_-. Since the line ℓ_- is contained in the smooth locus of X, considering the family of lines implies that exactly one of the Hermitian cone points specializes to x_-, whereas the remaining q must specialize to x_+. The multiplicities are now determined from 3.10.3. ∎

The scheme structure of the Fano scheme can now be similarly determined:

3.11.2. Proposition. — *Let ℓ be a line in X. Then*

$$\mathrm{mult}_{[\ell]}(\mathbf{F}_1(X)) = \begin{cases} q^4 & \text{if } \ell = \ell_+, \\ q(q^2+1) & \text{if } \ell = \ell_0, \text{ and} \\ 1 & \text{if } \ell = \ell_-. \end{cases}$$

Proof. — Since ℓ_- is contained in the smooth locus, it is a reduced point in $\mathbf{F}_1(X)$, see 3.8.3. For the remaining points, consider as in the proof of 3.11.1, a flat family of q-bic surfaces with general fibre of type $\mathbf{N}_2 \oplus \mathbf{1}^{\oplus 2}$ and special fibre X. Then the relative Fano scheme of lines is flat over the base. The lines that limit to ℓ_+ are those that join the singular point and one of the q Hermitian cone points limiting to x_+. By 3.10.2, each such line has multiplicity q^3 in the Fano scheme of the general fibre. Thus $[\ell_+]$ has multiplicity q^4 in $\mathbf{F}_1(X)$. The remaining multiplicity is deduced from the fact that $\deg(\mathbf{F}_1(X)) = (q+1)(q^3+1)$. ∎

The linear automorphisms of X are given by:

3.11.3. Proposition. — *Let (V, β) be a q-bic form of type $\mathbf{N}_4$. Then $\mathbf{Aut}(V, \beta)$ is isomorphic to the 2-dimensional closed subgroup scheme of $\mathbf{GL}_4$ consisting of*

$$\begin{pmatrix} \lambda & \epsilon_3 & t & \epsilon_1 \\ 0 & \lambda^{-q} & 0 & 0 \\ 0 & \epsilon_2 & \lambda^{q^2} & -\lambda^{-q^2(q-1)}\epsilon_2^q \\ 0 & -\lambda^{-q^2(q+1)}t^q & 0 & \lambda^{-q^3} \end{pmatrix}$$

where $\lambda \in \mathbf{G}_m$, $t \in \mathbf{G}_a$, and $\epsilon_i \in \boldsymbol{\alpha}_{q^i}$ for $i = 1, 2, 3$, and subject to the equation

$$\epsilon_2^q t^q - \lambda^{q(q^2-q+1)}\epsilon_2 - \lambda^{q^3}\epsilon_3^q = 0.$$

Proof. — Choose a basis $V = \langle e_0, e_1, e_2, e_3 \rangle$ such that $\mathrm{Gram}(\beta; e_0, e_1, e_2, e_3) = \mathbf{N}_4$. Then the $\perp$-filtration of V is given by $\langle e_0 \rangle \subset \langle e_0, e_2 \rangle \subset \langle e_0, e_2, e_3 \rangle$, and the first step of the $\mathrm{Fr}^*(\perp)$-filtration is given by $\langle e_3^{(q)} \rangle$, see **1.1.16** and **1.1.20**. Thus by **1.3.2**, $\mathbf{Aut}(V, \beta)$ is isomorphic to the closed subgroup scheme of $\mathbf{GL}_4$ consisting of matrices satisfying

$$\begin{pmatrix} a_{00}^q & 0 & 0 & 0 \\ a_{01}^q & a_{11}^q & a_{21}^q & a_{31}^q \\ a_{02}^q & 0 & a_{22}^q & 0 \\ 0 & 0 & 0 & a_{33}^q \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ 0 & a_{11} & 0 & 0 \\ 0 & a_{21} & a_{22} & a_{23} \\ 0 & a_{31} & 0 & a_{33} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $a_{03}^q = a_{23}^q = 0$. Expanding gives 7 equations: $a_{00}^q a_{11} = a_{11}^q a_{22} = a_{22}^q a_{33} = 0$ and

$$a_{01}^q a_{11} + a_{11}^q a_{21} + a_{21}^q a_{31} = a_{11}^q a_{23} + a_{21}^q a_{33} = a_{02}^q a_{11} + a_{22}^q a_{31} = 0.$$

The first three equations give the diagonal entries upon setting $\lambda := a_{00}$. The second and third displayed equations may be rearranged to give

$$a_{23} = -a_{21}^q a_{33}/a_{11}^q = -\lambda^{-q^2(q+1)}a_{21}^q \quad \text{and} \quad a_{31} = -a_{02}^q a_{11}/a_{22}^q = -\lambda^{-q^2(q-1)}a_{12}^q.$$

Substituting this into the remaining displayed equation and rearranging gives

$$\lambda^{-q(q^2+1)}(\lambda^{q^2}a_{01}^q + \lambda^{q(q^2-q+1)}a_{21} - a_{02}^q a_{21}^q) = 0.$$

Setting $t := a_{02}$, $\epsilon_1 := a_{03}$, $\epsilon_2 := a_{21}$, and $\epsilon_3 := a_{01}$ now gives the result. ∎

3.12. Type $\mathbf{N}_3 \oplus \mathbf{1}$

The next specialization is a q-bic surface X of type $\mathbf{N}_3 \oplus \mathbf{1}$. It's singular point x_+ and special point x_- span a line $\ell := \langle x_+, x_- \rangle$ which is contained in X. By 2.4.7 together with 1.2.5 and the computation of 1.2.10(iii), it follows that the locus of cone points of X is supported on this line ℓ. Moreover, the associated cone at each point is of type $\mathbf{0}^{\oplus 2} \oplus \mathbf{1}$ and consists of ℓ with multiplicity $q+1$. It follows that ℓ is the only line in X; this determines the Fano scheme:

3.12.1. Proposition. — $\mathbf{F}_1(X)$ *is supported on* $[\ell_0]$ *with multiplicity* $(q+1)(q^3+1)$. ■

The linear automorphisms of X are given by:

3.12.2. Proposition. — *Let* (V, β) *be a q-bic form of type* $\mathbf{N}_3 \oplus \mathbf{1}$. *Then* $\mathbf{Aut}(V, \beta)$ *is isomorphic to the* 3*-dimensional closed subgroup scheme of* $\mathbf{GL}_4$ *consisting of*

$$\begin{pmatrix} \lambda & t_1 & \epsilon & t_2 \\ 0 & \lambda^{-q} & 0 & 0 \\ 0 & -\lambda^{q(q-1)} t_1^q - \lambda^{-q} t_2^{q(q+1)} & \lambda^{q^2} & t_2^{q^2} \\ 0 & -\zeta \lambda^{-q} t_2^q & 0 & \zeta \end{pmatrix}$$

where $\lambda \in \mathbf{G}_m$, $t_1, t_2 \in \mathbf{G}_a$, $\zeta \in \boldsymbol{\mu}_{q+1}$, *and* $\epsilon \in \boldsymbol{\alpha}_q$.

Proof. — Choose a basis $V = \langle e_0, e_1, e_2, e_3 \rangle$ such that $\mathrm{Gram}(\beta; e_0, e_1, e_2, e_3) = \mathbf{N}_3 \oplus \mathbf{1}$. Then the $\perp$-filtration of V is $\langle e_0 \rangle \subset \langle e_0, e_2 \rangle \subset \langle e_0, e_2, e_3 \rangle$ and the first piece of the $\mathrm{Fr}^*(\perp)$-filtration is $\langle e_2^{(q)} \rangle$, see 1.1.16 and 1.1.20. Thus by 1.3.2, $\mathbf{Aut}(V, \beta)$ is isomorphic to the closed subgroup scheme of $\mathbf{GL}_4$ consisting of matrices satisfying

$$\begin{pmatrix} a_{00}^q & 0 & 0 & 0 \\ a_{01}^q & a_{11}^q & a_{21}^q & a_{31}^q \\ 0 & 0 & a_{22}^q & 0 \\ a_{03}^q & 0 & a_{23}^q & a_{33}^q \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ 0 & a_{11} & 0 & 0 \\ 0 & a_{21} & a_{22} & a_{23} \\ 0 & a_{31} & 0 & a_{33} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and $a_{02}^q = 0$. Expanding gives 6 more equations: $a_{00}^q a_{11} = a_{11}^q a_{22} = a_{33}^{q+1} = 1$, and

$$a_{03}^q a_{11} + a_{33}^q a_{31} = a_{11}^q a_{23} + a_{31}^q a_{33} = a_{01}^q a_{11} + a_{11}^q a_{21} + a_{31}^{q+1} = 0.$$

The first three equations yield the diagonal entries upon setting $\zeta := a_{33}$ and $\lambda := a_{00}$. Then rearranging the first two displayed equations gives

$$a_{31} = -a_{03}^q a_{11}/a_{33}^q = -\zeta\lambda^{-q}a_{03}^q \quad \text{and} \quad a_{23} = -a_{31}^q a_{33}/a_{11}^q = a_{03}^{q^2}.$$

The final equation then gives $a_{21} = -\lambda^{q(q-1)}a_{01}^q - \lambda^{-q}a_{03}^{q(q+1)}$. Setting $t_1 := a_{01}$, $t_2 := a_{03}$, and $\epsilon := a_{02}$ finishes the computation. ■

3.13. Type $\mathbf{0} \oplus \mathbf{1}^{\oplus 3}$

The final type of corank 1 surface is a cone over a smooth q-bic curve. Let $L := L_+ = L_-$ denote the radical of the underlying q-bic form, and let $x := x_+ = x_-$ denote the vertex of the surface X. As in 3.8.6, write $W := V/L$ and let $C \subset \mathbf{P}W$ be the smooth q-bic curve induced from (V, β); this forms the base of the cone.

Since C contains no lines, the lines in X must pass through the vertex. Thus the support of $\mathbf{F}_1(X)$ is canonically identified with C. Its schematic structure, however, is quite interesting. In the following, embed $\mathbf{P}W$ as the closed subscheme of $\mathbf{G}(2, V)$ parameterizing lines in $\mathbf{P}V$ through x.

3.13.1. Proposition. — *The Fano scheme $\mathbf{F}_1(X)$ satisfies*

$$\mathbf{F}_1(X)_{\mathrm{red}} = \mathbf{F}_1(X) \cap \mathbf{P}W = C$$

as subschemes of $\mathbf{G}(2, V)$, is generically a q-order neighbourhood of C, and has embedded points of multiplicity $q+1$ at the Hermitian points of C.

Proof. — The first statement follows from the preceding comments. For the latter statements, choose coordinates $(x_0 : x_1 : x_2 : x_3)$ on $\mathbf{P}V = \mathbf{P}^3$ so that

$$X = \mathrm{V}(x_1^q x_2 + x_1 x_2^q + x_3^{q+1}) \subset \mathbf{P}^3.$$

Then $\mathbf{F}_1(X)$ is covered by the Plücker charts $\mathrm{D}(p_{01})$ and $\mathrm{D}(p_{02})$; by symmetry, it suffices to consider $\mathrm{D}(p_{01})$, wherein $\mathbf{F}_1(X)$ has coordinates and equations

$$\begin{pmatrix} 1 & 0 & a_2 & a_3 \\ 0 & 1 & b_2 & b_3 \end{pmatrix} \qquad \begin{array}{rr} a_2 + a_2^q + a_3^{q+1} = 0, & b_2 + a_3^q b_3 = 0, \\ b_2^q + a_3 b_3^q = 0, & b_3^{q+1} = 0. \end{array}$$

Setting $b_2 = -a_3^q b_3$ and substituting it into the third equation gives

$$(a_3 - a_3^{q^2}) b_3^q = 0.$$

Therefore either $a_3 - a_3^{q^2} = b_3^{q+1} = 0$, yielding the embedded points of multiplicity $q+1$ along the Hermitian points of C, or $b_3^q = 0$, yielding the generic nonreduced structure of multiplicity q. ■

3.14. Type $\mathbf{N}_2^{\oplus 2}$

General q-bic surfaces of corank 2 are of type $\mathbf{N}_2^{\oplus 2}$, see 3.8.2. Let X be such a surface, and, as in 3.8.7, write ℓ_+ and ℓ_- for its singular and special lines.

Consider in detail the blowup $\tilde{X} \to X$ along ℓ_+, continuing the analysis started in 3.8.9. By 3.8.12, X does not contain a plane, so the discussion at the end of 3.8.10 gives a short exact sequence

$$0 \to \mathrm{Fr}^*(\mathcal{V})^\perp \to \mathcal{V} \xrightarrow{\beta_{\mathcal{V}}} \mathcal{O}_{\mathbf{P}W}(q) \to 0$$

of vector bundles on $\mathbf{P}W$; the kernel is furthermore isotropic for $\beta_{\mathcal{V}}$. By 3.8.11, $\tilde{X} \to \mathbf{P}W$ is the ruled surface associated with $\mathrm{Fr}^*(\mathcal{V})^\perp$. The kernel may be explicitly identified; the description arises by observing that the projection induces an isomorphism $\ell_- \cong \mathbf{P}W$.

3.14.1. Proposition. — *The blowup $\tilde{X} \to X$ along ℓ_+ is isomorphic to the projective bundle $\mathbf{P}\,\mathrm{Fr}^*(\mathcal{V})^\perp$ over $\mathbf{P}W$. There is a canonical split short exact sequence*

$$0 \to \mathrm{Fr}^*(\Omega^1_{\mathbf{P}W}(1)) \to \mathrm{Fr}^*(\mathcal{V})^\perp \to \mathcal{O}_{\mathbf{P}W}(-1) \to 0$$

and the exceptional divisor of $\tilde{X} \to X$ is given by the subbundle $\mathbf{P}\,\mathrm{Fr}^*(\Omega^1_{\mathbf{P}W}(1))$.

Proof. — The short exact sequence arises from the exact commutative diagram

$$\begin{tikzcd}
 & \mathcal{O}_{\mathbf{P}W}(-1) \arrow[r, equal] & \mathcal{O}_{\mathbf{P}W}(-1) & & \\
0 \arrow[r] & \mathrm{Fr}^*(\mathcal{V})^{\perp} \arrow[r] \arrow[u, two heads] & \mathcal{V} \arrow[r, "\beta_{\mathcal{V}}"] \arrow[u, two heads] & \mathcal{O}_{\mathbf{P}W}(q) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Fr}^*(\Omega^1_{\mathbf{P}W}(1)) \arrow[r] \arrow[u, hook] & L_{+,\mathbf{P}W} \arrow[r] \arrow[u, hook] & \mathrm{Fr}^*(\mathcal{O}_{\mathbf{P}W}(-1))^{\vee} \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

in which the bottom row is identified with the Fr^*-twisted Euler sequence by:

$$\mathrm{Fr}^*(\Omega^1_{\mathbf{P}W}(1)) \cong \ker\left(\mathrm{eu}_{\mathbf{P}W}^{\vee,(q)} \circ \beta : L_{+,\mathbf{P}W} \xrightarrow{\cong} \mathrm{Fr}^*(L_{-,\mathbf{P}W})^{\vee} \twoheadrightarrow \mathrm{Fr}^*(\mathcal{O}_{\mathbf{P}W}(-1))^{\vee}\right).$$

Since the vertical sequence for $\mathcal{V}$ is split, so is that for $\mathrm{Fr}^*(\mathcal{V})^{\perp}$. The exceptional divisor of $\tilde{X} \to X$ correspond to points through ℓ_+, and from the diagram above, these lie in the subbundle $\mathbf{P}\,\mathrm{Fr}^*(\Omega^1_{\mathbf{P}W}(1))$. ■

3.14.2. Cone points. — By 2.4.7 together with 1.2.5 and 1.2.10(iii), the set of cone points of X is given by $\ell_- \cup \ell_+$. The cone associated with a cone point $x = \mathbf{P}L$ is given by

$$\begin{cases} X \cap \mathbf{P}\,\mathrm{Fr}^{-1}(L^{\perp}) = \ell \cup q\ell_+ & \text{if } x \in \ell_+, \text{ and} \\ X \cap \mathbf{P}\,\mathrm{Fr}^*(L)^{\perp} = \ell_- \cup q\ell & \text{if } x \in \ell_-, \end{cases}$$

where the line ℓ contains x and intersects $\ell_{\mp}$ at the point corresponding to the 1-dimensional subspaces $\mathrm{Fr}^{-1}(L^{\perp}) \cap L_-$ and $\mathrm{Fr}^*(L)^{\perp} \cap L_+$, respectively.

Since every line passes through a cone point, this shows that the scheme $\mathbf{F}_1(X)$ has three components: two 0-dimensional components supported on the points $[\ell_{\pm}]$, and a 1-dimensional component parameterizing the lines ℓ whose support may be identified with the projective line $\mathbf{P}W$ by 3.8.13. The scheme structure is as follows:

3.14.3. Proposition. — *The Fano scheme $\mathbf{F}_1(X)$ has multiplicity*

(i) 1 *at the isolated point $[\ell_-]$,*

(ii) q^4 *at the isolated point $[\ell_+]$, and*

(iii) q *along the* 1*-dimensional component* $\mathbf{P}W$.

Proof. — Since ℓ_- is contained in the smooth locus of X, the corresponding point is reduced in $\mathbf{F}_1(X)$. For the remainder, choose coordinates $(x_0 : x_1 : x_2 : x_3)$ on $\mathbf{P}V = \mathbf{P}^3$ so that

$$X = \mathrm{V}(x_0^q x_2 + x_1^q x_3) \subset \mathbf{P}^3.$$

Then $\ell_+ = (0:0:*:*)$. The corresponding point is supported on the origin in the Plücker chart $\mathrm{D}(p_{23})$ in $\mathbf{G}(2,4)$, with coordinates and equations:

$$\begin{pmatrix} a_0 & a_1 & 1 & 0 \\ b_0 & b_1 & 0 & 1 \end{pmatrix} \qquad a_0^q = a_1^q = b_0^q = b_1^q = 0.$$

The one-dimensional component is covered by $\mathrm{D}(p_{03})$ and $\mathrm{D}(p_{12})$, and by symmetry, it suffices to consider the former. The coordinates and equations of $\mathbf{F}_1(X)$ there are

$$\begin{pmatrix} 1 & a_1 & a_2 & 0 \\ 0 & b_1 & b_2 & 1 \end{pmatrix} \qquad a_2 = b_2 + a_1^q = b_1^q = 0$$

verifying that it is a q-th order neighbourhood of an affine line. ■

The automorphism group scheme of the underlying q-bic form is obtained directly from the general computation of 1.3.7 with $a = 2$ and $b = 0$:

3.14.4. Proposition. — *Let (V,β) be a q-bic form of type $\mathbf{N}_2^{\oplus 2}$. Then $\mathbf{Aut}(V,\beta)$ is isomorphic to the 4-dimensional closed subgroup scheme of $\mathbf{GL}_4$ consisting of matrices*

$$\begin{pmatrix} a_{00} & a_{01} & \epsilon_{02} & \epsilon_{03} \\ a_{10} & a_{11} & \epsilon_{12} & \epsilon_{13} \\ 0 & 0 & a_{11}^q/\Delta^q & -a_{01}^q/\Delta^q \\ 0 & 0 & -a_{10}^q/\Delta^q & a_{00}^q/\Delta^q \end{pmatrix}$$

where $A := \left(\begin{smallmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{smallmatrix}\right) \in \mathbf{GL}_2$, $\Delta := \det(A)$, and $\epsilon_{02}, \epsilon_{03}, \epsilon_{12}, \epsilon_{13} \in \boldsymbol{\alpha}_q$.

Proof. — For a direct computation, choose a basis $\mathrm{Fr}^*(V)^\perp = \langle e_0, e_1 \rangle$, and let $V^\perp = \langle e_2^{(q)}, e_3^{(q)} \rangle$ be the dual basis with respect to β. Since automorphisms must preserve the two kernels, $\mathbf{Aut}(V,\beta)$ is isomorphic to the closed subgroup scheme of

$\mathbf{GL}_4$ consisting of block matrices that satisfy

$$\begin{pmatrix} A_-^{\vee,(q)} & 0 \\ 0 & A_+^{\vee,(q)} \end{pmatrix}\begin{pmatrix} 0 & I_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A_- & B \\ 0 & A_+ \end{pmatrix} = \begin{pmatrix} 0 & I_2 \\ 0 & 0 \end{pmatrix}$$

where $A_\pm \in \mathbf{GL}_2$, $B \in \mathbf{Mat}_{2\times 2}$ satisfies $B^{(q)} = 0$, and $I_2 := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Expanding shows that $A_-^{\vee,(q)} A_+ = I_2$, so rearranging and setting $A := A_-$ finishes the computation. ∎

3.15. Type $\mathbf{0} \oplus \mathbf{N}_2 \oplus \mathbf{1}$

The last reduced and irreducible q-bic surfaces are those of type $\mathbf{0} \oplus \mathbf{N}_2 \oplus \mathbf{1}$. The singular and special lines $\ell_\pm$ of such a surface X intersect at the vertex $x_0 := \mathbf{P}L_0$, with $L_0 := \mathrm{rad}(\beta)$. To continue the analysis of the blowup $\tilde{X} \to X$ along ℓ_+ from 3.8.9, note that X contains no planes by 3.8.11, so there is a short exact sequence

$$0 \to \mathrm{Fr}^*(\mathcal{V})^\perp \to \mathcal{V} \xrightarrow{\beta_{\mathcal{V}}} \mathcal{O}_{\mathbf{P}W}(q) \to 0$$

of vector bundles on $\mathbf{P}W$, and $\tilde{X}$ is isomorphic to $\mathbf{P}\,\mathrm{Fr}^*(\mathcal{V})^\perp$ over $\mathbf{P}W$. There is a distinguished closed point $\infty \in \mathbf{P}W$: Let $\bar{V} := V/L_0$ and write $\bar{L}_\pm := L_\pm/L_0$ for the images of the kernels. Then β passes to the quotient to yield a q-bic form $(\bar{V}, \beta_{\bar{V}})$ of type $\mathbf{N}_2 \oplus \mathbf{1}^{\oplus 1}$. By 1.3.6, there is a canonical orthogonal decomposition

$$(\bar{V}, \beta_{\bar{V}}) = (\bar{U}, \beta_{\bar{U}}) \perp (\bar{W}, \beta_{\bar{W}})$$

where $\bar{U} := \bar{L}_- \oplus \bar{L}_+$ has type $\mathbf{N}_2$ and its complement $\bar{W}$ has type $\mathbf{1}^{\oplus 2}$. Its image along the quotient map $\bar{V} \to W$ is 1-dimensional so it underlies a point $\infty \in \mathbf{P}W$.

3.15.1. Proposition. — *The blowup $\tilde{X} \to X$ along ℓ_+ is isomorphic to the projective bundle $\mathbf{P}\,\mathrm{Fr}^*(\mathcal{V})^\perp$ over $\mathbf{P}W$. There is a canonical split short exact sequence*

$$0 \to L_{0,\mathbf{P}W} \to \mathrm{Fr}^*\mathcal{V}^\perp \to \mathcal{O}_{\mathbf{P}W}(-q\infty) \otimes \mathcal{O}_{\mathbf{P}W}(-1) \to 0$$

and the exceptional divisor of $\tilde{X} \to X$ is the subbundle $\mathbf{P}L_{0,\mathbf{P}W}$.

Proof. — The first statements follow from the preceding discussion. The short exact sequence arises from the exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & L_{0,\mathbf{P}W} \arrow[r] \arrow[d, hook] & L_{+,\mathbf{P}W} \arrow[r, "\beta"'] \arrow[d, hook] & \mathrm{Fr}^*(\bar{L}_-)^\vee_{\mathbf{P}W} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathrm{Fr}^*(\mathcal{V})^\perp \arrow[r] \arrow[d, two heads] & \mathcal{V} \arrow[r, "\beta_{\mathcal{V}}"] \arrow[d, two heads] & \mathcal{O}_{\mathbf{P}W}(q) \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbf{P}W}(-q\infty) \otimes \mathcal{O}_{\mathbf{P}W}(-1) \arrow[r] & \mathcal{O}_{\mathbf{P}W}(-1) \arrow[r] & \mathcal{O}_{q\infty} \arrow[r] & 0
\end{tikzcd}$$

in which the right column arises by noting $W \cong \bar{W} \oplus \bar{L}_-$ and that the map to $\mathcal{O}_{\mathbf{P}W}(q)$ is the q-th power of the coordinate vanishing on $\bar{L}_-$. Since the exceptional divisor of $\tilde{X} \to X$ corresponds to points intersecting ℓ_+, it is the subbundle $\mathbf{P}L_{0,\mathbf{P}W}$. ■

That the point $\infty \in \mathbf{P}W$ is distinguished in regards to $\tilde{X} \to \mathbf{P}W$ can be alternatively seen upon considering the family of q-bic curves $X_{\mathbf{P}\psi} \to \mathbf{P}W$ from 3.8.9.

3.15.2. Proposition. — *The fibre $X_{\mathbf{P}\psi,y}$ over $y \in \mathbf{P}W$ is of type*

$$\mathrm{type}(X_{\mathbf{P}\psi,y}) = \begin{cases} \mathbf{0} \oplus \mathbf{N}_2 & \text{if } y \neq \infty, \text{ and} \\ \mathbf{0} \oplus \mathbf{1}^{\oplus 2} & \text{if } y = \infty. \end{cases}$$

Proof. — The quotient map $\bar{V} \to W$ induces linear projection $\mathbf{P}\bar{V} \dashrightarrow \mathbf{P}W$ at the point $\mathbf{P}\bar{L}_+$. Since this is the singular point of the q-bic curve C at the base of the cone, it induces an isomorphism $\pi \colon C \setminus \mathbf{P}\bar{L}_+ \to \mathbf{P}W \setminus \infty$. Given a point $y \in \mathbf{P}W \setminus \infty$, write ℓ_y for the line in X corresponding to the point $\pi^{-1}(y)$; write $\ell_\infty := \ell_+$. Then since planes through ℓ_+ in $\mathbf{P}V$ correspond to lines through $\mathbf{P}\bar{L}_+$ in $\mathbf{P}\bar{V}$, it follows that

$$X_{\mathbf{P}\psi,y} = \ell_y \cup q\ell_+ \quad \text{for all } y \in \mathbf{P}W. \qquad ■$$

3.15.3. Cone points. — As in 3.14.2, the set of cone points of X is given by $\ell_- \cup \ell_+$. Let $x := \mathbf{P}L$ be a cone point. If x is not the vertex of X, then there is a unique associated cone and it is given by

$$\begin{cases} X \cap \mathbf{P}\langle L_-, \bar{W}\rangle = (q+1)\ell_- & \text{if } x \in \ell_- \setminus x_0, \text{ and} \\ X \cap \mathbf{P}\langle L_+, \bar{W}\rangle = (q+1)\ell_+ & \text{if } x \in \ell_+ \setminus x_0. \end{cases}$$

When x is the vertex of X, then any plane through x will intersect X at a q-bic curve which is a cone, thereby witnessing it as a cone point.

Since the q-bic curve C at the base of the cone contains no lines, every line in X passes through the vertex. Embedding $\mathbf{P}\bar{V}$ as the subscheme of $\mathbf{G}(2,V)$ parameterizing lines through x_0, this gives the first statement of:

3.15.4. Proposition. — *The Fano scheme $\mathbf{F}_1(X)$ satisfies, as subschemes of $\mathbf{G}(2,V)$,*

$$\mathbf{F}_1(X)_{\mathrm{red}} = \mathbf{F}_1(X) \cap \mathbf{P}\bar{V} = C,$$

is generically a q-order neighbourhood of C, and has embedded points of multiplicities $q+1$ and $q^3(q+1)$ supported on $[\ell_-]$ and $[\ell_+]$, respectively.

Proof. — Choose coordinates $(x_0 : x_1 : x_2 : x_3)$ of $\mathbf{P}V = \mathbf{P}^3$ so that

$$X = \mathrm{V}(x_1^q x_2 + x_3^{q+1}) \subset \mathbf{P}^3.$$

Then $\mathbf{F}_1(X)$ is covered by the Plücker charts $\mathrm{D}(p_{01})$ and $\mathrm{D}(p_{02})$. Its coordinates and equations in the former are

$$\begin{pmatrix} 1 & 0 & a_2 & a_3 \\ 0 & 1 & b_2 & b_3 \end{pmatrix} \qquad a_3^{q+1} = a_3^q b_3 = a_2 + a_3 b_3^q = b_2 + b_3^{q+1} = 0.$$

Then $\mathbf{P}(V/L)$ is defined by $a_2 = a_3 = 0$ and $[\ell_-]$ is the origin. The last two equations eliminate a_2 and b_2. Equation two implies either $a_3^q = 0$ or $b_3 = 0$. The first case shows $\mathbf{F}_1(X)$ is a q-order neighbourhood of C. The second case gives $a_2 = b_2 = b_3 = a_3^{q+1} = 0$, yielding the embedded point of multiplicity $q+1$ on $[\ell_-]$.

The coordinates and equations of $\mathbf{F}_1(X)$ in $\mathrm{D}(p_{02})$ are given by

$$\begin{pmatrix} 1 & a_1 & 0 & a_3 \\ 0 & b_1 & 1 & b_3 \end{pmatrix} \qquad a_3^{q+1} = a_1^q + a_3^q b_3 = a_3 b_3^q = b_1^q + b_3^{q+1} = 0.$$

Then $\mathbf{P}(V/L)$ is defined by $a_1 = a_3 = 0$ and $[\ell_+]$ is the origin. The third equation implies either $a_3 = 0$ or $b_3^q = 0$. The first case implies $a_3 = a_1^q = 0$, again witnessing the q-order thickening of C. The second case implies $b_1^q = b_3^q = a_3^{q+1} = a_1^q + a_3^q b_3 = 0$, giving the embedded point of multiplicity $q^3(q+1)$ supported on $[\ell_+]$. ∎

Chapter 4

q-bic Threefolds

Whereas general geometric features of q-bic hypersurfaces described in Chapter 2 largely take on a quality akin to that of quadrics, finer geometric properties begin to resemble those of cubic hypersurfaces. Hints of this appeared in Chapter 3: there are three types of q-bic points, see 3.1.2; smooth q-bic curves have a collection of q^3+1 special points analogous to the flex points of smooth plane cubics, see 3.5.2 and 3.5.3; and smooth q-bic surfaces contain exactly $(q+1)(q^3+1)$ lines and are purely inseparably unirational, see 3.9.1 and 3.9.2.

Lines on threefolds lend further support to this analogy: as with cubic threefolds, q-bic threefolds have a smooth surface of lines and a certain intermediate Jacobian of the hypersurface is closely related to the Albanese variety of the surface of lines. This result and a basic geometric study of the surface of lines on a smooth q-bic threefold are the main objects of this Chapter: see Section 4.7, especially 4.7.21, 4.7.1, 4.7.7, and 4.7.27.

Smooth q-bic threefolds, however, are quite complicated and invariants of its surface of lines are difficult to access directly. Instead, computations are made indirectly via a degeneration method: study the family of Fano surfaces obtained via a carefully chosen degeneration of a smooth q-bic threefold to one of type $\mathbf{N}_2 \oplus \mathbf{1}^{\oplus 3}$; this is made possible using the global methods developed in Chapter 2. Thus the bulk of this Chapter is devoted to setting up this degeneration method, see Sections 4.2 and 4.3, and studying the geometry of q-bic threefolds of type $\mathbf{N}_2 \oplus \mathbf{1}^{\oplus 3}$, see Sections 4.4, 4.5, and 4.6.

Throughout this Chapter, $\mathbf{k}$ is an algebraically closed field of characteristic $p > 0$, X is the q-bic threefold associated with a q-bic form (V, β) of dimension 5, and $S := \mathbf{F}_1(X)$ denotes the Fano scheme of lines in X.

4.1. Generalities on q-bic threefolds

This Section begins with general comments on the geometry of q-bic threefolds, especially in relation to their scheme of lines. Further features in types $\mathbf{N}_2 \oplus \mathbf{1}^{\oplus 3}$ and $\mathbf{1}^{\oplus 5}$ will be given in Sections 4.4 and 4.7, respectively.

4.1.1. Scheme of lines. — The Fano scheme S of lines in X is a closed subscheme of the Grassmannian $\mathbf{G}(2, V)$ of expected dimension 2, see 2.7.3. When $\dim S = 2$, it is Cohen–Macaulay and its dualizing bundle is computed in 2.7.11 to be

$$\omega_S \cong \mathcal{O}_S(2q-3) \otimes \det(V^\vee)^{\otimes 2}$$

where $\mathcal{O}_S(1)$ is the Plücker line bundle. The degree of the Plücker bundle is obtained by taking $n = 4$ in the formula given in 2.7.19, yielding

$$\deg(\mathcal{O}_S(1)) = (q+1)^2(q^2+1).$$

The next statement computes the Chern numbers of S in the case X is smooth. It can be used to show that, if $q > 2$, then S does not lift to characteristic 0 as a lift would violate the Bogomolov–Miyaoka–Yau inequality of [Miy77, Theorem 4]. See, however, 4.7.3 below for a stronger result.

4.1.2. Proposition. — *Let X be a smooth q-bic threefold and let S be its Fano scheme of lines. Then $\mathcal{T}_S \cong \mathcal{S}^\vee \otimes \mathcal{O}_S(1-q)$ and the Chern numbers of S are*

$$\begin{aligned} \int_S c_1(\mathcal{T}_S)^2 &= (q+1)^2(q^2+1)(2q-3)^2, \text{ and} \\ \int_S c_2(\mathcal{T}_S) &= (q+1)^2(q^4-3q^3+4q^2-4q+3). \end{aligned}$$

Proof. — By 2.7.16, $\mathcal{T}_S \cong (\mathrm{Fr}^*(\mathcal{S})^\perp/\mathcal{S}) \otimes \mathcal{S}^\vee$ where the first tensor factor is the line bundle characterized by the short exact sequence

$$0 \to \mathrm{Fr}^*(\mathcal{S})^\perp/\mathcal{S} \to \mathcal{Q} \xrightarrow{\beta} \mathrm{Fr}^*(\mathcal{S})^\vee \to 0.$$

Taking determinants gives $\mathrm{Fr}^*(\mathcal{S})^\perp/\mathcal{S} \cong \mathcal{O}_S(1-q)$, and this gives the identification of the tangent bundle of S. The first Chern number is now easily obtained upon using the computation of $\deg(\mathcal{O}_S(1))$ recalled above in 4.1.1:

$$\int_S c_1(\mathcal{T}_S)^2 = \int_S c_1(\mathcal{O}_S(3-2q))^2 = (2q-3)^2 \deg(\mathcal{O}_S(1)) = (q+1)^2(q^2+1)(2q-3)^2.$$

For the second Chern number, observe

$$\begin{aligned} c_2(\mathcal{T}_S) &= c_2(\mathcal{S}^\vee) + c_1(\mathcal{S}^\vee)c_1(\mathcal{O}_S(1-q)) + c_1(\mathcal{O}_S(1-q))^2 \\ &= c_2(\mathcal{S}^\vee) + (q-2)(q-1)c_1(\mathcal{O}_S(1))^2. \end{aligned}$$

A general section of $\mathcal{S}^\vee$ cuts out the subscheme consisting of lines contained in a general hyperplane section of X. But such a slice is a smooth q-bic surface by 2.1.4. Thus the degree of $c_2(\mathcal{S}^\vee)$ on S is $(q+1)(q^3+1)$ by 3.8.3. Therefore

$$\begin{aligned} \int_S c_2(\mathcal{T}_S) &= (q+1)(q^3+1) + (q-2)(q-1)(q+1)^2(q^2+1) \\ &= (q+1)^2(q^4-3q^3+4q^2-4q+3) \end{aligned}$$

upon once again using the computation of the degree of $\mathcal{O}_S(1)$. ■

The Chern numbers can now be used to compute the Euler characteristic of the structure sheaf $\mathcal{O}_S$ whenever S is smooth; this holds more generally, whenever S has expected dimension, by comparing with the Koszul resolution given in 2.7.8.

4.1.3. Proposition. — *Let X be a q-bic threefold and assume that its Fano scheme S of lines is of expected dimension* 2*. Then*

$$\chi(S, \mathcal{O}_S) = \frac{1}{12}(q+1)^2(5q^4 - 15q^3 + 17q^2 - 16q + 12).$$

Proof. — When X is a smooth q-bic threefold, S is a smooth surface by 2.7.16, so Noether's formula, see [Ful98, Example 15.2.2], gives the first equality in

$$\chi(S,\mathcal{O}_S) = \frac{1}{12}\int_S c_1^2(\mathcal{T}_S) + c_2(\mathcal{T}_S) = \sum\nolimits_{i=0}^4 (-1)^i \chi(\mathbf{G}(2,V), \wedge^i \mathrm{Fr}^*(\mathcal{S}) \otimes \mathcal{S}).$$

Substituting the Chern number computations from 4.1.2 gives the formula in the statement. The second equality arises from the Koszul resolution from 2.7.8. Since the Koszul resolution persists whenever $\dim S = 2$, this gives the general case. ■

4.2. Cone Situation for threefolds

The goal of this Section is to identify and study a geometric situation in which there is a canonical rational map $S \dashrightarrow C$ from the Fano scheme S to a certain q-bic curve C. A canonical resolution of this rational map is constructed in 4.2.21, and equations for the intervening spaces are constructed in 4.2.15 and 4.2.26. The most important cases of the general situation are further studied in 4.3.

4.2.1. Cone Situation. — Let $(X,\infty,\mathbf{P}W)$ be a triple consisting of a q-bic threefold X, a point $\infty = \mathbf{P}L$ of X, and a hyperplane $\mathbf{P}W$ such that $X \cap \mathbf{P}W$ is a cone with vertex ∞ over a reduced q-bic curve $C \subset \mathbf{P}(W/L)$. This may sometimes be considered with some of the following additional assumptions:

- (i) there does not exist a $\mathbf{P}^2$ contained in X which passes through ∞; or
- (ii) ∞ is a smooth point of X; or
- (iii) C is a smooth q-bic curve.

Conditions (ii) and (iii) together imply (i). Condition (ii) is easy to appreciate:

4.2.2. Lemma. — *Let $(X,\infty,\mathbf{P}W)$ be a Cone Situation 4.2.1. Then*

$$W \subseteq \mathrm{Fr}^{-1}(L^\perp) \cap \mathrm{Fr}^*(L)^\perp.$$

In particular, $\mathbf{P}W \subseteq \mathbf{T}_{X,\infty}$ with equality if and only if $(X,\infty,\mathbf{P}W)$ satisfies 4.2.1(ii).

Proof. — The containment follows from 2.4.3. Then 2.2.9 implies $\mathbf{P}W$ is contained in the embedded tangent space of X at ∞. Equality holds if and only if $\mathbf{T}_{X,\infty}$ is a hyperplane, and this is precisely when ∞ is a smooth point of X. ■

The Cone Situation identifies the locus $C_{\infty,\mathbf{P}W}$ in S parameterizing the lines in $\mathbf{P}W$ which pass through ∞ with the curve C. With some of the further conditions on the Cone Situation, this identification can be made even more precise, as follows:

4.2.3. Lemma. — *Let $(X,\infty,\mathbf{P}W)$ be a Cone Situation over the q-bic curve C. Then there exists a canonical closed immersion $C \hookrightarrow S$ identifying C with the closed subscheme*

$$C_{\infty,\mathbf{P}W} := \{\,[\ell] \in S \mid \infty \in \ell \subset \mathbf{P}W\,\}$$

of lines in X containing ∞ and contained in $\mathbf{P}W$. Moreover, the Plücker line bundle $\mathcal{O}_S(1)$ restricts to the given polarization $\mathcal{O}_C(1)$. If $(X,\infty,\mathbf{P}W)$ furthermore satisfies

(i) 4.2.1(i), *then, as closed subsets of S,*

$$C_{\infty,\mathbf{P}W} = \{\,[\ell] \in S \mid \infty \in \ell\,\} = \{\,[\ell] \in S \mid \ell \subset \mathbf{P}W\,\};$$

(ii) 4.2.1(ii), *then $C_{\infty,\mathbf{P}W}$ coincides with the subscheme C_∞ of lines in X through ∞;*

(iii) 4.2.1(ii) *and* 4.2.1(iii), *then $\mathcal{N}_{C/S} \cong \mathcal{O}_C(-q+1)$.*

Proof. — The locus of lines in $\mathbf{G}(2,V)$ contained in $\mathbf{P}W$ and passing through ∞ is given by a linearly embedded $\mathbf{P}(W/L)$; in terms of the Plücker embedding, this is given by $-\wedge L \colon \mathbf{P}(W/L) \hookrightarrow \mathbf{P}(\wedge^2 V)$. The restriction of the tautological subbundle of $\mathbf{G}(2,V)$ to $\mathbf{P}(W/L)$ fits into a short exact sequence

$$0 \to L_{\mathbf{P}(W/L)} \to \mathcal{S}|_{\mathbf{P}(W/L)} \to \mathcal{O}_{\mathbf{P}(W/L)}(-1) \to 0.$$

Moreover, $\mathcal{S}|_{\mathbf{P}(W/L)}$ is a subbundle of $W_{\mathbf{P}(W/L)}$. In particular, the restriction of the q-bic form β contains L in its kernel. Thus the restriction to $\mathbf{P}(W/L)$ of the equations of S in $\mathbf{G}(2,V)$ factors through the quotient by L and yields the equations of C in $\mathbf{P}(W/L)$. This establishes the first statements.

Statement (i) will be established through its contrapositive. If there were a line $\ell \subset X \cap \mathbf{P}W$ that did not contain ∞, then since $X \cap \mathbf{P}W$ is a cone over ∞, the plane

$\langle \infty, \ell \rangle$ spanned by ∞ and ℓ is contained in X. Likewise, if there were a line ℓ which contains ∞ but not contained in $\mathbf{P}W$, then 4.2.2 implies that

$$\mathrm{Fr}^{-1}(L^{\perp}) = \mathrm{Fr}^{*}(L)^{\perp} = V.$$

The characterization 2.4.1 of q-bics which are cones implies that X is a cone over a q-bic surface with vertex ∞. But, by 2.7.3, every q-bic surface contains a line, so taking the span of any such line with ∞ yields a plane in X through ∞.

For (ii), consider the $\mathbf{P}(V/L)$ embedded linearly in $\mathbf{G}(2,V)$ parameterizing lines through ∞. Then $L_{\mathbf{P}(V/L)}$ is a subbundle of $\mathcal{S}|_{\mathbf{P}(V/L)}$. View S as the vanishing locus of the morphism $\mathcal{S} \to \mathrm{Fr}^{*} \mathcal{S}^{\vee}$ induced by β. Upon restriction to $\mathbf{P}(V/L)$, it follows that the morphism

$$\mathcal{S} \to \mathrm{Fr}^{*} \mathcal{S}^{\vee} \to \mathrm{Fr}^{*} L^{\vee}$$

must vanish; in other words, $\mathcal{S}|_{\mathbf{P}(V/L)}$ is contained in the subspace $\mathrm{Fr}^{*}(L)^{\perp} \subset V$. But 4.2.2 implies that $W = \mathrm{Fr}^{*}(L)^{\perp}$, and so the first statement of the Lemma implies $\mathbf{P}(V/L) \cap S = C$ as schemes.

For (iii), consider the normal bundle sequence

$$0 \to \mathcal{T}_{C} \to \mathcal{T}_{S}|_{C} \to \mathcal{N}_{C/S} \to 0.$$

This is exact since the assumptions together with 2.7.7 imply that S is smooth along C. Taking determinants and applying 2.7.11 then gives the normal bundle. ∎

4.2.4. Examples. — The following are some examples which illustrate the import of the various additional assumptions.

(i) Let X be a smooth q-bic threefold and let $\infty \in X$ be a cone point as in 2.4.8. Then $(X, \infty, \mathbf{T}_{X,\infty})$ is a Cone Situation, and all Cone Situations for smooth X are obtained this way. All the conditions (i), (ii), and (iii) are satisfied.

In the next three examples, let X be a q-bic threefold of type $\mathbf{N}_2 \oplus \mathbf{1}^{\oplus 3}$. For concreteness, choose coordinates so that

$$X = \mathrm{V}(x_0^q x_1 + x_2^{q+1} + x_3^q x_4 + x_3 x_4^q) \subset \mathbf{P}^4.$$

Set $C := \mathrm{V}(x_2^{q+1} + x_3^q x_4 + x_3 x_4^q)$ in the $\mathbf{P}^2$ in which x_0 and x_1 are projected out.

(ii) Let $x_- := (1:0:0:0:0)$. Then $(X, x_-, \mathbf{T}_{X,x_-})$ is a Cone Situation over the curve C satisfying each of the conditions (i), (ii), and (iii).

(iii) Let $x_+ := (0:1:0:0:0)$. Then $(X, x_+, \mathrm{V}(x_0))$ is a Cone Situation over the curve C satisfying (i) and (iii), but not (ii).

(iv) Let $\infty := (0:0:0:0:1)$. Then $(X, \infty, \mathbf{T}_{X,\infty})$ is a Cone Situation over the curve $\mathrm{V}(x_0^q x_1 + x_2^{q+1})$ satisfying (i) and (ii), but not (iii).

The remaining examples illustrate the generality of the Cone Situation.

(v) Let $X := \mathrm{V}(x_0^q x_1 + x_1^q x_2 + x_3^q x_4)$ and $\infty := (0:0:0:1:0)$. Then $(X, \infty, \mathbf{T}_{X,\infty})$ is a Cone Situation over $\mathrm{V}(x_0^q x_1 + x_1^q x_2)$ satisfying (ii), but not (i) nor (iii).

(vi) Let X be of type $\mathbf{0} \oplus \mathbf{1}^{\oplus 4}$, let ∞ be the singular point of X, and let $\mathbf{P}W$ be a general hyperplane through ∞. Then $(X, \infty, \mathbf{P}W)$ is a Cone Situation satisfying (iii), but not (i) nor (ii).

(vii) Let X be of type $\mathbf{0}^{\oplus 2} \oplus \mathbf{N}_2 \oplus \mathbf{1}$, let ∞ be any point of the vertex of X, and let $\mathbf{P}W$ be any hyperplane intersecting the vertex of X exactly at ∞. Then $(X, \infty, \mathbf{P}W)$ is a Cone Situation satisfying none of (i), (ii), nor (iii).

A Cone Situation $(X, \infty, \mathbf{P}W)$ gives a canonical way to transform lines contained in X to points of C. Namely, a line $\ell \subset X$ neither contained in $\mathbf{P}W$ nor containing ∞ determines a point of C in the following two equivalent ways:

$$\mathrm{proj}_\infty(\ell \cap \mathbf{P}W) = \mathrm{proj}_\infty(\ell) \cap \mathbf{P}(W/L) \in C.$$

Here $\mathrm{proj}_\infty \colon \mathbf{P}V \to \mathbf{P}(V/L)$ is the linear projection away from $\infty = \mathbf{P}L$. Using projective geometry of the ambient Grassmannains, this construction may be refined to a rational map $S \dashrightarrow C$ from the Fano scheme S. The constructions of **A.3** then provide geometric methods to resolve this map, essentially by tracking the intermediate points $\ell \cap \mathbf{P}W$ and lines $\mathrm{proj}_\infty(\ell)$ created in the process.

4.2.5. Cone Situation to Subquotient Situation. — Let $(X, \infty, \mathbf{P}W)$ be a Cone Situation **4.2.1** over a q-bic curve C. The data produces a Subquotient Situation

A.3.1 for the ambient Grassmannian:

$$\begin{tikzcd} V \arrow[r, two heads] & V/L \\ W \arrow[u, hook] \arrow[r, two heads] & W/L \arrow[u, hook] \end{tikzcd} \quad \text{yielding maps} \quad \begin{tikzcd} S \subset \mathbf{G}(2,V) \arrow[r, dashed] \arrow[d, dashed] & \mathbf{G}(2,V/L) \arrow[d, dashed] \\ \mathbf{P}W \arrow[r, dashed] & \mathbf{P}(W/L) \supset C \end{tikzcd}$$

where the horizontal maps are given by linear projection, and the vertical maps are given by intersection with a hyperplane; compare with A.3.2. Let $\mathbf{P}W^\circ := \mathbf{P}W \setminus \{\infty\}$,

$$\mathbf{G}(2,V)^\circ := \{\, [\ell] \mid \infty \notin \ell \text{ and } \ell \not\subset \mathbf{P}W \,\}, \text{ and } \mathbf{G}(2,V/L)^\circ := \{\, [\ell_0] \mid \ell_0 \not\subset \mathbf{P}(W/L) \,\},$$

so that the rational maps restrict to morphisms

$$\mathbf{G}(2,V)^\circ \to \mathbf{P}W^\circ \times_{\mathbf{P}(W/L)} \mathbf{G}(2,V/L)^\circ \to \mathbf{P}(W/L).$$

Since $X \cap \mathbf{P}W$ is a cone over ∞ with base C, restricting this to $S^\circ := S \cap \mathbf{G}(2,V)^\circ$ yields a morphism $S^\circ \to C$. In many cases, this gives a rational map $S \dashrightarrow C$:

4.2.6. Lemma. — *Assume the Cone Situation $(X,\infty,\mathbf{P}W)$ satisfies 4.2.1(i). Then there is a rational map $\varphi \colon S \dashrightarrow C$ given on points $[\ell] \in S^\circ$ by*

$$\varphi([\ell]) = \mathrm{proj}_\infty(\ell \cap \mathbf{P}W) = \mathrm{proj}_\infty(\ell) \cap \mathbf{P}(W/L)$$

where $\mathrm{proj}_\infty \colon \mathbf{P}V \dashrightarrow \mathbf{P}(V/L)$ is linear projection away from ∞.

Proof. — That φ exists follows from 4.2.3(i), as it implies that the general line in X neither contains ∞ nor is contained in $\mathbf{P}W$. ■

Some hypothesis on $(X,\infty,\mathbf{P}W)$ is necessary to obtain a rational map $S \dashrightarrow C$. Indeed, $S^\circ \to C$ does not induce a rational map $S \dashrightarrow C$ in example 4.2.4(vi), since no open subset of the irreducible component of S parameterizing lines through ∞ is contained in S°.

4.2.7. — Towards a resolution of this map, let

$$\mathbf{P}^\circ := (\mathbf{P}W^\circ \times_{\mathbf{P}(W/L)} \mathbf{G}(2,V/L)^\circ)|_C$$

and view its points as triples $(y \mapsto y_0 \in \ell_0)$ consisting of points $y \in \mathbf{P}W^\circ$, $y_0 \in C$, and $[\ell_0] \in \mathbf{G}(2, V/L)^\circ$ such that $\mathrm{proj}_\infty(y) = y_0$ and $y_0 \in \ell_0$. Let

$$T^\circ := \mathrm{image}(S^\circ \to \mathbf{P}^\circ).$$

Given a line $\ell_0 \subset \mathbf{P}(V/L)$, write $P_{\ell_0} := \langle \ell_0, \infty \rangle := \mathrm{proj}_\infty^{-1}(\ell_0)$ for the plane in $\mathbf{P}V$ spanned by ℓ_0 and ∞. Then $X_{\ell_0} := X \cap P_{\ell_0}$ is a q-bic curve, possibly defined by the zero form. The points of T° may be characterized in terms of the geometry of X_{ℓ_0}:

4.2.8. Lemma. — $T^\circ = \left\{ (y \mapsto y_0 \in \ell_0) \in \mathbf{P}^\circ \;\middle|\; X_{\ell_0} \textit{ is a cone over } y \right\}$.

Proof. — Consider a point $(y \mapsto y_0 \in \ell_0)$. If $P_{\ell_0} \subset X$, then any line $\ell \subset P_{\ell_0} \setminus \{\infty\}$ through $y \in X$ witnesses the inclusion $(y \mapsto y_0 \in \ell_0) \in T^\circ$. In the case $P_{\ell_0} \not\subset X$, the intersection X_{ℓ_0} is a q-bic curve that contains the line $\langle y, \infty \rangle$ as an irreducible component. But

$$\langle y, \infty \rangle = \mathrm{proj}_\infty^{-1}(y_0) = \mathrm{proj}_\infty^{-1}(\ell_0 \cap \mathbf{P}(W/L)) = P_{\ell_0} \cap \mathbf{P}W$$

so $(y \mapsto y_0 \in \ell_0)$ is contained in T° if and only if the residual curve $X_{\ell_0} - \langle y, \infty \rangle$ contains a line passing through y. By the classification of q-bic curves in 3.4.1, this happens if and only if X_{ℓ_0} is a cone and y is the vertex of the cone. ■

This analysis also gives a geometric criterion for when $S^\circ \to T^\circ$ has finite fibres:

4.2.9. Lemma. — *Assume the Cone Situation $(X, \infty, \mathbf{P}W)$ satisfies 4.2.1(i). Then the morphism $S^\circ \to T^\circ$ is quasi-finite of degree q.*

Proof. — The proof of 4.2.8 shows that the fibre of $S^\circ \to T^\circ$ over a point $(y \mapsto y_0 \in \ell_0)$ is the scheme parameterizing lines in $X_{\ell_0} = X \cap P_{\ell_0}$ which are distinct from $\langle y, \infty \rangle$. If X does not contain any planes passing through ∞, then X_{ℓ_0} is a cone with vertex over q-bic points and the scheme of lines in question is canonically the scheme of q-bic points with a single point removed. ■

4.2.10. The scheme P. — The rational maps of 4.2.5 can be resolved following the methods of A.3. To begin, by A.3.5, the restriction of the rational maps from $\mathbf{P}W$ and

$\mathbf{G}(2, V/L)$ to C are resolved on the product of projective bundles $\mathbf{P} := \mathbf{P}\mathscr{V}_1 \times_C \mathbf{P}\mathscr{V}_2$ over C, where $\mathscr{V}_1$ and $\mathscr{V}_2$ may be defined via the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{V}_1 \arrow[rr] \arrow[dr] \arrow[dd, two heads] & & V_C \arrow[rr] \arrow[dr, two heads] & & \mathscr{V}_2 \arrow[r] & 0 \\
& & W_C \arrow[ur, hook] \arrow[dr, two heads] & & (V/L)_C \arrow[ur, two heads] & & \\
0 \arrow[r] & \mathscr{O}_C(-1) \arrow[rr] & & (W/L)_C \arrow[rr] \arrow[ur, hook] & & \mathscr{T}_{\mathbf{P}(W/L)}(-1)|_C \arrow[r] \arrow[uu, hook] & 0
\end{tikzcd}$$

as the pullback of the left square and pushout of the right square, respectively. Thus they fit into split short exact sequences

$$0 \to L_C \to \mathscr{V}_1 \to \mathscr{O}_C(-1) \to 0 \quad \text{and} \quad 0 \to \mathscr{T}_{\mathbf{P}(W/L)}(-1)|_C \to \mathscr{V}_2 \to (V/W)_C \to 0.$$

Comparing the description of the functor represented by $\mathbf{P}$ given in A.3.3 with the definition of $\mathbf{P}^\circ$ in 4.2.7 shows that $\mathbf{P}^\circ$ is a dense open affine subbundle of $\mathbf{P}$, and its complement is the projective subbundle

$$\mathbf{P} \setminus \mathbf{P}^\circ := \{ (\infty \mapsto y_0 \in \ell_0) \mid \ell_0 \subset \mathbf{P}(W/L) \} = \mathbf{P}L_C \times_C \mathbf{P}(\mathscr{T}_{\mathbf{P}(W/L)}(-1)|_C).$$

To give equations for this closed complement, let $\pi_i \colon \mathbf{P}\mathscr{V}_i \to C$ and $\pi \colon \mathbf{P} \to C$ be the structure morphisms, and for any $\mathscr{O}_{\mathbf{P}}$-module $\mathscr{F}$ and $a, b \in \mathbf{Z}$, write

$$\mathscr{F}(a,b) := \mathscr{F} \otimes \mathrm{pr}_1^* \mathscr{O}_{\pi_1}(a) \otimes \mathrm{pr}_2^* \mathscr{O}_{\pi_2}(b).$$

4.2.11. Lemma. — *Consider the morphisms of line bundles on* $\mathbf{P}$

$$u_1 \colon \mathscr{O}_{\mathbf{P}}(-1,0) \hookrightarrow \pi^*\mathscr{V}_1 \twoheadrightarrow \pi^*\mathscr{O}_C(-1)$$
$$u_2 \colon \mathscr{O}_{\mathbf{P}}(0,-1) \hookrightarrow \pi^*\mathscr{V}_2 \twoheadrightarrow (V/W)_{\mathbf{P}}$$

obtained by composing the Euler sections and the quotient map from the construction of $\mathscr{V}_1$ *and* $\mathscr{V}_2$*. Let*

$$u := \left(\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right) \colon \mathscr{O}_{\mathbf{P}} \to \mathscr{O}_{\mathbf{P}}(1,0) \otimes \pi^*\mathscr{O}_C(-1) \oplus \mathscr{O}_{\mathbf{P}}(0,1) \otimes (V/W)_{\mathbf{P}} =: \mathscr{E}_2.$$

Then $\mathbf{P} \setminus \mathbf{P}^\circ = \mathrm{V}(u)$.

Proof. — Consider a point $(y \mapsto y_0 \in \ell_0) \in \mathbf{P}$. Then u_1 vanishes on this point if and only if y is the point corresponding to the subbundle $L \subset \mathscr{V}_1$, which is precisely the point ∞ from 4.2.1. Similarly, u_2 vanishes on this point if and only if ℓ_0 is the line in $\mathbf{P}(V/L)$ spanned by y_0 and a direction at y_0 within $\mathbf{P}(W/L)$, so that $\ell_0 \subset \mathbf{P}(W/L)$. ■

4.2.12. Closure of T°. — The next few statements aim to find equations for the Zariski closure of T° in $\mathbf{P}$. A first approximation, using the geometric description of T° given in 4.2.8, is given in the next statement. Consider the locally free $\mathscr{O}_{\mathbf{P}}$-module $\mathscr{P}$ of rank 3 fitting into the diagram

$$\begin{tikzcd}
0 \arrow[r] & \pi^*\mathscr{V}_1 \arrow[r] & V_{\mathbf{P}} \arrow[r] & \pi^*\mathscr{V}_2 \arrow[r] & 0 \\
0 \arrow[r] & \pi^*\mathscr{V}_1 \arrow[r] \arrow[u, equal] & \mathscr{P} \arrow[r] \arrow[u, hook] & \mathscr{O}_{\mathbf{P}}(0,-1) \arrow[r] \arrow[u, hook, "\mathrm{eu}_{\pi_2}"] & 0
\end{tikzcd}$$

where the upper sequence is the pullback of that from 4.2.10. With the notation of 4.2.7, the fibre at a point $(y \mapsto y_0 \in \ell_0) \in \mathbf{P}$ of

- the bundle $\mathscr{P}$ is the subspace of V underlying the plane $P_{\ell_0} := \langle \infty, \ell_0 \rangle$,
- the subbundle $\pi^*\mathscr{V}_1$ is the line $\ell_{0,\infty} := \langle \infty, y_0 \rangle$, and
- the tautological subbundle $\mathscr{O}_{\mathbf{P}}(-1,0) \hookrightarrow \pi^*\mathscr{V}_1$ is the point y.

Let $\beta_{\mathscr{P}} \colon \mathrm{Fr}^*(\mathscr{P}) \otimes \mathscr{P} \to \mathscr{O}_{\mathbf{P}}$ be the restriction of the q-bic form β to $\mathscr{P}$. Since $\mathscr{V}_1$ parameterizes lines contained in X, it is isotropic for β and the restriction of the adjoints of $\beta_{\mathscr{P}}$ to $\mathscr{V}_1$ may be viewed as maps

$$\beta_{\mathscr{P}}|\pi^*\mathscr{V}_1 \xrightarrow{\beta|_{\mathscr{V}_1}} \pi^*\,\mathrm{Fr}^*\,\mathscr{V}_2^\vee \xrightarrow{\mathrm{eu}_{\pi_2}^{(q)}} \mathscr{O}_{\mathbf{P}}(0,q) \quad \text{and} \quad \beta_{\mathscr{P}}^\vee|\pi^*\,\mathrm{Fr}^*\,\mathscr{V}_1 \xrightarrow{\beta^\vee|_{\mathscr{V}_1}} \pi^*\mathscr{V}_2 \xrightarrow{\mathrm{eu}_{\pi_2}} \mathscr{O}_{\mathbf{P}}(0,1).$$

4.2.13. Lemma. — *The closed subscheme of $\mathbf{P}$ given by*

$$T' := \{\, (y \mapsto y_0 \in \ell_0) \,|\, X_{\ell_0} \textit{ is a cone over } y \,\}$$

satisfies the following:

(i) $T^\circ = T' \cap \mathbf{P}^\circ$,

(ii) $\mathbf{P} \setminus \mathbf{P}^\circ \subset T'$, *and*

(iii) T' *is the zero locus of a section* $v \colon \mathscr{O}_{\mathbf{P}} \to \mathscr{O}_{\mathbf{P}}(q,1) \oplus \mathscr{O}_{\mathbf{P}}(1,q)$.

Proof. — Item (i) follows directly from the geometric characterization of the points of T° from **4.2.8**. To see (ii), by **4.2.11**, a point of $\mathbf{P} \setminus \mathbf{P}^\circ$ is of the form $(\infty \mapsto y_0 \in \ell_0)$ where $\infty \in P_{\ell_0} \subset \mathbf{P}W$. Thus $X_{\ell_0} = X \cap P_{\ell_0}$ is a cone over ∞, so $\mathbf{P} \setminus \mathbf{P}^\circ \subset T'$. Finally, as for the equations of T' promised in (iii), the characterization of q-bic hypersurfaces which are cones, **2.4.1**, shows that T' is the locus in $\mathbf{P}$ on which $\mathcal{O}_{\mathbf{P}}(-1,0) \hookrightarrow \pi^*\mathcal{V}_1 \hookrightarrow \mathcal{P}$ is contained in the kernel of $\beta_{\mathcal{P}}$. The discussion in **4.2.12** shows that this happens precisely when the following two morphisms vanish:

$$v_1 := \beta_{\mathcal{P}}^\vee \circ \mathrm{eu}_{\pi_1}^{(q)} \colon \mathcal{O}_{\mathbf{P}}(-q,0) \hookrightarrow \mathrm{Fr}^*\pi^*\mathcal{V}_1 \twoheadrightarrow \mathcal{O}_{\mathbf{P}}(0,1)$$
$$v_2 := \beta_{\mathcal{P}} \circ \mathrm{eu}_{\pi_1} \colon \mathcal{O}_{\mathbf{P}}(-1,0) \hookrightarrow \pi^*\mathcal{V}_1 \twoheadrightarrow \mathcal{O}_{\mathbf{P}}(0,q).$$

Then $v = \binom{v_1}{v_2}$ is the sought after section. ∎

Thus T' contains the closure of T° in $\mathbf{P}$. However, **4.2.13**(ii) shows that T' contains $\mathbf{P} \setminus \mathbf{P}^\circ$, and this generally will be an irreducible component of T' not contained in the closure of T° in $\mathbf{P}$. The following factors the equations of $\mathbf{P} \setminus \mathbf{P}^\circ$ from those of T':

4.2.14. Lemma. — *The section $v \colon \mathcal{O}_{\mathbf{P}} \to \mathcal{O}_{\mathbf{P}}(q,1) \oplus \mathcal{O}_{\mathbf{P}}(1,q)$ factors through u as*

$$v \colon \mathcal{O}_{\mathbf{P}} \xrightarrow{u} \mathcal{E}_2 \xrightarrow{v'} \mathcal{O}_{\mathbf{P}}(q,1) \oplus \mathcal{O}_{\mathbf{P}}(1,q)$$

for some morphism v' of rank 2 vector bundles.

Proof. — On the one hand, **4.2.13**(ii) shows that $\mathbf{P} \setminus \mathbf{P}^\circ$ is a closed subscheme of T'; on the other hand, **4.2.11** and **4.2.13**(iii) show that both schemes are vanishing loci of a section of a vector bundle. Thus the factorization $v = v' \circ u$ exists by expressing local generators of the ideal of T' in terms of those of $\mathbf{P} \setminus \mathbf{P}^\circ$. ∎

The following gives a candidate for the Zariski closure of T° in $\mathbf{P}$:

4.2.15. Proposition. — *Let $T := T' \cap \mathrm{V}(\det(v'))$. Then T*

(i) *contains the Zariski closure of T° in $\mathbf{P}$, and*

(ii) *is the degeneracy locus of the map*

$$\phi := \binom{v'}{\wedge u} \colon \mathcal{E}_2 \to \mathcal{E}_1 := \mathcal{O}_{\mathbf{P}}(q,1) \oplus \mathcal{O}_{\mathbf{P}}(1,q) \oplus \det(\mathcal{E}_2)$$

where $\mathscr{E}_2 := \mathscr{O}_{\mathbf{P}}(1,0) \otimes \pi^*\mathscr{O}_C(-1) \oplus \mathscr{O}_{\mathbf{P}}(0,1) \otimes (V/W)$*.*

(iii) *If furthermore* $\dim S^\circ = 2$*, then* $\dim T = 2$*,* T *is connected, Cohen–Macaulay, and there is an exact complex of sheaves on* $\mathbf{P}$ *given by*

$$0 \longrightarrow \mathscr{E}_2(-q-1,-q-1) \xrightarrow{\phi} \mathscr{E}_1(-q-1,-q-1) \xrightarrow{\wedge^2\phi^\vee} \mathscr{O}_{\mathbf{P}} \longrightarrow \mathscr{O}_T \longrightarrow 0.$$

Proof. — For (i), by 4.2.13(i), it suffices to see that $\det(v')$ vanishes on T°. Since $u|_{T^\circ} \neq 0$ By 4.2.11 whereas $v|_{T^\circ} = 0$, the factorization $v = v' \circ u$ of 4.2.14 implies $v'|_{T^\circ}$ has rank at most 1. Thus $\det(v')|_{T^\circ} = 0$.

To express T as the degeneracy locus in (ii), observe first that the vanishing locus of $\det(v')$ is precisely the top degeneracy locus of $v' \colon \mathscr{E}_2 \to \mathscr{O}_{\mathbf{P}}(q,1) \oplus \mathscr{O}_{\mathbf{P}}(1,q)$. Next, observe that there is a short exact sequence

$$0 \to \mathscr{O}_{\mathbf{P}} \xrightarrow{u} \mathscr{E}_2 \xrightarrow{\wedge u} \det(\mathscr{E}_2) \to 0.$$

The factorization $v = v' \circ u$ from 4.2.14 means v vanishes on $\mathrm{V}(\det(v'))$ if and only if $\mathrm{image}(u) \subseteq \ker(v')$; by the short exact sequence, this is equivalent to $\ker(\wedge u) \subseteq \ker(v')$; finally, this is equivalent to the degeneracy of the map $\phi := \left(\begin{smallmatrix} v' \\ \wedge u \end{smallmatrix}\right)$.

Now to establish (iii). To see that $\dim T = 2$, note that its manifestation as a degeneracy locus already shows $\dim T \geq 2$; since $\mathbf{P} \setminus \mathbf{P}^\circ$ is of dimension 2, it suffices to see that $\dim T^\circ \leq 2$. But T° is an image of S° by 4.2.7, so this certainly is the case when $\dim S^\circ = 2$. Since C is assumed to be reduced, see 4.2.1, it is Cohen–Macaulay, and thus so is T by [**HE70**, Theorem 1]; see also [**Kem71**, Section 3, p.9] and [**Ful98**, Theorem 14.3(c)]. The complex in question is the Eagon–Northcott complex associated with ϕ; see [**Laz04**, Appendix B.2, sequence (EN_0)]. Exactness of the complex follows from [**Laz04**, Theorem B.2.2(ii)] since T is of expected dimension 2. Finally, connectedness of T follows from that of S, see 2.7.15. ■

The following gives some sufficient conditions in terms of the Cone Situation for the conclusion $\dim S^\circ = 2$, as in the hypothesis of 4.2.15(iii). It is not a complete characterization, however: a cone over a smooth q-bic surface has a 2-dimensional Fano scheme but does not carry a Cone Situation satisfying the hypotheses below.

4.2.16. Lemma. — *Assume the Cone Situation* $(X, \infty, \mathbf{P}W)$ *satisfies* 4.2.1(i) *and* 4.2.1(ii). *Then* $\dim S^\circ = \dim S = 2$.

Proof. — The assumptions imply that X does not have a vertex, that is, a point $x \in X$ such that $\langle x, y \rangle \subseteq X$ for all $y \in X$. Indeed, if x were such a point, then $x \neq \infty$ since ∞ is a smooth point. Choose a line $\ell \subset X \cap \mathbf{P}W$ through ∞ and which does not pass through x. Since x is a vertex, $\langle x, \ell \rangle$ would be a plane contained in X passing through ∞. Thus the vertex of X is empty. This also implies, via 2.4.2, that $\dim \mathrm{Sing}(X) = \mathrm{corank}(X) - 1 \leq 1$.

Consider any point $x = \mathbf{P}L \in \mathrm{Sing}(X)$. The lines in X through x are contained in

$$X \cap \mathbf{P}\,\mathrm{Fr}^*(L)^\perp \cap \mathbf{P}\,\mathrm{Fr}^{-1}(L^\perp)$$

see 2.4.4. Since x is a singular point, $\mathrm{Fr}^*(L)^\perp = V$ by 2.2.9. Since x is not a vertex, however, $\mathrm{Fr}^{-1}(L^\perp)$ is a hyperplane by 2.4.1. Thus lines in X through x are contained in the surface $X \cap \mathbf{P}\,\mathrm{Fr}^{-1}(L^\perp)$. By 2.4.7(i), this is a cone with vertex x over a q-bic curve. Therefore the subscheme of S parameterizing lines through x is of dimension 1. Since the singular locus of X is of dimension at most 1, this implies that the closed subscheme of S parameterizing lines in X through $\mathrm{Sing}(X)$ is dimension at most 2. Then 2.7.13 implies S has expected dimension 2. ∎

4.2.17. The morphism ν'. — Describing further properties of T will depend on understanding the morphism

$$\nu' := \begin{pmatrix} \nu'_{11} & \nu'_{12} \\ \nu'_{21} & \nu'_{22} \end{pmatrix} : \mathcal{O}_{\mathbf{P}}(1,0) \otimes \pi^*\mathcal{O}_C(-1) \oplus \mathcal{O}_{\mathbf{P}}(0,1) \otimes (V/W)_{\mathbf{P}} \to \mathcal{O}_{\mathbf{P}}(q,1) \oplus \mathcal{O}_{\mathbf{P}}(1,q)$$

constructed in 4.2.14. First, the components of ν may be written as

$$\nu_1 := \beta_{\mathscr{P}}^\vee \circ \mathrm{eu}_{\pi_1}^{(q)} = \mathrm{eu}_{\pi_2}^\vee \circ \beta^\vee \circ \mathrm{eu}_{\pi_1}^{(q)} \quad \text{and} \quad \nu_2 := \beta_{\mathscr{P}} \circ \mathrm{eu}_{\pi_1} = \mathrm{eu}_{\pi_2}^{\vee,(q)} \circ \beta \circ \mathrm{eu}_{\pi_1}$$

thanks to their construction in 4.2.13 together with the discussion of 4.2.12. Thus v_1 fits into a commutative diagram with exact columns given by

$$\begin{tikzcd}
 & \pi^*\mathscr{O}_C(-q) \arrow[r] & \pi^*(\mathscr{T}_{\mathbf{P}(W/L)}(-1)|_C^\vee) & \\
\mathscr{O}_{\mathbf{P}}(-q,0) \arrow[ur, "u_1^q"] \arrow[r, "\mathrm{eu}_{\pi_1}^{(q)}"] & \pi^*\operatorname{Fr}^*\mathscr{V}_1 \arrow[u, two heads] \arrow[r, "\beta^\vee"] & \pi^*\mathscr{V}_2^\vee \arrow[u, two heads] \arrow[r, "\mathrm{eu}_{\pi_2}^\vee"] & \mathscr{O}_{\mathbf{P}}(0,1) \\
 & \operatorname{Fr}^* L_{\mathbf{P}} \arrow[u, hook] \arrow[r] & (V/W)_{\mathbf{P}}^\vee \arrow[u, hook] \arrow[ur, "u_2"'] &
\end{tikzcd}$$

in which u_1 and u_2 are the components of u from 4.2.11; that $\beta_L^\vee$ factors through $(V/L)_{\mathbf{P}}^\vee$ is because the fibres of $\mathscr{T}_{\mathbf{P}(W/L)}(-1)$ are quotients of $W \subseteq \operatorname{Fr}^*(L)^\perp$. Similarly, v_2 fits into

$$\begin{tikzcd}
 & \pi^*\mathscr{O}_C(-1) \arrow[r, "\sigma_C"] & \pi^*\operatorname{Fr}^*(\mathscr{T}_{\mathbf{P}(W/L)}(-1)|_C^\vee) & \\
\mathscr{O}_{\mathbf{P}}(-1,0) \arrow[ur, "u_1"] \arrow[r, "\mathrm{eu}_{\pi_1}"] & \pi^*\mathscr{V}_1 \arrow[u, two heads] \arrow[r, "\beta"] & \pi^*\operatorname{Fr}^*\mathscr{V}_2^\vee \arrow[u, two heads] \arrow[r, "\mathrm{eu}_{\pi_2}^{\vee,(q)}"] & \mathscr{O}_{\mathbf{P}}(0,q) \\
 & L_{\mathbf{P}} \arrow[u, hook] \arrow[r] & \operatorname{Fr}^*(V/W)_{\mathbf{P}}^\vee \arrow[u, hook] \arrow[ur, "u_2^q"'] &
\end{tikzcd}$$

in which the morphism on top induced by β_C is the morphism σ_C from 2.2.12: indeed, the map β is restricted from V, and, comparing with the diagram of 4.2.10, the quotient $\mathscr{O}_C(-1)$ includes via the Euler section into V.

Each column in the diagram is split exact, see 4.2.10. Fix a splitting and write

$$\mathrm{eu}_{\pi_1}|_L \colon \mathscr{O}_{\mathbf{P}}(-1,0) \to L_{\mathbf{P}} \quad \text{and} \quad \mathrm{eu}_{\pi_2}|_{\mathscr{T}_{\mathbf{P}(W/L)}(-1)} \colon \mathscr{O}_{\mathbf{P}}(0,-1) \to \pi^*(\mathscr{T}_{\mathbf{P}(W/L)}(-1))$$

for the projection of the Euler sections to the subbundles of $\mathscr{V}_1$ and $\mathscr{V}_2$. The components of v' may now be described in terms of the morphisms appearing in the diagrams above:

4.2.18. Lemma. — *The components of* $v' \colon \mathscr{E} \to \mathscr{O}_{\mathbf{P}}(q,1) \oplus \mathscr{O}_{\mathbf{P}}(1,q)$ *are given by:*

$$\begin{aligned}
v'_{11} &= (\mathrm{eu}_{\pi_2}|_{\mathscr{T}_{\mathbf{P}(W/L)}(-1)})^\vee \circ \beta^\vee \circ u_1^{q-1} \colon \mathscr{O}_{\mathbf{P}}(1,0) \otimes \pi^*\mathscr{O}_C(-1) \to \mathscr{O}_{\mathbf{P}}(q,1),\\
v'_{12} &= \beta^\vee \circ (\mathrm{eu}_{\pi_1}|_L)^{(q)} \colon \mathscr{O}_{\mathbf{P}}(0,1) \otimes (V/W)_{\mathbf{P}} \to \mathscr{O}_{\mathbf{P}}(q,1),\\
v'_{21} &= (\mathrm{eu}_{\pi_2}|_{\mathscr{T}_{\mathbf{P}(W/L)}(-1)})^{\vee,(q)} \circ \sigma_C \colon \mathscr{O}_{\mathbf{P}}(1,0) \otimes \pi^*\mathscr{O}_C(-1) \to \mathscr{O}_{\mathbf{P}}(1,q),\\
v'_{22} &= u_2^{q-1} \circ \beta \circ (\mathrm{eu}_{\pi_1}|_L) \colon \mathscr{O}_{\mathbf{P}}(0,1) \otimes (V/W)_{\mathbf{P}} \to \mathscr{O}_{\mathbf{P}}(q,1).
\end{aligned}$$

Proof. — This follows from the factorization 4.2.14 and the diagrams of 4.2.17. ∎

The proof of 4.2.15(i) shows that $T^\circ = T \cap \mathbf{P}^\circ$. Thus a natural closed complement of T° in T is given by

$$T \setminus T^\circ := T \cap (\mathbf{P} \setminus \mathbf{P}^\circ) = \mathrm{V}(\det(v')) \cap (\mathbf{P} \setminus \mathbf{P}^\circ).$$

The computation of the components of v' shows that, at least when $\infty \in X$ is a smooth point, $T \setminus T^\circ$ is generically a purely inseparable multisection of $T \to C$:

4.2.19. Lemma. — *Assume the Cone Situation* $(X, \infty, \mathbf{P}W)$ *satisfies* 4.2.1(ii). *Then* $T \setminus T^\circ$ *is connected, of pure dimension* 1*, and there is a commutative diagram*

$$\begin{tikzcd}
 & T \setminus T^\circ \arrow[d, "\pi"] \\
C \arrow[ur, "\sigma_C"] \arrow[r, "\mathrm{Fr}"'] & C
\end{tikzcd}$$

in which σ_C *is a closed immersion and an isomorphism away from* $C \cap \mathbf{P}\,\mathrm{Fr}^*(W/L)^\perp$.

Proof. — The scheme $\mathbf{P} \setminus \mathbf{P}^\circ$ is the vanishing locus of u_1 and u_2 by 4.2.11. Thus the computation of v' from 4.2.18 shows that $\det(v')|_{\mathbf{P}\setminus\mathbf{P}^\circ}$ is, up to sign, the section

$$(\mathrm{eu}_{\pi_2}|_{\mathscr{T}_{\mathbf{P}(W/L)}(-1)})^{\vee,(q)} \circ \sigma_C \circ \beta^\vee \circ (\mathrm{eu}_{\pi_1}|_L)^{(q)} \colon \mathscr{O}_{\mathbf{P}\setminus\mathbf{P}^\circ} \to (\pi^*\mathscr{O}_C(-1) \otimes V/W)(q,q)|_{\mathbf{P}\setminus\mathbf{P}^\circ}.$$

Since ∞ is a smooth point, 2.2.7 implies that $\beta^\vee|_L \colon \mathrm{Fr}^* L_{\mathbf{P}} \to (V/W)^\vee_{\mathbf{P}}$ is an isomorphism. Since σ_C is also nonzero by 2.2.12, $\det(v')|_{\mathbf{P}\setminus\mathbf{P}^\circ}$ is a nonzero section of an ample line bundle so $T \setminus T^\circ = \mathrm{V}(\det(v')) \cap (\mathbf{P} \setminus \mathbf{P}^\circ)$ is of pure dimension 1 and, by [**Stacks**, 0FD9], is connected. The commutative diagram now comes from 2.2.12, together with the canonical isomorphism

$$\mathcal{T}_{\mathbf{P}(W/L)}(-1) \cong \Omega^1_{\mathbf{P}(W/L)}(2)$$

of rank 2 vector bundles arising from the wedge product pairing. ■

4.2.20. Corollary. — *Assume the Cone Situation* $(X, \infty, \mathbf{P}W)$ *satisfies* 4.2.1(ii) *and that* $\dim S^\circ = 2$. *Then* T *is the Zariski closure of* T° *in* $\mathbf{P}$.

Proof. — By 4.2.15(i), if T were not the Zariski closure in $\mathbf{P}$ of T°, then some irreducible component of $T \setminus T^\circ$ is an irreducible component of T. This is impossible: on the one hand, $T \setminus T^\circ$ is of pure dimension 1 by 4.2.19; on the other hand, T is connected and Cohen–Macaulay by 4.2.15(iii), and is therefore equidimension 2 by [**Stacks**, 00OV]. ■

4.2.21. — Assume the Cone Situation $(X, \infty, \mathbf{P}W)$ satisfies 4.2.1(i). Then, as in 4.2.6, the morphism $S^\circ \to T^\circ$ from 4.2.7 induces a rational map $S \dashrightarrow T$. This may be resolved as follows: Write $\mathcal{T}_{\pi_i}$ for the pullback to $\mathbf{P}$ of the relative tangent bundle of $\pi_i \colon \mathbf{P}\mathcal{V}_i \to C$, and let $\mathcal{P}$ be as in 4.2.12. Set

$$\mathcal{V} := \mathcal{H}\big(\mathcal{O}_{\mathbf{P}}(-1,0) \hookrightarrow V_{\mathbf{P}} \twoheadrightarrow \mathcal{T}_{\pi_2}(0,-1)\big) \cong \operatorname{coker}\big(\mathcal{O}_{\mathbf{P}}(-1,0) \hookrightarrow \pi^*\mathcal{V}_1 \hookrightarrow \mathcal{P}\big).$$

This is a bundle of rank 2 and points of its projective bundle are described as

$$\mathbf{P}\mathcal{V} = \big\{ \big((y \in \ell) \mapsto (y_0 \in \ell_0)\big) \,\big|\, (y \mapsto y_0 \in \ell_0) \in \mathbf{P} \text{ and } \ell \text{ a line in } P_{\ell_0} \big\}.$$

There exists a morphism $\mathbf{P}\mathcal{V} \to \mathbf{G}(2, V)$ given by projection onto the line ℓ which, by A.3.7, is an isomorphism onto its image away from the locus

$$\{ [\ell] \in \mathbf{G}(2, V) \mid \infty \in \ell \text{ or } \ell \subset \mathbf{P}W \}.$$

By 4.2.6, the Fano scheme S is contained in the image of $\mathbf{P}\mathcal{V}$ and is not completely contained in the non-isomorphism locus. Thus its strict transform $\tilde{S}$ along $\mathbf{P}\mathcal{V} \to \mathbf{G}(2, V)$ is well-defined, and the resulting map $\tilde{S} \to T$ resolves $S \dashrightarrow T$. In summary, there is a commutative diagram

$$\begin{tikzcd} S \arrow[dr, dashed] & \tilde{S} \arrow[l] \arrow[r, hook] \arrow[d] & \mathbf{P}\mathscr{V} \arrow[d] \\ & T \arrow[r, hook] \arrow[dr] & \mathbf{P} \arrow[d, "\pi"] \\ & & C\,. \end{tikzcd}$$

The next few paragraphs construct equations for $\tilde{S}$ in some cases by realizing it as a bundle of q-bic points over T. Let $\mathscr{V}_T$ be the restriction of $\mathscr{V}$ to T, so that $\mathbf{P}\mathscr{V} \times_{\mathbf{P}} T = \mathbf{P}\mathscr{V}_T$, and write $\rho \colon \mathbf{P}\mathscr{V}_T \to T$ for the projection. Then 4.2.9 implies that $\tilde{S}$ is a codimension 1 closed subscheme of $\mathbf{P}\mathscr{V}_T$. Let $(\mathscr{P}_T, \beta_{\mathscr{P}_T})$ be the restriction to T of the q-bic form $(\mathscr{P}, \beta_{\mathscr{P}})$ from 4.2.12.

4.2.22. Lemma. — *The q-bic form $\beta_{\mathscr{P}_T}$ induces a q-bic form $\beta_{\mathscr{V}_T} \colon \mathrm{Fr}^*(\mathscr{V}_T) \otimes \mathscr{V}_T \to \mathscr{O}_T$. The q-bic equation induced by $\beta_{\mathscr{V}_T}$, that is the map of line bundles on $\mathbf{P}\mathscr{V}_T$ given by*

$$\beta_{\mathscr{V}_T}(\mathrm{eu}_\rho^{(q)}, \mathrm{eu}_\rho) \colon \mathscr{O}_\rho(-q-1) \hookrightarrow \mathrm{Fr}^* \rho^* \mathscr{V}_T \otimes \rho^* \mathscr{V}_T \to \mathscr{O}_{\mathbf{P}\mathscr{V}_T},$$

vanishes at the point $((y \in \ell) \mapsto (y_0 \in \ell_0)) \in \mathbf{P}\mathscr{V}_T$ if and only if ℓ is an isotropic line for β. In particular, this section vanishes on the strict transform $\tilde{S}$ of S.

Proof. — By 4.2.8, for every point $(y \mapsto y_0 \in \ell_0) \in T^\circ$, the plane section $X_{\ell_0} = X \cap P_{\ell_0}$ is a cone over y; since being a cone is a closed condition, this holds for all points of T. Comparing the description of the tautological bundles given in 4.2.12 with the characterization of q-bic hypersurfaces which are cones given in 2.4.1, it follows that the tautological subbundle $\mathscr{O}_T(-1,0) \hookrightarrow \mathscr{P}_T$ lies in the kernel of the form $\beta_{\mathscr{P}_T}$. Since $\mathscr{V}_T \cong \mathscr{P}_T / \mathscr{O}_T(-1,0)$, as noted in 4.2.21, it passes to the quotient and induces the desired q-bic form $\beta_{\mathscr{V}_T}$. The statement about the zero locus of the resulting section comes from unwinding the construction. ■

4.2.23. — By A.3.7(ii), there is a canonical short exact sequence

$$0 \to \mathscr{T}_{\pi_1}(-1,0) \to \mathscr{V} \to \mathscr{O}_{\mathbf{P}}(0,-1) \to 0$$

in which the subbundle may be identified as

$$\begin{aligned}\mathcal{T}_{\pi_1}(-1,0) &= \operatorname{coker}(\mathrm{eu}_{\pi_1}\colon \mathcal{O}_{\mathbf{P}}(-1,0) \to \pi^*\mathcal{V}_1) \\ &\cong \det(\pi^*\mathcal{V}_1)(1,0) \cong \mathcal{O}_{\mathbf{P}}(1,0) \otimes \pi^*\mathcal{O}_C(-1) \otimes L.\end{aligned}$$

Its inverse image under the quotient map $\mathcal{P} \to \mathcal{V}$ is the subbundle $\pi^*\mathcal{V}_1 \subset \mathcal{P}$, so its points are those of $\mathbf{P}\mathcal{V}$ in which the line ℓ is that spanned by y_0 and ∞:

$$\mathbf{P}(\mathcal{T}_{\pi_1}(-1,0)) = \left\{ \big((y \in \ell) \mapsto (y_0 \in \ell_0)\big) \in \mathbf{P}\mathcal{V} \;\middle|\; \ell = \langle y_0, \infty\rangle \right\}.$$

Since $\ell = \langle y_0, \infty\rangle \subset \mathbf{P}W$ whenever $(y \mapsto y_0 \in \ell_0) \in T$, this subbundle is isotropic for $\beta_{\mathcal{V}_T}$ by 4.2.22, yielding the following observation:

4.2.24. Lemma. — *The q-bic equation $\beta_{\mathcal{V}_T}(\mathrm{eu}_\rho^{(q)}, \mathrm{eu}_\rho)$ from 4.2.22 vanishes on the subbundle $\mathbf{P}(\mathcal{T}_{\pi_1}(-1,0)|_T) \subset \mathbf{P}\mathcal{V}_T$ and so it factors through the section*

$$v_3 := u_3^{-1}\beta_{\mathcal{V}_T}(\mathrm{eu}_\rho^{(q)}, \mathrm{eu}_\rho)\colon \mathcal{O}_{\mathbf{P}\mathcal{V}_T} \to \mathcal{O}_\rho(q) \otimes \rho^*\mathcal{O}_T(0,1)$$

where $u_3\colon \mathcal{O}_\rho(-1) \to \rho^*\mathcal{V}_T \to \rho^*\mathcal{O}_T(0,-1)$ is the equation of the subbundle. ■

This also allows for a fine description of the type of $\beta_{\mathcal{V}_T}$ upon restriction to fibres:

4.2.25. Lemma. — *Assume the Cone Situation $(X, \infty, \mathbf{P}W)$ satisfies 4.2.1(i). The restriction of $\beta_{\mathcal{V}_T}$ to the fibre over $t = (y \mapsto y_0 \in \ell_0) \in T$ is of type*

(i) $\mathbf{1}^{\oplus 2}$ *if $P_{\ell_0} \not\subset \mathbf{P}\,\mathrm{Fr}^*(L)^\perp$ and $P_{\ell_0} \not\subset \mathbf{P}\,\mathrm{Fr}^{-1}(L^\perp)$;*

(ii) $\mathbf{N}_2$ *with $\mathbf{P}(\mathcal{T}_{\pi_1}(-1,0)|_t)$ reduced if $P_{\ell_0} \not\subset \mathbf{P}\,\mathrm{Fr}^*(L)^\perp$ and $P_{\ell_0} \subset \mathbf{P}\,\mathrm{Fr}^{-1}(L^\perp)$;*

(iii) $\mathbf{N}_2$ *with $\mathbf{P}(\mathcal{T}_{\pi_1}(-1,0)|_t)$ multiple if $P_{\ell_0} \subset \mathbf{P}\,\mathrm{Fr}^*(L)^\perp$ and $P_{\ell_0} \not\subset \mathbf{P}\,\mathrm{Fr}^{-1}(L^\perp)$; and*

(iv) $\mathbf{0} \oplus \mathbf{1}$ *if $P_{\ell_0} \subset \mathbf{P}\,\mathrm{Fr}^*(L)^\perp$ and $P_{\ell_0} \subset \mathbf{P}\,\mathrm{Fr}^{-1}(L^\perp)$.*

Proof. — The proof of 4.2.22 implies that the restriction $\beta_{\mathcal{V}_t}$ of $\beta_{\mathcal{V}_T}$ to the fibre over t is the q-bic form underlying the q-bic points obtained by projecting $X_{\ell_0} = X \cap P_{\ell_0}$ from its cone point y; note that the assumption on the Situation ensures that X_{ℓ_0} is of dimension 1. The subbundle $\bar{L}_t := \mathcal{T}_{\pi_1}(-1,0)|_t$ corresponds to the image of the line $\langle y_0, \infty\rangle \subset X_{\ell_0}$; equivalently, this is the image of the subspace L in the fibre $\mathcal{V}_t$.

The type of $\beta_{\mathscr{V}_t}$ may now be identified using classification of q-bic points 3.1.1 by examining the orthogonals of $\bar{L}_t$. In case (i), both its orthogonals of $\bar{L}_t$ are nontrivial and so $\beta_{\mathscr{V}_t}$ is of type $\mathbf{1}^{\oplus 2}$. In cases (ii) and (iii), exactly one of the orthogonals of $\bar{L}_t$ is nontrivial so $\beta_{\mathscr{V}_t}$ is of type $\mathbf{N}_2$; whether $\bar{L}_t$ underlies the smooth point is now determined by the identification of tangent spaces given in 2.2.9. Finally, in case (iv), $\bar{L}_t$ lies in the kernel of $\beta_{\mathscr{V}_t}$ and so the form is of type $\mathbf{0} \oplus \mathbf{1}$. ■

4.2.26. Lemma. — *Assume the Cone Situation* $(X, \infty, \mathbf{P}W)$ *satisfies* 4.2.1(i). *Then*

$$\tilde{S} \subseteq \mathrm{V}(v_3) \subseteq \mathbf{P}\mathscr{V}_T$$

and $\tilde{S} \to T$ *is finite flat of degree* q *onto its image.*

Proof. — The discussion of 4.2.21 and 4.2.23 together with 4.2.3(i) imply that the intersection of $\tilde{S}$ with the exceptional locus of $\mathbf{P}\mathscr{V} \to \mathbf{G}(2, V)$ is contained in $\mathbf{P}(\mathscr{T}_{\pi_1}(-1,0)|_T)$. Therefore v_3 vanishes on $\tilde{S}$ if and only if it vanishes on $S^\circ \cong \tilde{S} \setminus \mathbf{P}(\mathscr{T}_{\pi_1}(-1,0)|_T)$. Since u_3 is invertible on the latter open subscheme, v_3 vanishes on S° by 4.2.22.

Since $S^\circ \to T$ is quasi-finite of degree q by 4.2.9, the final statement follows upon showing that $\mathrm{V}(v_3) \to T$ is finite flat of degree q. Since v_3 is degree q on each fibre of $\mathbf{P}\mathscr{V}_T \to T$ by 4.2.24, it suffices to see that v_3 does not vanish on an entire fibre. But if v_3 did vanish on the fibre over $(y \mapsto y_0 \in \ell_0) \in T$, 4.2.22 would imply that all lines $\ell \subset P_{\ell_0}$ passing through y are isotropic for β, and hence P_{ℓ_0} would be contained in X. This is impossible with condition 4.2.1(i). ■

4.2.27. Corollary. — *Assume the Cone Situation* $(X, \infty, \mathbf{P}W)$ *satisfies* 4.2.1(i) *and* 4.2.1(ii). *Then*

(i) $\tilde{S} \to T$ *is surjective and finite flat of degree* q,

(ii) $\tilde{S} = \mathrm{V}(v_3)$, *and*

(iii) *there is a short exact sequence of bundles on* T *given by*

$$0 \to \mathscr{O}_T \to \rho_*\mathscr{O}_{\tilde{S}} \to \mathrm{Div}^{q-2}(\mathscr{V}_T)(1,-2) \otimes \pi^*\mathscr{O}_C(-1) \otimes L \to 0$$

and so $\rho_*\mathscr{O}_{\tilde{S}}$ has an increasing filtration whose graded pieces are

$$\mathrm{gr}_i(\rho_*\mathscr{O}_{\tilde{S}}) = \begin{cases} \mathscr{O}_T & \text{if } i = 0, \\ (\pi^*\mathscr{O}_C(-q+i)\otimes L^{\otimes q-i})(q-i,-i-1) & \text{if } 1 \leq i \leq q-1. \end{cases}$$

Proof. — The assumptions on the Cone Situation imply, by 4.2.16, that $\dim S = 2$. So 4.2.20 applies to show that T is the Zariski closure of T° in $\mathbf{P}$. Thus $\tilde{S} \to T$ is proper and dominant, whence surjective. That it is finite flat of degree q then follows from 4.2.26, establishing (i). This implies (ii) and, with 4.2.23, gives a short exact sequence of sheaves on $\mathbf{P}\mathscr{V}_T$:

$$0 \to \mathscr{O}_\rho(-q)\otimes\rho^*\mathscr{O}_T(0,-1) \xrightarrow{v_3} \mathscr{O}_{\mathbf{P}\mathscr{V}_T} \to \mathscr{O}_{\tilde{S}} \to 0.$$

Pushing this along ρ gives a short exact sequence of $\mathscr{O}_T$-modules

$$0 \to \mathscr{O}_T \to \rho_*\mathscr{O}_{\tilde{S}} \to \mathbf{R}^1\rho_*\mathscr{O}_\rho(-q)\otimes\mathscr{O}_T(0,-1) \to 0.$$

The Euler sequence gives $\omega_\rho \cong \rho^*\det(\mathscr{V}_T^\vee)\otimes\mathscr{O}_\rho(-2)$ and so by Grothendieck duality

$$\begin{aligned} \mathbf{R}^1\rho_*\mathscr{O}_\rho(-q) &\cong \mathbf{R}\rho_*\mathbf{R}\mathscr{H}om_{\mathscr{O}_{\mathbf{P}\mathscr{V}_T}}(\mathscr{O}_\rho(q)\otimes\omega_\rho,\omega_\rho) \\ &\cong \mathbf{R}\mathscr{H}om_{\mathscr{O}_T}(\mathbf{R}\rho_*\mathscr{O}_\rho(q-2)\otimes\det(\mathscr{V}_T^\vee),\mathscr{O}_T) \\ &\cong \mathrm{Div}^{q-2}(\mathscr{V}_T)\otimes\det(\mathscr{V}_T). \end{aligned}$$

The discussion of 4.2.23 shows $\det(\mathscr{V}_T) \cong (\pi^*\mathscr{O}_C(-1)\otimes L)(1,-1)$, and this gives the exact sequence of (iii). The filtration comes from applying divided powers to the short exact sequence for $\mathscr{V}_T$ in 4.2.23. ■

4.2.28. Equivariance. — The Subquotient Situation in 4.2.5 admits an action by the subgroup

$$\mathbf{Aut}(L \subset W \subset V) \subset \mathbf{GL}(V)$$

of linear automorphisms of V which preserve the flag $L \subset W \subset V$. Let

$$\mathbf{Aut}(L \subset W \subset V, \beta) := \mathbf{Aut}(L \subset W \subset V) \cap \mathbf{Aut}(V, \beta)$$

be the subgroup of automorphisms that further preserve the q-bic form β, see 1.3.1. Its linear action on the Subquotient Situation induces an action on each of the schemes X, S, $\tilde{S}$, T, and C.

In good situations, such as that considered in the following, the morphism $\tilde{S} \to T$ is a quotient map by the action by a finite subgroup scheme of $\mathbf{Aut}(L \subset W \subset V, \beta)$. Namely, let

$$\mathbf{Aut}_{\mathrm{uni}}(W \subset V) \subset \mathbf{Aut}(L \subset W \subset V) \subset \mathbf{GL}(V)$$

be the unipotent subgroup which induces the identity on W and V/W, and let

$$\mathbf{Aut}_{\mathrm{uni}}(W \subset V, \beta) := \mathbf{Aut}_{\mathrm{uni}}(W \subset V) \cap \mathbf{Aut}(V, \beta)$$

be the subgroup that preserves β. This is a subgroup scheme of $\mathbf{Aut}(L \subset W \subset V, \beta)$. Since this subgroup acts trivially on W, it acts trivially on C.

4.2.29. Lemma. — *Assume the Cone Situation $(X, \infty, \mathbf{P}W)$ satisfies 4.2.1(i) and 4.2.1(ii). Then*

$$\mathbf{Aut}_{\mathrm{uni}}(W \subset V, \beta) \cong \begin{cases} \boldsymbol{\alpha}_q & \text{if } L^\perp = \mathrm{Fr}^*(V), \text{ and} \\ \mathbf{F}_q & \text{if } L^\perp \neq \mathrm{Fr}^*(V). \end{cases}$$

Proof. — Consider a point g of $\mathbf{Aut}_{\mathrm{uni}}(W \subset V)$. Since g induces the identity on W, the endomorphism $g - \mathrm{id}_V$ factors through the quotient V/W; since g also induces the identity on V/W, the image of $g - \mathrm{id}_V$ is contained in W. Thus $g - \mathrm{id}_V$ induces a linear map $\delta_g : V/W \to W$ and the map $g \mapsto \delta_g$ yields an isomorphism of linear algebraic groups

$$\delta : \mathbf{Aut}_{\mathrm{uni}}(W \subset V) \to \mathbf{Hom}(V/W, W)$$

where $\mathbf{Hom}(V/W, W)$ is viewed as a vector group.

Observe that δ maps $\mathbf{Aut}_{\mathrm{uni}}(W \subset V, \beta)$ into $\mathbf{Hom}(V/W, L)$. Indeed, let A be any $\mathbf{k}$-algebra. Then for any $g \in \mathbf{Aut}_{\mathrm{uni}}(W \subset V, \beta)(A)$,

$$\beta(v^{(q)}, w) = \beta((g \cdot v)^{(q)}, w) \quad \text{and} \quad \beta(w^{(q)}, v) = \beta(w^{(q)}, g \cdot v)$$

for all $w \in W \otimes_{\mathbf{k}} A$ and $v \in V \otimes_{\mathbf{k}} A$. Rearranging the equations shows that

$$\delta_g(v) \in (\mathrm{Fr}^*(W)^\perp \cap \mathrm{Fr}^{-1}(W^\perp)) \otimes_{\mathbf{k}} A.$$

The assumptions on the Situation imply that the intersection is $L \otimes_{\mathbf{k}} A$, as required.

To construct equations of $\mathbf{Aut}_{\mathrm{uni}}(W \subset V, \beta)$ in $\mathbf{Hom}(V/W, L)$, fix a nonzero $w \in L$, and choose $v \in V$ such that its image $\bar{v} \in V/W$ is nonzero. This induces an isomorphism

$$\mathbf{Hom}(V/W, L) \to \mathbf{G}_a : (\bar{v} \mapsto tw) \mapsto t.$$

Let $g \in \mathbf{Aut}_{\mathrm{uni}}(W \subset V, \beta)(A)$ be mapped to $t \in \mathbf{G}_a(A)$. Then

$$0 = \beta((g \cdot v)^{(q)}, g \cdot v) - \beta(v^{(q)}, v) = \beta(w^{(q)}, v)t^q + \beta(v^{(q)}, w)t.$$

Since ∞ is a smooth point, $\mathrm{Fr}^*(L)^\perp = W$ by 2.2.9 and so $\beta(w^{(q)}, v)$ is a nonzero scalar. Now if $L^\perp = \mathrm{Fr}^*(V)$, then $\beta(v^{(q)}, w) = 0$ and $\mathbf{Aut}_{\mathrm{uni}}(W \subset V, \beta) \cong \boldsymbol{\alpha}_q$. If $L^\perp \neq \mathrm{Fr}^*(V)$, however, then $L^\perp = \mathrm{Fr}^*(W)$ by 4.2.2, and so $\beta(v^{(q)}, w)$ is a nonzero scalar. This implies $\mathbf{Aut}_{\mathrm{uni}}(W \subset V, \beta) \cong \mathbf{F}_q$. ■

The proof of 4.2.29 shows a bit more:

4.2.30. Corollary. — *Assume the Cone Situation $(X, \infty, \mathbf{P}W)$ satisfies 4.2.1(i) and 4.2.1(ii). Then the action of $\mathbf{Aut}_{\mathrm{uni}}(W \subset V, \beta)$*

(i) *is trivial on T, and*

(ii) *preserves the short exact sequence of $\mathcal{O}_T$-modules*

$$0 \to \mathcal{T}_{\pi_1}(-1, 0)|_T \to \mathcal{V}_T \to \mathcal{O}_T(0, -1) \to 0,$$

and acts via the identity on the sub and quotient line bundles.

Proof. — In fact, $G := \mathbf{Aut}_{\mathrm{uni}}(W \subset V, \beta)$ acts trivially on the bundle $\mathbf{P} = \mathbf{P}\mathcal{V}_1 \times_C \mathbf{P}\mathcal{V}_2$ containing T. Indeed, G acts diagonally on the product, so it suffices to see that G acts trivially on each of $\mathbf{P}\mathcal{V}_1$ and $\mathbf{P}\mathcal{V}_2$. On the one hand, the proof of 4.2.29 shows that the nontrivial component of the action of G maps V/W into L; on the other hand, the short exact sequences of 4.2.10 show that L is only contained in $\mathcal{V}_1$ and V/W is only a quotient of $\mathcal{V}_2$. Thus G acts trivially on both $\mathcal{V}_1$ and $\mathcal{V}_2$. This gives (i).

Item (ii) now follows from the same considerations upon examining the construction of $\mathscr{V}$ from 4.2.21. ■

4.3. Smooth Cone Situation

This Section is devoted to studying a Cone Situation $(X,\infty,\mathbf{P}W)$ which satisfies the conditions 4.2.1(ii) and 4.2.1(iii); that is, when $\infty \in X$ is a smooth point, and the base C of the cone $X \cap \mathbf{P}W$ is smooth q-bic curve. These conditions turn out to be quite restrictive and the possibilities for $(X,\infty,\mathbf{P}W)$ are classified in 4.3.2. The properties of the constructions made in 4.2 in this situation are summarized in 4.3.3; the additional properties that hold are further summarized in 4.3.9. This Section ends by constructing in 4.3.12 a distinguished degeneration of the Smooth Cone Situation and applying the constructions of the previous Section in families to yield 4.3.16.

4.3.1. — Let $(X,\infty,\mathbf{P}W)$ be a Cone Situation satisfying 4.2.1(ii) and 4.2.1(iii). Note that these conditions together imply 4.2.1(i). Further, since ∞ is a smooth point, 4.2.2 implies that the hyperplane $\mathbf{P}W$ must be the embedded tangent space $\mathbf{T}_{X,\infty}$ and thus will be dropped from the notation. To reiterate and fix notation, a *Smooth Cone Situation* is a pair (X,∞) consisting of a q-bic threefold $X \subset \mathbf{P}V$ together with a smooth point ∞ such that $X \cap \mathbf{T}_{X,\infty}$ is a cone over a smooth q-bic curve C.

The conditions of the Smooth Cone Situation turn out to be quite restrictive and all possibilities have appeared amongst the examples of 4.2.4:

4.3.2. Proposition. — *A Smooth Cone Situation* (X,∞) *is projectively equivalent to either* 4.2.4(i) *or* 4.2.4(ii). *In particular,* X *is of type* $\mathbf{1}^{\oplus 5}$ *or* $\mathbf{N}_2 \oplus \mathbf{1}^{\oplus 3}$.

Proof. — If X is smooth, then 2.4.8 shows that (X,∞) must be as in 4.2.4(i). Thus it remains to show that when X is singular, then (X,∞) is as in 4.2.4(ii): X is of type $\mathbf{N}_2 \oplus \mathbf{1}^{\oplus 3}$, and ∞ is the smooth point from the subform of type $\mathbf{N}_2$.

Since $\infty = \mathbf{P}L$ is a smooth point and the base of the cone $X \cap \mathbf{P}W$ is smooth, with $W = \mathrm{Fr}^*(L)^\perp$ the linear subspace underlying $\mathbf{T}_{X,\infty}$, $\mathbf{P}W$ is disjoint from the singular locus $\mathrm{Sing}(X)$ of X. By 2.2.7, $\mathrm{Sing}(X) = \mathbf{P}L'$ with $L' := \mathrm{Fr}^{-1}(V^{\perp})$, so

$$W \cap L' = \{0\} \quad \text{so} \quad \mathrm{corank}(X) = \dim(L') \leq 1.$$

Since X is assumed not to be smooth, it must have corank 1. Since $L \subset W$, this also shows that $L' \neq L$; moreover, the natural map $L' \to \mathrm{Fr}^*(L)^\vee$ is an isomorphism, as it is a nonzero map between 1-dimensional spaces. Let β be the q-bic form underlying X. By 1.1.8, there is an exact sequence

$$0 \to \mathrm{Fr}^*(V)^{\perp} \to V \xrightarrow{\beta} \mathrm{Fr}^*(V)^\vee \to \mathrm{Fr}^*(L')^\vee \to 0.$$

Then $\mathrm{Fr}^*(V)^{\perp} = L$: write $V = W \oplus L'$ and use the above facts to see

$$\mathrm{Fr}^*(V)^{\perp} = \ker(W \oplus L' \to \mathrm{Fr}^*(W)^\vee) = \ker(W \to \mathrm{Fr}^*(W/L)^\vee) = L.$$

Thus the restriction β_U of β to $U := L \oplus L'$ is of type $\mathbf{N}_2$ and

$$\mathrm{Fr}^*(U)^{\perp} = \mathrm{Fr}^*(L)^{\perp} \cap \mathrm{Fr}^*(L')^{\perp} = \mathrm{Fr}^*(L)^{\perp},$$
$$\mathrm{Fr}^{-1}(U^{\perp}) = \mathrm{Fr}^{-1}(L^{\perp}) \cap \mathrm{Fr}^{-1}(L'^{\perp}) = \mathrm{Fr}^{-1}(L'^{\perp}),$$

are distinct hyperplanes in V and so their intersection V_0 is an orthogonal complement of U in V. This shows that X is of type $\mathbf{N}_2 \oplus \mathbf{1}^{\oplus 3}$ and $\infty = \mathbf{P}L$ is the cone point arising from the smooth point of the subform $\mathbf{N}_2$. ■

4.3.3. Summary of general properties. — The constructions made in 4.2 have very good properties in the Smooth Cone Situation. So far:

- the scheme $T \subset \mathbf{P}$ constructed in 4.2.15 is a Cohen–Macaulay surface whose structure sheaf admits a short resolution by vector bundles on $\mathbf{P}$;
- T is the Zariski closure of T° in $\mathbf{P}$ by 4.2.20, and the complement $T \setminus T^\circ$ maps to C via a purely inseparable morphism of degree q by 4.2.19;
- by 4.2.27 the dominant rational map $S \dashrightarrow T$ is resolved to a finite flat morphism $\tilde{S} \to T$ of degree q in which $\tilde{S}$ is a hypersurface in a $\mathbf{P}^1$-bundle over T.

In the Smooth Cone Situation, more can be said about the scheme $\tilde{S}$ resolving the rational map $S \dashrightarrow T$. To begin, the birational morphism $\tilde{S} \to S$ may be explicitly identified. In the following statement, $C_\infty \subset S$ is the divisor—abstractly isomorphic to the q-bic curve C—parameterizing lines in X through ∞, see 4.2.3(ii).

4.3.4. Lemma. — *Let (X, ∞) be a Smooth Cone Situation. Then $\tilde{S} \to S$ is the blowup along the Hermitian points of $C_\infty \subset S$.*

Proof. — Consider in detail the construction of $\tilde{S}$ as the strict transform of S along the resolution of the rational map $\mathbf{G}(2,V) \dashrightarrow \mathbf{P}(W/L)$ given by the subquotient situation in 4.2.5. By A.3.8, this is resolved on a blowup $\mathbf{H} \to \mathbf{G}(2,V)$ that factors as

$$\mathbf{H} \to \mathbf{H}_1 \times_{\mathbf{G}(2,V)} \mathbf{H}_2 \to \mathbf{G}(2,V)$$

in which

- $\mathbf{H}_1 := \{\, (y \in \ell) \in \mathbf{P}W \times \mathbf{G}(2,V) \,\}$ is the scheme constructed in A.2.6 resolving the intersection map $\mathbf{G}(2,V) \dashrightarrow \mathbf{P}W$, and by A.2.9(i), $\mathbf{H}_1 \to \mathbf{G}(2,V)$ is the blowup along the subscheme of lines contained in $\mathbf{P}W$; and
- $\mathbf{H}_2 := \{\, (\ell \mapsto \ell_0) \in \mathbf{G}(2,V) \times \mathbf{G}(2,V/L) \,\}$ is the scheme constructed in A.2.1 resolving linear projection $\mathbf{G}(2,V) \dashrightarrow \mathbf{G}(2,V/L)$, and by A.2.4(i), $\mathbf{H}_2 \to \mathbf{G}(2,V)$ is the blowup along the subscheme of lines containing ∞.

Let S_i be the strict transform of S along $\mathbf{H}_i \to \mathbf{G}(2,V)$, and let $\tilde{S}' := S_1 \times_S S_2$. Then there is a factorization $\tilde{S} \to \tilde{S}' \to S$. Now $S_2 \to S$ is the blowup along the subscheme of lines in X containing ∞, which by 4.2.3(ii), is the effective Cartier divisor $C_\infty \subset S$ parameterizing the lines in the cone $X \cap \mathbf{P}W$; thus $S_2 \to S$ is an isomorphism. On the other hand, $S_1 \to S$ is the blowup along the subscheme of lines in X contained in $\mathbf{P}W$, which by 3.13.1 is a 1-dimensional subscheme of S with divisorial part qC_∞, but also embedded points along the Hermitian points of C_∞; therefore $S_1 \to S$ is the blowup of S along the Hermitian points of C_∞.

This shows that $\tilde{S}' \to S$ is the blowup along the Hermitian points of C_∞; in particular, the fibre product is smooth. Consider the morphism $\tilde{S} \to \tilde{S}'$. In general, by the description of points of $\mathbf{H}$ from A.3.6,

$$\tilde{S} \subset \big\{\, \big((y \in \ell) \mapsto (y_0 \in \ell_0)\big) \,\big|\, \ell \subset X, \ell_0 \subset \mathbf{P}(V/L) \,\big\}$$

and the map to $\tilde{S}'$ is that which omits y_0. By 4.2.6, $\tilde{S}$ lies in the closed subscheme in which $y_0 \in C$, whence $\tilde{S} \to \tilde{S}'$ is finite. By A.3.9, this is an isomorphism away from the exceptional divisor of $\tilde{S}' \to S$. But $\tilde{S}'$ is smooth therein, so Zariski's Main Theorem, as in [Har77, Corollary III.11.4], implies $\tilde{S} \to \tilde{S}'$ is an isomorphism. ■

The next two statements describe the boundary divisors of T and $\tilde{S}$, respectively.

4.3.5. Lemma. — *Let (X, ∞) be a Smooth Cone Situation. Then T is smooth along the closed subscheme $T \setminus T^\circ$, and its points are given by*

$$\{ (\infty \mapsto y_0 \in \ell_0) \in \mathbf{P} \mid y_0 = \mathbf{P}L_0 \in C \text{ and } \ell_0 = \mathbf{P}\operatorname{Fr}^{-1}(L_0^\perp) \},$$

Proof. — The description of the points of $T \setminus T^\circ$ follows directly from 4.2.19 and 2.2.12. For smoothness of T therein, 4.2.27 together with 4.3.4 imply that $T \setminus T^\circ$ is the image of the strict transform $\tilde{C}_\infty$ of C_∞ along $\tilde{S} \to S$. Since $\tilde{S} \to T$ is flat by 4.2.27(i) and smoothness descends along flat morphisms, see [Stacks, 05AW], it suffices to show that $\tilde{S}$ is smooth along $\tilde{C}_\infty$. For this, note that the hypotheses of the Smooth Cone Situation together with 2.7.7 imply that S is smooth along C_∞. Since $\tilde{S} \to S$ is a blowup along smooth points by 4.3.4, $\tilde{S}$ is smooth along $\tilde{C}_\infty$. ■

4.3.6. Lemma. — *Let (X, ∞) be a Smooth Cone Situation. The strict transform of $C_\infty \subset S$ in $\tilde{S}$ is the closed subscheme*

$$\tilde{C}_\infty := \{ ((\infty \in \ell) \mapsto (y_0 \in \ell_0)) \mid y_0 = \mathbf{P}L_0 \in C, \ell_0 = \mathbf{P}\operatorname{Fr}^{-1}(L_0^\perp), \ell = \langle y_0, \infty \rangle \}.$$

The exceptional divisor of $\tilde{S} \to S$ is the closed subscheme

$$\{ ((y \in \ell) \mapsto (y_0 \in \ell_0)) \mid y_0 = \mathbf{P}L_0 \in C \text{ Hermitian}, \ell_0 = \mathbf{P}\operatorname{Fr}^{-1}(L_0^\perp), \ell = \langle y_0, \infty \rangle \}.$$

Proof. — As noted in the proof of 4.3.5, $\tilde{C}_\infty$ maps to $T \setminus T^\circ$ along the map $\tilde{S} \to T$. Thus the description of the points of $\tilde{C}_\infty$ follows from that of $T \setminus T^\circ$ together with the discussion of 4.2.23. The description of the points of the exceptional divisor follows from 4.3.4. ■

4.3.7. Corollary. — *Let (X, ∞) be a Smooth Cone Situation. Then*

(i) *the rational map $S \dashrightarrow T$ is defined away from the Hermitian points of C_∞; and*

(ii) *the rational map $S \dashrightarrow C$ extends to a morphism.*

Proof. — For (i), this is because $\tilde{S} \to S$ is an isomorphism away from the Hermitian points of C_∞ by 4.3.4. Since the rational map to C factors as $S \dashrightarrow T \to C$, this implies that $S \dashrightarrow C$ is also defined away from the Hermitian points of C_∞. Then (ii) follows from the description given in 4.3.6 of the points of the exceptional divisors of $\tilde{S} \to S$, as it implies they get contracted along $\tilde{S} \to C$. ∎

4.3.8. Lemma. — *Let (X, ∞) be a Smooth Cone Situation. Then the morphism $\tilde{S} \to T$ is a quotient map for* $\mathbf{Aut}_{\mathrm{uni}}(W \subset V, \beta)$.

Proof. — The scheme S° may be identified as the open subscheme of $\tilde{S}$ parameterizing lines that neither contain ∞ nor are contained in $\mathbf{P}W$. To conclude, it suffices to show that $S^\circ \to T^\circ$ is a torsor for $G := \mathbf{Aut}_{\mathrm{uni}}(W \subset V, \beta)$. Indeed, this would imply that the canonical morphism $\tilde{S}/G \to T$, coming from G-invariance of T established in 4.2.30(i), is an isomorphism over T°. But 4.2.27(i) implies that $\tilde{S}/G \to T$ is finite; since T is regular along $T \setminus T^\circ$ by 4.3.5, Zariski's Main Theorem as in [**Stacks**, 02LR] implies that $\tilde{S}/G \to T$ is also an isomorphism away from T°.

So consider $S^\circ \to T^\circ$. The characterization given in 4.2.25 of the type of $\beta_{\mathcal{V}_T}$ on fibres implies that, for every $t \in T^\circ$,

$$\mathrm{type}(\beta_{\mathcal{V}_t}) = \begin{cases} \mathbf{1}^{\oplus 2} & \text{if } (X,\infty) \text{ is as in 4.2.4(i) so that } \mathrm{Fr}^{-1}(L^\perp) \neq V, \\ \mathbf{N}_2 & \text{if } (X,\infty) \text{ is as in 4.2.4(ii) so that } \mathrm{Fr}^{-1}(L^\perp) = V, \end{cases}$$

such that the subspace $\bar{L}_t := \mathcal{T}_{\pi_1}(-1,0)|_t$ of $\mathcal{V}_t$ underlies a smooth point. The fibre of $S^\circ \to T^\circ$ is then the degree q scheme obtained by removing the point corresponding to $\bar{L}_t$. The action of G on $(\mathcal{V}_t, \beta_{\mathcal{V}_t})$ factors through $\mathbf{Aut}_{\mathrm{uni}}(\bar{L}_t \subset \mathcal{V}_t, \beta_{\mathcal{V}_t})$, and the computations of 3.2.4 and 3.3.1 show that this action is simply transitive on the degree q scheme complementary to the point given by $\bar{L}_t$. ∎

The following summarizes the additional properties of the Smooth Cone Situation.

4.3.9. Proposition. — *Let (X, ∞) be a Smooth Cone Situation. Then there is a canonical commutative diagram of morphisms*

$$\begin{tikzcd} & \tilde{S} \arrow[dl, "b"'] \arrow[dr, "\rho"] & \\ S \arrow[dr, "\varphi"'] & & T \arrow[dl, "\pi"] \\ & C & \end{tikzcd}$$

of schemes over $\mathbf{k}$ *such that*

(i) $b\colon \tilde{S} \to S$ *is the blowup along the Hermitian points of* $C_\infty \subset S$*,*

(ii) $\rho\colon \tilde{S} \to T$ *is a quotient by* $\mathbf{F}_q$ *if* X *is smooth and* $\boldsymbol{\alpha}_q$ *if* X *is singular, and*

(iii) $\varphi\colon S \to C$ *and* $\pi\colon T \to C$ *are proper, surjective, and flat of relative dimension* 1.

Proof. — That the commutative diagram exists follows from the constructions of **4.2**; see also **4.2.21**. Note that, *a priori*, the construction of **4.2.6** yields only a rational map $S \dashrightarrow C$, but this extends to a morphism here by **4.3.7**(ii). Item (i) is **4.3.4**, and (ii) is **4.3.8** together with **4.2.29**. To see (iii), note first that both S and T are Cohen–Macaulay by **4.2.16** and **4.2.15**(iii). Since C is regular, flatness follows from Miracle Flatness, see [**Stacks**, 00R4]. ■

The following gives some basic structure to the fibres of S, $\tilde{S}$, and T over C:

4.3.10. Lemma. — *Let (X, ∞) be a Smooth Cone Situation. Then*

(i) *for every $x \in C$, there are isomorphisms*

$$\kappa(x) \cong \mathrm{H}^0(T_x, \mathcal{O}_{T_x}) \cong \mathrm{H}^0(\tilde{S}_x, \mathcal{O}_{\tilde{S}_x}) \cong \mathrm{H}^0(S_x, \mathcal{O}_{S_x}),$$

(ii) $\varphi_*\mathcal{O}_S \cong (\pi \circ \rho)_*\mathcal{O}_{\tilde{S}} \cong \pi_*\mathcal{O}_T \cong \mathcal{O}_C$*, and*

(iii) $\mathbf{R}^1\varphi_*\mathcal{O}_S$ *is locally free and carries a filtration with graded pieces*

$$\mathrm{gr}_i(\mathbf{R}^1\varphi_*\mathcal{O}_S) \cong \begin{cases} \mathbf{R}^1\pi_*\mathcal{O}_T & \text{if } i = 0, \text{ and} \\ \mathcal{O}_C(-q+i) \otimes L^{\otimes q-i} \otimes \mathbf{R}^1\pi_*\mathcal{O}_T(q-i, -i-1) & \text{if } 1 \leq i \leq q-1. \end{cases}$$

Proof. — By 4.2.15(ii), every fibre T_x is the degeneracy locus in $\mathbf{P}_x := \mathbf{P}^1_{\kappa(x)} \times \mathbf{P}^2_{\kappa(x)}$ of a morphism

$$\phi_x \colon \mathscr{O}(1,0) \oplus \mathscr{O}(0,1) \to \mathscr{O}(q,1) \oplus \mathscr{O}(1,q) \oplus \mathscr{O}(1,1)$$

where $\mathscr{O}(a,b) := \mathscr{O}_{\mathbf{P}_x}(a,b)$ for integers a and b. By 4.3.9(iii), T_x is of expected codimension 2 and so, as in 4.2.15(iii), $\mathscr{O}_{T_x}$ has a resolution by $\mathscr{O}_{\mathbf{P}_x}$-modules given by

$$\mathscr{O}(-q,-q-1) \oplus \mathscr{O}(-q-1,-q) \to \mathscr{O}(-1,-q) \oplus \mathscr{O}(-q,-1) \oplus \mathscr{O}(-q,-q) \to \mathscr{O}.$$

The associated spectral sequence computing global sections of $\mathscr{O}_{T_x}$ has only one nonzero contribution and gives

$$\mathrm{H}^0(T_x, \mathscr{O}_{T_x}) \cong \mathrm{H}^0(\mathbf{P}_x, \mathscr{O}) \cong \kappa(x).$$

By 4.2.27(iii), the sheaf $\rho_{x,*}\mathscr{O}_{\tilde{S}_x}$ has a filtration with graded pieces $\mathscr{O}_{T_x}$ and $\mathscr{O}_{T_x}(q-i,-i-1)$ for $1 \leq i \leq q-1$. The latter pieces are resolved by

$$\begin{aligned} \mathscr{O}(-i,-q-i-2) \oplus \mathscr{O}(-i-1,-q-i-1) \;&\to\; \mathscr{O}(q-i-1,-q-i-1) \oplus \\ \mathscr{O}(-i,-i-2) \oplus \mathscr{O}(-i,-q-i-1) \;&\to\; \mathscr{O}(q-i,-i-1). \end{aligned}$$

Since the degree in the second factor is negative, cohomology only appears in either H^2 or H^3; moreover, since the degree in the first factor is also negative for the left-most sheaves, cohomology for them only appears in H^3. Thus the spectral sequence computing $\mathrm{H}^0(T_x, \mathscr{O}_{T_x}(q-i,-i-1))$ has no nonzero contributions, so

$$\mathrm{H}^0(\tilde{S}_x, \mathscr{O}_{\tilde{S}_x}) \cong \mathrm{H}^0(T_x, \mathscr{O}_{T_x}) \cong \kappa(x).$$

Since $b_{x,*}\mathscr{O}_{\tilde{S}_x} \cong \mathscr{O}_{S_x}$ by 4.3.9(i), $\mathrm{H}^0(S_x, \mathscr{O}_{S_x}) \cong \mathrm{H}^0(\tilde{S}_x, \mathscr{O}_{\tilde{S}_x}) \cong \kappa(x)$. This shows (i) and implies (ii). That $\mathbf{R}^1\varphi_*\mathscr{O}_S$ is locally free now follows from Cohomology and Base Change, see [Har77, Theorem III.12.11]; the filtration arises the filtration of $\rho_*\mathscr{O}_{\tilde{S}}$ from 4.2.27(iii) via the isomorphism $\mathbf{R}^1\varphi_*\mathscr{O}_S \cong \mathbf{R}^1\pi_*(\rho_*\mathscr{O}_{\tilde{S}})$ obtained from 4.3.9. ■

4.3.11. Families of Smooth Cone Situations. — The remainder of this Section is devoted to constructing a special 1-parameter family of Smooth Cone Situations with smooth general fibre and one singular special fibre. In general, a *family of Smooth Cone Situations* over a base scheme S is a pair $(\mathscr{X}, \sigma)$ consisting of a q-bic threefold bundle $\pi \colon \mathscr{X} \to S$ and a section $\sigma \colon S \to \mathscr{X}$ such that $(\mathscr{X}_s, \sigma(s))$ is a Smooth Cone Situation for every closed point $s \in S$. The following constructs a special family of Smooth Cone Situations starting from a smooth q-bic threefold and two appropriately chosen cone points:

4.3.12. Lemma. — *Let X be a smooth q-bic threefold and let $x_-, x_+ \in X$ be cone points with $\langle x_-, x_+ \rangle \not\subset X$. Then there exists a q-bic threefold $\mathscr{X} \subset \mathbf{P}V \times \mathbf{A}^1$ over $\mathbf{A}^1$ such that*

(i) *the constant sections $x_\pm \colon \mathbf{A}^1 \to \mathbf{P}V \times \mathbf{A}^1$ factor through $\mathscr{X}$;*
(ii) *$(\mathscr{X}, x_-)$ is a family of Smooth Cone Situations;*
(iii) *the projection $\pi \colon \mathscr{X} \to \mathbf{A}^1$ is smooth away from $0 \in \mathbf{A}^1$ and $X = \pi^{-1}(1)$; and*
(iv) *$X_0 := \pi^{-1}(0)$ is of type $\mathbf{N}_2 \oplus \mathbf{1}^{\oplus 3}$ with singular point x_+.*

Moreover, there exists a choice of coordinates $(x_0 : x_1 : x_2 : x_3 : x_4)$ such that

$$\mathscr{X} = \mathrm{V}(x_0^q x_1 + t x_0 x_1^q + x_2^{q+1} + x_3^{q+1} + x_4^{q+1}) \subset \mathbf{P}^4 \times \mathbf{A}^1,$$

$x_- = (1:0:0:0:0)$, and $x_+ = (0:1:0:0:0)$.

Proof. — Let $x_- = \mathbf{P}L_-$, $x_+ = \mathbf{P}L_+$, and set $U := L_- \oplus L_+$. Let (V, β) be a q-bic form defining X. Since (X, x_-) is a Smooth Cone Situation by 4.2.4(i), the assumption that $\langle x_-, x_+ \rangle = \mathbf{P}U \not\subset X$ is equivalent to the fact that the restricted q-bic form (U, β_U) is of type $\mathbf{1}^{\oplus 2}$. Since $x_\pm$ are cone points, 2.4.8 implies that U is a Hermitian subspace of V and thus has an orthogonal complement (W, β_W) by 1.2.7.

Set $V[t] := V \otimes_{\mathbf{k}} \mathbf{k}[t]$, and similarly for $U[t]$ and $W[t]$. Let

$$\beta_W[t] := \beta_W \otimes \mathrm{id}_{\mathbf{k}[t]} \colon \mathrm{Fr}^*(W[t]) \otimes_{\mathbf{k}[t]} W[t] \to \mathbf{k}[t]$$

denote the q-bic form on $W[t]$ given by the constant extension of β_W. Applying the construction of 3.1.4 with the decomposition $U = L_- \oplus L_+$ yields a q-bic form

$$\beta_U^{L_\pm} \colon \mathrm{Fr}^*(U[t]) \otimes_{\mathbf{k}[t]} U[t] \to \mathbf{k}[t]$$

so that the restriction to $t = 1$ is β_U, and the restriction to $t = 0$ is of type $\mathbf{N}_2$. Set

$$(V[t], \beta^{L_\pm}) := (U[t], \beta_U^{L_\pm}) \perp (W[t], \beta_W[t]).$$

Let $\mathscr{X} \subset \mathbf{P}V \times \mathbf{A}^1$ be the q-bic threefolds over $\mathbf{A}^1 := \operatorname{Spec}(\mathbf{k}[t])$ defined by $(V[t], \beta^{L_\pm})$. By 3.1.4(i), the constant sections $x_\pm \colon \mathbf{A}^1 \to \mathbf{P}V \times \mathbf{A}^1$ factor through $\mathscr{X}$, verifying (i). By 3.1.4(ii) and 2.2.7, $x_- \colon \mathbf{A}^1 \to \mathscr{X}$ lands in the smooth locus of each fibre. Then, together with 3.1.4(iii) and 2.4.6, x_- is a cone point in each fibre and hence $(\mathscr{X}, x_-)$ is a family of Smooth Cone Situations over $\mathbf{A}^1$, verifying (ii). These also imply that $\pi \colon \mathscr{X} \to \mathbf{A}^1$ is smooth away from $0 \in \mathbf{A}^1$, that $X = \pi^{-1}(1)$ from 3.1.4(iv), and that $X_0 := \pi^{-1}(0)$ is of type $\mathbf{N}_2 \oplus \mathbf{1}^{\oplus 3}$, showing properties (iii) and (iv). Finally, the explicit equation of $\mathscr{X}$ can be realized by choosing standard coordinates for the form (W, β_W) of type $\mathbf{1}^{\oplus 3}$, and choosing coordinates (U, β_U) adapted to the decomposition $U = L_- \oplus L_+$ as in 3.2.1. ∎

4.3.13. — Consider the special family of q-bic threefolds $\pi \colon \mathscr{X} \to \mathbf{A}^1$ and its underlying q-bic form over $\mathbf{k}[t]$,

$$(V[t], \beta^{L_\pm}) = (U[t], \beta_U^{L_\pm}) \perp (W[t], \beta_W[t])$$

as constructed in 4.3.12. Then $\mathscr{X}$ admits actions from two group schemes:

First, consider the linear $\mathbf{G}_m$-action on $\mathbf{P}V \times \mathbf{A}^1$ with weights

$$\operatorname{wt}(W) = 0, \quad \operatorname{wt}(L_-) = -1, \quad \operatorname{wt}(L_+) = q, \quad \operatorname{wt}(t) = q^2 - 1.$$

By 3.1.6, this $\mathbf{G}_m$-action leaves $\beta^{L_\pm} = \beta_U^{L_\pm} \perp \beta_W[t]$ invariant, so $\mathscr{X}$ is preserved.

Second, $\mathscr{X}$ admits an action over $\mathbf{A}^1$ by the automorphism group scheme $\mathbf{Aut}(V[t], \beta^{L_\pm})$ of the q-bic form over $\mathbf{k}[t]$ which is furthermore equivariant for the action of $\mathbf{G}_m$. By 1.3.3, this contains the group scheme

$$\mathbf{Aut}(U[t], \beta_U^{L_\pm}) \times_{\mathbf{A}^1} \mathbf{Aut}(W[t], \beta_W[t]) \subseteq \mathbf{Aut}(V[t], \beta^{L_\pm}),$$

which contains the finite flat group scheme $\mathscr{G} := \mathscr{G}_U \times_{\mathbf{k}} \mathrm{U}(V_0, \beta_0)$ over $\mathbf{A}^1$, where

$$\mathscr{G}_U \cong \left\{ \begin{pmatrix} \lambda & \epsilon \\ 0 & \lambda^{-q} \end{pmatrix} \,\middle|\, \lambda \in \boldsymbol{\mu}_{q^2-1}, \epsilon^q + t\lambda^{q-1}\epsilon = 0 \right\} \quad \text{and} \quad \mathrm{U}(W, \beta_W) \cong \mathrm{U}_3(q)$$

are the subgroup of automorphisms of $(U[t], \beta_U^{L_\pm})$ identified in 3.1.7, and the finite unitary group associated with (W, β_W) as in 1.3.5, Together with 3.1.8, this gives:

4.3.14. Lemma. — *In the setting of 4.3.13, there exists a $\mathbf{G}_m$-equivariant diagram*

$$\begin{tikzcd} \mathscr{G} \times_{\mathbf{A}^1} \mathscr{X} \arrow[rr, "\mathrm{act}"] \arrow[dr] & & \mathscr{X} \arrow[dl] \\ & \mathbf{A}^1 & \end{tikzcd}$$

The section $x_- \colon \mathbf{A}^1 \to \mathscr{X}$ is invariant under both the action of $\mathscr{G}$ and $\mathbf{G}_m$. ■

4.3.15. — Let $\mathscr{S} \to \mathbf{A}^1$ be the relative Fano scheme of lines of the family of q-bic threefolds $\pi \colon \mathscr{X} \to \mathbf{A}^1$ constructed in 4.3.12. The projective constructions of A.2 and A.3 work in families, and the constructions of 4.2 may be applied to the family of Smooth Cone Situations $(\mathscr{X}, x_-)$. Namely, let

$$C_{\mathbf{A}^1} := C \times_{\mathbf{k}} \mathbf{A}^1, \quad \mathbf{P}_{\mathbf{A}^1} := \mathbf{P} \times_{\mathbf{k}} \mathbf{A}^1, \quad \mathbf{P}\mathscr{V}_{\mathbf{A}^1} := \mathbf{P}\mathscr{V} \times_{\mathbf{k}} \mathbf{A}^1$$

where $\mathbf{P}$ is as from 4.2.10, and $\mathbf{P}\mathscr{V}$ is as from 4.2.21. Let $\mathscr{T} \subset \mathbf{P}_{\mathbf{A}^1}$ be the degeneracy locus as in 4.2.15 formed using $\beta^{L_\pm}$; then let $\tilde{\mathscr{S}} \subset \mathbf{P}\mathscr{V}_{\mathbf{A}^1}|_{\mathscr{T}}$ be the hypersurface defined by the section induced by $\beta^{L_\pm}$ as in 4.2.24. Then by 4.2.27, $\tilde{\mathscr{S}} \to \mathscr{T}$ is a finite flat cover of degree q and the direct image of $\mathscr{O}_{\tilde{\mathscr{S}}}$ has an increasing filtration with graded pieces as described in 4.2.27(iii). Further properties of these schemes are as follows:

4.3.16. Proposition. — *Let X be a smooth q-bic threefold. Then every choice of cone points $x_-, x_+ \in X$ such that $\langle x_-, x_+ \rangle \not\subset X$ induces a commutative diagram*

$$\begin{tikzcd} & \tilde{\mathscr{S}} \arrow[dl] \arrow[dr] & \\ \mathscr{S} \arrow[dr] & & \mathscr{T} \arrow[dl] \\ & C_{\mathbf{A}^1} & \end{tikzcd}$$

of morphisms of schemes over $\mathbf{A}^1$ satisfying:

(i) *each of $\mathscr{S}$, $\tilde{\mathscr{S}}$, and $\mathscr{T}$ are flat projective surfaces over $\mathbf{A}^1$, smooth away from 0;*

(ii) *the morphisms $\mathscr{S} \to C_{\mathbf{A}^1} \leftarrow \mathscr{T}$ are flat, projective, and of relative dimension 1;*

(iii) *$\tilde{\mathscr{S}} \to \mathscr{S}$ is a blowup along $q^3 + 1$ sections of $\mathscr{S} \to C_{\mathbf{A}^1}$;*

(iv) *the diagram admits a linear action of* $\mathbf{G}_m$ *and a* $\mathbf{G}_m$*-equivariant action of* $\mathscr{G}$*; and*

(v) $\tilde{\mathscr{S}} \to \mathscr{T}$ *is the quotient by the group scheme* $\mathscr{G}_U$*.*

Proof. — The existence of the diagram together with properties (i)–(iii) and (v) follow from applying **4.3.9** to the family of q-bic threefolds from **4.3.12**. The group scheme actions in (iv) come from the discussion of **4.3.13** and **4.3.14**. ■

These special families relate the coherent cohomology of the smooth Fano surface $S := \mathbf{F}_1(X)$ with that of the singular Fano surface $S_0 := \mathbf{F}_1(X_0)$. To formulate the next statement, let $\varphi \colon S \to C$ and $\varphi_0 \colon S_0 \to C$ be the morphisms induced on the Fano surfaces by taking the fibre of $\mathscr{S} \to C_{\mathbf{A}^1}$ from **4.3.16** over the points 1 and 0 of $\mathbf{A}^1$, respectively. The sheaves $\mathbf{R}^1\varphi_*\mathscr{O}_S$ and $\mathbf{R}^1\varphi_{0,*}\mathscr{O}_{S_0}$ are locally free $\mathscr{O}_C$-modules that carry q-step filtrations by subbundles, see **4.3.10**(iii). Finally, it follows from **4.4.3** and **4.3.16**(iv) that S_0 admits a $\mathbf{G}_m$-action over C, and so the sheaf $\mathbf{R}^1\varphi_{0,*}\mathscr{O}_{S_0}$ carries a natural $\mathbf{Z}$-grading which is compatible with its aforementioned filtration.

4.3.17. Proposition. — *Let X be a smooth q-bic threefold. Then every choice of cone points $x_-, x_+ \in X$ such that $\langle x_-, x_+\rangle \not\subset X$ induces a weight decomposition*

$$\mathbf{R}^1\varphi_*\mathscr{O}_S = \bigoplus\nolimits_{\alpha \in \mathbf{Z}/(q^2-1)\mathbf{Z}} (\mathbf{R}^1\varphi_*\mathscr{O}_S)_\alpha$$

by filtered subbundles. Each weight component carries a further filtration $\mathrm{Fil}_\bullet$ *by filtered subbundles such that there are canonical isomorphisms*

$$\mathrm{gr}_i^{\mathrm{Fil}}(\mathbf{R}^1\varphi_*\mathscr{O}_S)_\alpha \cong (\mathbf{R}^1\varphi_{0,*}\mathscr{O}_{S_0})_{\alpha+i(q^2-1)}$$

of filtered $\mathscr{O}_C$-modules for each $\alpha = 1,2,\ldots,q^2-1$ and $i \in \mathbf{Z}$.

Proof. — By **4.3.16**(iv), $\varphi \colon S \to C$ is equivariant for the action of the group scheme

$$\mathscr{G}_1 \cong (\boldsymbol{\mu}_{q^2-1} \cdot \mathbf{F}_q) \times \mathrm{U}_3(q) \cong \mathbf{Aut}(L_- \subset U, \beta_U) \times \mathbf{Aut}(W, \beta_W).$$

The subgroup $\boldsymbol{\mu}_{q^2-1}$ acts trivially on C, so its action on $\mathbf{R}^1\varphi_*\mathscr{O}_S$ induces the claimed weight decomposition. This respects the filtration from **4.3.10**(iii) because the morphisms $\tilde{S} \to T \to C$ are equivariant for $\boldsymbol{\mu}_{q^2-1}$ by **4.3.16**(iv).

Consider the second statement. Applying 4.3.16(iii), the commutative diagram of 4.3.16, and 4.3.16(v) successively gives isomorphisms

$$\begin{aligned}\mathbf{R}(\mathscr{S} \to C_{\mathbf{A}^1})_*\mathcal{O}_{\mathscr{S}} &\cong \mathbf{R}(\mathscr{S} \to C_{\mathbf{A}^1})_*\mathbf{R}(\tilde{\mathscr{S}} \to \mathscr{S})_*\mathcal{O}_{\tilde{\mathscr{S}}} \\ &\cong \mathbf{R}(\mathscr{T} \to C_{\mathbf{A}^1})_*\mathbf{R}(\tilde{\mathscr{S}} \to \mathscr{T})_*\mathcal{O}_{\tilde{\mathscr{S}}} \cong \mathbf{R}(\mathscr{T} \to C_{\mathbf{A}^1})_*(\tilde{\mathscr{S}} \to \mathscr{T})_*\mathcal{O}_{\tilde{\mathscr{S}}}.\end{aligned}$$

Taking cohomology sheaves then gives an isomorphism

$$\mathbf{R}^1(\mathscr{S} \to C_{\mathbf{A}^1})_*\mathcal{O}_{\mathscr{S}} \cong \mathbf{R}^1(\mathscr{T} \to C_{\mathbf{A}^1})_*(\tilde{\mathscr{S}} \to \mathscr{T})_*\mathcal{O}_{\tilde{\mathscr{S}}}$$

between locally free $\mathcal{O}_{C_{\mathbf{A}^1}}$-modules. As in 4.3.15, the $\mathcal{O}_{\mathscr{T}}$-module $(\tilde{\mathscr{S}} \to \mathscr{T})_*\mathcal{O}_{\tilde{\mathscr{S}}}$ carries a q-step filtration by subbundles, and this isomorphism induces the filtration on $\mathbf{R}^1(\mathscr{S} \to C_{\mathbf{A}^1})_*\mathcal{O}_{\mathscr{S}}$ globalizing the filtration from 4.3.10(iii).

It now follows from 4.3.16(iv) that the $\mathcal{O}_{C_{\mathbf{A}^1}}$-module $\mathbf{R}^1(\mathscr{S} \to C_{\mathbf{A}^1})_*\mathcal{O}_{\mathscr{S}}$, together with its filtration and its action by $\boldsymbol{\mu}_{q^2-1}$, are all equivariant for an action of $\mathbf{G}_m$, in which $\mathbf{G}_m$ acts on $\mathbf{A}^1$ with weight q^2-1. The Rees construction, see [Sim91, Lemma 19], endows $\mathbf{R}^1\varphi_*\mathcal{O}_S$ with a filtration $\mathrm{Fil}_\bullet$ so that its graded pieces are the graded pieces of $\mathbf{R}^1\varphi_{0,*}\mathcal{O}_{S_0}$, these being the base change of $\mathbf{R}^1(\mathscr{S} \to C_{\mathbf{A}^1})_*\mathcal{O}_{\mathscr{S}}$ to 1 and 0, respectively. Moreover, $\mathrm{Fil}_\bullet$ is compatible with the filtrations constructed above and the $\boldsymbol{\mu}_{q^2-1}$-action, and so the second statement now follows. ■

4.4. Type $\mathbf{N}_2 \oplus \mathbf{1}^{\oplus 3}$

This Section pertains to the geometry of q-bic threefolds X of type $\mathbf{N}_2 \oplus \mathbf{1}^{\oplus 3}$, with a particular emphasis on the geometry of lines on X. The variety S of lines on X is a surface which is singular along the curve C_+ of lines through the unique singular point $x_+ = \mathbf{P}L_+$ of X. Its normalization $\nu \colon S^\nu \to S$ is constructed in 4.4.17 by resolving the rational map $S \dashrightarrow C$ obtained by applying 4.2.6 to the Cone Situation $(X, x_+, \mathbf{P}\,\mathrm{Fr}^{-1}(L_+^\perp))$ from 4.2.4(iii); this constructs S^ν as a ruled surface over the smooth q-bic curve C. The resulting morphism $\tilde{\varphi}_+ \colon S^\nu \to C$ compatible with the morphism $\varphi_- \colon S \to C$ coming from 4.3.7(ii) applied to the Smooth Cone Situation

(X, x_-) from 4.2.4(ii) in that there is a commutative square

$$\begin{tikzcd} S^\nu \arrow[r, "\nu"'] \arrow[d, "\tilde\varphi_+"'] & S \arrow[d, "\varphi_-"] \\ C \arrow[r, "\phi_C"] & C \end{tikzcd}$$

see 4.4.12. This diagram is used to relate the groups $\mathrm{H}^i(S, \mathcal{O}_S)$ to the cohomology of a sheaf $\mathscr{F}$ on C in 4.4.28; the structure of $\mathscr{F}$ is made explicit in 4.5, and its cohomology is computed in the case $q = p$ in 4.6. See 4.6.16 for a summary.

Throughout this Section, X is a q-bic threefold of type $\mathbf{N}_2 \oplus \mathbf{1}^{\oplus 3}$ associated with a q-bic form (V, β). Write

$$L_- := \mathrm{Fr}^*(V)^\perp \quad \text{and} \quad L_+ := \mathrm{Fr}^{-1}(V^\perp)$$

for the two kernels of β, and let $x_\pm := \mathbf{P}L_\pm$ be the corresponding points of X. By 2.2.7, x_+ is the unique singular point of X.

4.4.1. Orthogonal decomposition. — As in 1.3.6, the q-bic form (V, β) admits a canonical orthogonal decomposition

$$(V, \beta) = (U, \beta_U) \perp (W, \beta_W)$$

where $U := L_- \oplus L_+$ is the span of the two kernels, and

$$W := \mathrm{Fr}^*(U)^\perp \cap \mathrm{Fr}^{-1}(U^\perp) = \mathrm{Fr}^*(L_-)^\perp \cap \mathrm{Fr}^{-1}(L_+^\perp)$$

is its unique orthogonal complement. In particular, $C := X \cap \mathbf{P}W$ is a distinguished smooth q-bic curve contained inside X.

4.4.2. Automorphisms. — Consider the automorphism group scheme of the q-bic form (V, β) of type $\mathbf{N}_2 \oplus \mathbf{1}^{\oplus 3}$. The orthogonal decomposition

$$(V, \beta) \cong (U, \beta_U) \perp (W, \beta_W) \cong (\mathbf{k}^{\oplus 2}, \mathbf{N}_2) \perp (\mathbf{k}^{\oplus 3}, \mathbf{1}^{\oplus 3})$$

constructed in 4.4.1 furnishes, via 1.3.3, a subgroup

$$\mathbf{Aut}(V, \beta) \supseteq \mathbf{Aut}(U, \beta_U) \times \mathbf{Aut}(W, \beta_W) \cong \mathbf{Aut}(\mathbf{k}^{\oplus 2}, \mathbf{N}_2) \times \mathrm{U}_3(q)$$

where the former factor was computed in 3.3.1. Furthermore, 2.3.3 implies that the Lie algebra of $\mathbf{Aut}(V,\beta)$ is of dimension 5. The full automorphism group scheme can be described upon specializing 1.3.7 with $a=1$ and $b=3$:

4.4.3. Proposition. — *With the notation above, $\mathbf{Aut}(V,\beta)$ is the 1-dimensional closed subgroup scheme of $\mathbf{GL}(V) \cong \mathbf{GL}(\mathbf{k}^{\oplus 2} \oplus W)$ consisting of matrices of the form*

$$\left(\begin{array}{cc|c} \lambda & \epsilon & \mathbf{y}^\vee \\ 0 & \lambda^{-q} & 0 \\ \hline 0 & \mathbf{x} & A \end{array}\right)$$

where $A \in \mathrm{U}_3(q)$, $\lambda \in \mathbf{G}_m$, $\epsilon \in \boldsymbol{\alpha}_q$, $\mathbf{x} \in \boldsymbol{\alpha}_q^3$, and $\mathbf{y} \in \boldsymbol{\alpha}_{q^2}^3$, subject to the equation

$$\mathbf{x} = \lambda^{-q}(A^{\vee,(q)}\beta_W)^{-1}\mathbf{y}^{(q)} \qquad \blacksquare$$

As in 2.3.1, the linear action of $\mathbf{Aut}(V,\beta)$ on $\mathbf{P}V$ restricts to an action on X. In particular, the torus $\mathbf{G}_m$ acts on X and its fixed locus is seen to be the following:

4.4.4. Corollary. — *$\mathbf{G}_m$ acts on X with fixed locus $\{x_+, x_-\} \cup C$.* $\blacksquare$

4.4.5. Lines. — By 2.7.14(ii), 2.7.10, 2.7.12, and 2.7.15, the Fano scheme S of X is a connected, Cohen–Macaulay surface which is singular along the curve C_+, where

$$C_\pm := \{\, [\ell] \in S \mid x_\pm \in \ell \,\}.$$

Since the $x_\pm$ are cone points over the smooth q-bic curve C, both $C_\pm$ are isomorphic to C. Furthermore, it follows from 4.4.4 that the action of the torus $\mathbf{G}_m$ from the action of $\mathbf{Aut}(V,\beta)$ on S has fixed locus consisting of the curves $C_\pm$.

4.4.6. Lemma. — *$\mathbf{G}_m$ acts on S with fixed locus $C_+ \cup C_-$.*

Proof. — Consider first a line ℓ through either x_+ or x_-. Then ℓ intersects C at a unique point. But all of C, x_+, and x_- are fixed for the action of $\mathbf{G}_m$ on X by 4.4.4, so ℓ contains at least two fixed points and thus must itself be fixed. If ℓ misses both x_+ and x_- on X, then ℓ is not contained in $X \cap \mathbf{P}\,\mathrm{Fr}^{-1}(L_+^\perp)$ and they intersect at a unique point away from x_+. Since $\mathbf{G}_m$ scales along the cone, ℓ is not fixed. $\blacksquare$

4.4.7. Cone points and situations. — The orthogonal decomposition of (V,β) given in 4.4.1 together with the compatibility of Hermitian points with orthogonal decompositions from 1.2.5 neatly determines the cone points, in the sense of 2.4.5, of X. In terms of the types given in 2.4.7, they are:

(i) the singular point $x_+ := \mathbf{P}L_+$;
(ii) the special point $x_- := \mathbf{P}L_-$; and
(iii) the Hermitian points of C.

The points $x_\pm$ give rise to two Cone Situations

$$(X, x_+, \mathbf{P}\,\mathrm{Fr}^{-1}(L_+^\perp)) \quad \text{and} \quad (X, x_-, \mathbf{P}\,\mathrm{Fr}^*(L_-)^\perp)$$

as already seen in 4.2.4(ii) and 4.2.4(iii). Both Cone Situations satisfy the conditions 4.2.1(i) and 4.2.1(iii) and so, by 4.2.6, give rise to rational maps from S to a smooth q-bic curve. The corresponding cones

$$X_+ := X \cap \mathbf{P}\,\mathrm{Fr}^{-1}(L_+^\perp) \quad \text{and} \quad X_- := X \cap \mathbf{P}\,\mathrm{Fr}^*(L_-)^\perp$$

may be canonically viewed as cones over the curve $C \subset \mathbf{P}W$ determined by β_W. Therefore the two Cone Situations give a pair of rational maps

$$\varphi_\pm \colon S \dashrightarrow C$$

which are *a priori* defined away from the curve $C_\pm \subset S$ parameterizing lines through $x_\pm$, see 4.2.3. Since φ_- arises from a Smooth Cone Situation, see 4.3.2, it extends to a morphism $\varphi_- \colon S \to C$ by 4.3.7(ii).

4.4.8. Projection from U. — The maps $\varphi_\pm$ may be more directly described via the specific geometry of X. The orthogonal decomposition of 4.4.1 identifies linear projection of X away from $\mathbf{P}U$ as a rational map

$$\mathrm{proj}_{\mathbf{P}U} \colon X \dashrightarrow \mathbf{P}W.$$

As in A.2.4, this is resolved by a diagram

$$\begin{tikzcd} \tilde{X} \arrow[r, hook] \arrow[d] & \mathbf{P}\mathscr{U} \arrow[d] \\ X \arrow[r, dashed, "\mathrm{proj}_{\mathbf{P}U}"] & \mathbf{P}W \end{tikzcd}$$

in which $\mathscr{U} \cong U_{\mathbf{P}W} \oplus \mathscr{O}_{\mathbf{P}W}(-1)$. The blowup $\tilde{X}$ of X along $X \cap \mathbf{P}U$ inherits the structure of a bundle of q-bic curves over $\mathbf{P}W$ given by the q-bic form

$$\beta_{\mathscr{U}} \colon \mathrm{Fr}^*(\mathscr{U}) \otimes \mathscr{U} \subset \mathrm{Fr}^*(V)_{\mathbf{P}W} \otimes V_{\mathbf{P}W} \to \mathscr{O}_{\mathbf{P}W}.$$

The fibres of $\tilde{X} \to \mathbf{P}W$ are all singular q-bic curves. In the following, view the cones $X_{\pm}$ as having base C:

4.4.9. Lemma. — *The fibre $\tilde{X}_{y_0}$ over a closed point y_0 of $\mathbf{P}W$ is a q-bic curve of type*

$$\mathrm{type}(\tilde{X}_{y_0}) = \begin{cases} \mathbf{N}_2 \oplus \mathbf{1} & \text{if } y_0 \in \mathbf{P}W \setminus C, \text{ and} \\ \mathbf{0} \oplus \mathbf{N}_2 & \text{if } y_0 \in C. \end{cases}$$

In fact, if $y_0 \in C$, then $\tilde{X}_{y_0} = \langle y_0, x_- \rangle \cup q\langle y_0, x_+ \rangle$.

Proof. — The fibre $\tilde{X}_{y_0}$ is the q-bic curve obtained by intersecting X with the plane spanned by $\mathbf{P}U$ and y_0. Since each such plane intersects the singular point x_+, 2.2.7 implies that each fibre is a singular curve. Since $\mathbf{P}U \cap X = \{x_+, x_-\}$, the only lines contracted by $\mathrm{proj}_{\mathbf{P}U}$ are those in the cones $X_{\pm}$ over C. So if $y_0 \in \mathbf{P}W \setminus C$, then $\tilde{X}_{y_0}$ is a singular q-bic curve that contains no lines, and hence is of type $\mathbf{N}_2 \oplus \mathbf{1}$ by 3.4.1. On the other hand, if $y_0 \in C$, the fact that x_+ is a cone point of multiplicity q implies that $\tilde{X}_{y_0}$ must be the union of the lines $\langle y_0, x_- \rangle$ and $\langle y_0, x_+ \rangle$, the latter appearing with multiplicity q; in particular, $\tilde{X}_{y_0}$ is of type $\mathbf{0} \oplus \mathbf{N}_2$. ■

4.4.10. Corollary. — *Let $\ell \subset X$ be a line not passing through either $x_{\pm}$. Then*

$$\ell_0 := \mathrm{proj}_{\mathbf{P}U}(\ell) \subset \mathbf{P}W$$

is the tangent to C at $\mathrm{proj}_{\mathbf{P}U}(\ell \cap X_+)$ *with residual intersection* $\mathrm{proj}_{\mathbf{P}U}(\ell \cap X_-)$*. Thus*

$$\varphi_+([\ell]) = \textit{point of tangency between } \ell_0 \textit{ and } C,$$

$$\varphi_-([\ell]) = \textit{residual point of intersection between } \ell_0 \textit{ and } C.$$

Proof. — Since the only lines contracted by $\mathrm{proj}_{\mathbf{P}U}$ are those through $x_\pm$, for a line ℓ as in the statement, $\ell_0 := \mathrm{proj}_{\mathbf{P}U}(\ell)$ is a line in $\mathbf{P}W$. Observe that

$$C \cap \ell_0 = \mathrm{proj}_{\mathbf{P}U}\big(\ell \cap \mathrm{proj}_{\mathbf{P}U}^{-1}(C)\big) = \mathrm{proj}_{\mathbf{P}U}\big(\ell \cap (X_- \cup qX_+)\big)$$

with the second equality due to 4.4.9. Thus ℓ_0 and C are tangent at $\mathrm{proj}_{\mathbf{P}U}(\ell \cap X_+)$ and have residual point of intersection at $\mathrm{proj}_{\mathbf{P}U}(\ell \cap X_-)$. On the other hand, the definition of $\varphi_\pm$ from 4.2.6 gives

$$\varphi_\pm([\ell]) = \mathrm{proj}_{x_\pm}(\ell \cap X_\pm) = \mathrm{proj}_{\mathbf{P}U}(\ell \cap X_\pm)$$

with the second equality because $x_\mp \notin X_\pm$. The result now follows. ■

This description of φ_- gives an alternate proof that it extends to a morphism:

4.4.11. Corollary. — *The rational map φ_- extends to a morphism $S \to C$.*

Proof. — By 4.4.10, φ_- takes a $\ell \subset X$ not contained in either cone $X_\pm$ to the residual intersection point with C to its tangent line at $\mathrm{proj}_{\mathbf{P}U}(\ell \cap X_+)$. This description makes sense for $\ell \subset X_-$, thereby extending φ_- over C_-. ■

The descriptions of $\varphi_\pm$ from 4.4.10 yield a direct relationship between them:

4.4.12. Corollary. — *There is a commutative square*

$$\begin{tikzcd} S \arrow[r, "\mathrm{id}_S"'] \arrow[d, dashed, "\varphi_+"'] & S \arrow[d, "\varphi_-"] \\ C \arrow[r, "\phi_C"] & C \end{tikzcd}$$

where $\phi_C \colon C \to C$ is the endomorphism sending a point $x \in C$ to the residual intersection point between C and its tangent line at x, as described in 3.5.2. ■

The next task is to resolve φ_+. By 4.4.10, this can be done by constructing a proper family of lines in X parameterized by C with the property that the general line lying over $y_0 \in C$ projects via $\mathrm{proj}_{\mathbf{P}U}$ to the tangent line $\mathbf{T}_{C,y_0}$. The universal such family is obtained by taking the pencil of hyperplane sections of X parameterized by C such that the fibre over $y_0 \in C$ is the intersection of X with the hyperplane

$$\mathscr{W}_{y_0} := \langle \mathbf{T}_{C,y_0}, \mathbf{P}U \rangle \subset \mathbf{P}V.$$

This gives a family of q-bic surfaces over C in which each fibre has a ruling by lines. Taking this family of rulings appropriately yields the sought-after resolution.

4.4.13. Tangent pencil. — To proceed with the construction, let $\mathscr{T}_C^{\mathrm{e}} \subset W_C$ be the embedded tangent bundle of C in $\mathbf{P}W$, as in 2.2.8, and let

$$\mathscr{W} := U_C \oplus \mathscr{T}_C^{\mathrm{e}} \subset U_C \oplus W_C \subset V_C.$$

The projective bundle $\mathbf{P}\mathscr{W}$ admits a canonical morphism to $\mathbf{P}V$ so that the image of the fibre $\mathbf{P}\mathscr{W}_{y_0}$ over a point $y_0 \in C$ is the hyperplane spanned by $\mathbf{P}U$ and $\mathbf{T}_{C,y_0} \subset \mathbf{P}W \subset \mathbf{P}V$. The corresponding family $X_{\mathbf{P}\mathscr{W}} := X \times_{\mathbf{P}V} \mathbf{P}\mathscr{W}$ of hyperplane sections of X is then the family of q-bic surfaces over C determined by the q-bic form

$$\beta_{\mathscr{W}} \colon \mathrm{Fr}^*(\mathscr{W}) \otimes \mathscr{W} \subset \mathrm{Fr}^*(V)_C \otimes V_C \xrightarrow{\beta} \mathscr{O}_C.$$

The splitting $(V,\beta) = (U,\beta_U) \perp (W,\beta_W)$ of 4.4.1 gives an orthogonal decomposition

$$(\mathscr{W},\beta_{\mathscr{W}}) = (U_C,\beta_U) \perp (\mathscr{T}_C^{\mathrm{e}},\beta_{W,\tan})$$

where $\beta_{W,\tan}$ is the tangent form associated with C, as in 2.2.13. This implies that the fibres of $X_{\mathbf{P}\mathscr{W}} \to C$ are certain singular q-bic surfaces:

4.4.14. Lemma. — *The fibre $X_{\mathbf{P}\mathscr{W},y_0}$ over $y_0 \in C$ is a q-bic surface of type*

$$\mathrm{type}(X_{\mathbf{P}\mathscr{W},y_0}) = \begin{cases} \mathbf{N}_2^{\oplus 2} & \text{if } y_0 \in C \text{ is not a Hermitian point,} \\ \mathbf{0} \oplus \mathbf{N}_2 \oplus \mathbf{1} & \text{if } y_0 \in C \text{ is a Hermitian point.} \end{cases}$$

The singular locus of $X_{\mathbf{P}\mathscr{W}} \to C$ is supported on the subbundle

$$\mathbf{P}\mathscr{W}' = \mathbf{P}(L_{+,C} \oplus \mathscr{O}_C(-1)) = \{ (y_0,x) \mid y_0 \in C, x \in \langle y_0, x_+ \rangle \subset \mathbf{P}\mathscr{W}_{y_0} \} \subset \mathbf{P}\mathscr{W}.$$

The subbundle $\mathscr{W}'$ is characterized by the property that $\mathrm{Fr}^*(\mathscr{W}') = \mathscr{W}^{\perp}$.

Proof. — The identification of types follows from the orthogonal decomposition of $\beta_{\mathcal{W}}$: β_U is a constant q-bic form of type $\mathbf{N}_2$, whereas 3.5.2 implies that $\beta_{W,\tan}$ is of type $\mathbf{0}\oplus\mathbf{1}$ over Hermitian points of C and $\mathbf{N}_2$ otherwise. By 2.2.7, the singular locus of the morphism $X_{\mathbf{P}\mathcal{W}} \to C$ is given by the image of

$$\mathcal{W}^{\perp} = U_C^{\perp_{\beta_U}} \oplus (\mathcal{T}_C^{\mathrm{e}})^{\perp_{\beta_{W,\tan}}} = \mathrm{Fr}^*(L_+)_C \oplus \mathrm{Fr}^*(\mathcal{O}_C(-1))$$

by 4.4.1 and 2.2.14. This gives the remaining statements. ∎

The analyses from 3.14.2 and 3.15.3 show that q-bic surfaces of types $\mathbf{N}_2^{\oplus 2}$ and $\mathbf{0}\oplus\mathbf{N}_2\oplus\mathbf{1}$ are ruled by a family of lines transversal to the singular line, and that this family of lines can be constructed by projecting away from the singular locus. Since the singular lines in the fibres of $X_{\mathbf{P}\mathcal{W}} \to C$ form a subbundle over C, this construction can be performed in this relative setting and it will yield the sought-after family $\tilde{X}_{\mathbf{P}\mathcal{W}} \to S^\vee$ of lines in X projecting to tangent lines to C.

4.4.15. Construction of $\tilde{X}_{\mathbf{P}\mathcal{W}}$. — Let $\mathcal{W}'' := \mathcal{W}/\mathcal{W}'$. Then linear projection over C of $X_{\mathbf{P}\tilde{\mathcal{W}}} \subset \mathbf{P}\mathcal{W}$ along the subbundle $\mathbf{P}\mathcal{W}'$ yields a rational map to the surface

$$S^\vee := \mathbf{P}\mathcal{W}'' \xrightarrow{\tilde{\varphi}_+} C.$$

Blowing up along $\mathbf{P}\mathcal{W}'$ resolves this into a morphism $\tilde{X}_{\mathbf{P}\mathcal{W}} \to S^\vee$, which can be identified as a $\mathbf{P}^1$-bundle as follows: The blowup of $\mathbf{P}\mathcal{W}$ along $\mathbf{P}\mathcal{W}'$ is identified in A.2.4 as the $\mathbf{P}^2$-bundle over $S^\vee$ associated with $\tilde{\mathcal{W}}$ formed in the pullback diagram

$$\begin{tikzcd}
0 \arrow[r] & \tilde{\varphi}_+^*\mathcal{W}' \arrow[r] \arrow[d, equal] & \tilde{\mathcal{W}} \arrow[r] \arrow[d, hook] & \mathcal{O}_{\tilde{\varphi}_+}(-1) \arrow[r] \arrow[d, hook, "\mathrm{eu}_{\tilde{\varphi}_+}"] & 0 \\
0 \arrow[r] & \tilde{\varphi}_+^*\mathcal{W}' \arrow[r] & \tilde{\varphi}_+^*\mathcal{W} \arrow[r] & \tilde{\varphi}_+^*\mathcal{W}'' \arrow[r] & 0
\end{tikzcd}$$

and so the exceptional divisor of $\mathbf{P}\tilde{\mathcal{W}} \to \mathbf{P}\mathcal{W}$ is the subbundle $\mathbf{P}(\tilde{\varphi}_+^*\mathcal{W}') \subset \mathbf{P}\tilde{\mathcal{W}}$. The inverse image $X_{\mathbf{P}\tilde{\mathcal{W}}} := X_{\mathbf{P}\mathcal{W}} \times_{\mathbf{P}\mathcal{W}} \mathbf{P}\tilde{\mathcal{W}}$ of $X_{\mathbf{P}\mathcal{W}}$ along this blowup is the bundle of q-bic curves over $S^\vee$ defined by the q-bic form

$$\beta_{\tilde{\mathcal{W}}} \colon \mathrm{Fr}^*(\tilde{\mathcal{W}}) \otimes \tilde{\mathcal{W}} \subset \mathrm{Fr}^*(\tilde{\varphi}_+^*\mathcal{W}) \otimes \tilde{\varphi}_+^*\mathcal{W} \xrightarrow{\beta_{\mathcal{W}}} \mathcal{O}_{S^\vee}.$$

Since $\mathrm{Fr}^*(\mathscr{W}') = \mathscr{W}^\perp$ by 4.4.14, $\tilde{\varphi}_+^* \mathrm{Fr}^*(\mathscr{W}') = \tilde{\mathscr{W}}^\perp$ and there is an exact sequence

$$0 \to \mathscr{K} \to \tilde{\mathscr{W}} \xrightarrow{\beta_{\tilde{\mathscr{W}}}} \mathrm{Fr}^* \tilde{\mathscr{W}}^\vee \to \mathrm{Fr}^*(\tilde{\varphi}_+^* \mathscr{W}')^\vee \to 0$$

where $\mathscr{K}$ is a rank 2 subbundle such that $\tilde{\mathscr{W}}/\mathscr{K} \cong \mathscr{O}_{\tilde{\varphi}_+}(q)$ via $\beta_{\tilde{\mathscr{W}}}$. The sequences implies that $X_{\mathbf{P}\tilde{\mathscr{W}}} \to S^\nu$ is generically of type $\mathbf{0} \oplus \mathbf{N}_2$ with its two irreducible components given by $\mathbf{P}\mathscr{K}$ and $\mathbf{P}(\tilde{\varphi}_+^* \mathscr{W}')$, appearing with multiplicities 1 and q, respectively. Since $\mathbf{P}(\tilde{\varphi}_+^* \mathscr{W}')$ is the exceptional divisor, the strict transform is $\tilde{X}_{\mathbf{P}W} = \mathbf{P}\mathscr{K}$.

Therefore $\tilde{X}_{\mathbf{P}\mathscr{W}} \to S^\nu$ is the family of lines in X given by the isotropic subbundle $\mathscr{K} \subset V_{S^\nu}$. This induces a morphism $\nu \colon S^\nu \to S$. Let

$$\mathscr{W}'' = L_{-,C} \oplus \mathscr{T}_C(-1), \quad C_-^\nu := \mathbf{P}L_{-,C}, \quad C_+^\nu := \mathbf{P}(\mathscr{T}_C(-1))$$

be the splitting induced by the orthogonal decomposition $\mathscr{W} = U_C \oplus \mathscr{T}_C^{\mathrm{e}}$ and the corresponding subbundles of S^ν.

4.4.16. Fibres of $\mathscr{K}$. — To understand the morphism ν, consider the fibre of $\mathscr{K}$ at a point $x \in S^\nu$. Set $y := \tilde{\varphi}_+(x) \in C$. The plane $\mathbf{P}\tilde{\mathscr{W}}_x$ is that spanned in $\mathbf{P}V$ by the line $\ell_y := \langle x_+, y\rangle = \mathbf{P}\mathscr{W}'_y$ and a point $z \neq y$ on the cone X_-. Then $X \cap \mathbf{P}\tilde{\mathscr{W}}_x$ is a q-bic curve containing ℓ_y with multiplicity q together with a residual line $\ell = \mathbf{P}\mathscr{K}_x$. The behaviour of $\mathscr{K}_x$ splits into two main cases:

- If $x \in C_+^\nu$, then $\mathbf{P}\tilde{\mathscr{W}}_x = \langle x_+, \mathbf{T}_{C,y}\rangle$ and $X \cap \mathbf{P}\tilde{\mathscr{W}}_x = (q+1)\ell_y$, so $\mathscr{K}_x = \mathscr{W}'_y$.
- If $x \notin C_+^\nu$, then $\mathbf{P}\tilde{\mathscr{W}}_x = \langle x_+, y, z\rangle$ for $z \in X_- \setminus \mathbf{P}W$, so $z \in \ell$ and $\mathscr{K}_x \neq \mathscr{W}'_y$.

Since $\nu(x) = [\ell]$, this shows that $\nu(C_\pm^\nu) = C_\pm$ and implies that ν is bijective on points. In fact, much more is true:

4.4.17. Proposition. — *The morphism $\nu \colon S^\nu \to S$ fits into a commutative diagram*

$$\begin{tikzcd} S^\nu \arrow[rr, "\nu"] \arrow[dr, "\tilde{\varphi}_+"'] & & S \arrow[dl, dashed, "\varphi_+"] \\ & C, & \end{tikzcd}$$

is bijective, restricts to an isomorphism $S^\nu \setminus C_+^\nu \to S \setminus C_+$, and so is the normalization.

That the diagram commutes on closed points can be seen by comparing the description of φ_+ from 4.4.10 with the fact that the fibres $S^\nu_{y_0}$, for $y_0 \in C$, parameterizes the lines contained in the hyperplane section $X_{\mathbf{P}\mathscr{W}_{y_0}} = X \cap \langle \mathbf{T}_{C,y_0}, \mathbf{P}U \rangle$. Bijectivity of ν is as above or can then be seen fibrewise via 3.8.13. However, the crucial point is separability of ν. The following argument will directly construct an inverse to $\nu \colon S^\nu \setminus C^\nu_+ \to S \setminus C_+$.

Proof. — Let $S^{\nu,\circ} := S^\nu \setminus C^\nu_+$ and $S^\circ := S \setminus C_+$. The discussion of 4.4.15 together with 4.4.16 gives an exact commutative diagram of locally free modules on $S^{\nu,\circ}$ given by

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{K}' \arrow[r] \arrow[d, hook] & \mathscr{K}|_{S^{\nu,\circ}} \arrow[r] \arrow[d, hook] & \mathscr{O}_{\tilde{\varphi}_+}(-1)|_{S^{\nu,\circ}} \arrow[r] \arrow[d, hook, "\mathrm{eu}_{\tilde{\varphi}_+}"] & 0 \\ 0 \arrow[r] & \tilde{\varphi}_+^*\mathscr{W}'|_{S^{\nu,\circ}} \arrow[r] & \tilde{\varphi}_+^*\mathscr{W}|_{S^{\nu,\circ}} \arrow[r] & \tilde{\varphi}_+^*\mathscr{W}''|_{S^{\nu,\circ}} \arrow[r] & 0\,. \end{tikzcd}$$

Moreover, since $\ell = \mathbf{P}\mathscr{K}_x$ never passes through x_+ for $x \in S^{\nu,\circ}$, the composition

$$\mathscr{K}' \hookrightarrow \tilde{\varphi}_+^*\mathscr{W}'|_{S^{\nu,\circ}} = L_{+,S^{\nu,\circ}} \oplus \tilde{\varphi}_+^*\mathscr{O}_C(-1)|_{S^{\nu,\circ}} \twoheadrightarrow \tilde{\varphi}_+^*\mathscr{O}_C(-1)|_{S^{\nu,\circ}}$$

is an isomorphism. Thus $\tilde{\varphi}_+ \colon S^{\nu,\circ} \to C$ is determined by the map $\mathscr{K}' \to W_{S^{\nu,\circ}}$.

On the other hand, the description $\varphi_+([\ell]) = \mathrm{proj}_{x_+}(\ell \cap X_+)$ from 4.2.6 shows that $\varphi_+ \colon S^\circ \to C$ is determined by a map $\mathscr{S}' \to W_{S^\circ}$ from a line subbundle $\mathscr{S}' \subset \mathscr{S}$ fitting into the exact commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{S}' \arrow[r] \arrow[d, hook] & \mathscr{S}|_{S^\circ} \arrow[r] \arrow[d, hook] & L_{-,S^\circ} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & (L_+ \oplus W)_{S^\circ} \arrow[r] & V_{S^\circ} \arrow[r] & L_{-,S^\circ} \arrow[r] & 0 \end{tikzcd}$$

where $\mathscr{S}|_{S^\circ} \to L_{-,S^\circ}$ is surjective by definition of S°. Since the composite

$$\mathscr{O}_{\tilde{\varphi}_+}(-1)|_{S^{\nu,\circ}} \xrightarrow{\mathrm{eu}_{\tilde{\varphi}_+}} \tilde{\varphi}_+^*\mathscr{W}''|_{S^{\nu},\circ} = L_{-,S^{\nu,\circ}} \oplus \tilde{\varphi}_+^*\mathscr{T}_C(-1)|_{S^{\nu,\circ}} \twoheadrightarrow L_{-,S^{\nu,\circ}}$$

is an isomorphism on $S^{\nu,\circ}$, comparing the two diagrams shows that

$$\nu^*(\mathscr{S}' \to W_{S^\circ}) = (\mathscr{K}' \to W_{S^{\nu,\circ}}),$$

which proves that $\tilde{\varphi}_+ = \varphi_+ \circ \nu$ on $S^{\nu,\circ}$.

To construct the inverse to $\nu\colon S^{\nu,\circ} \to S^\circ$, observe that $\mathscr{S}|_{S^\circ} \subset \varphi_+^*\mathscr{W}$ by 4.4.10, and via this inclusion, $\mathscr{S}'$ includes into $\varphi_+^*\mathscr{W}'$. Set

$$\mathscr{S}'' := \mathrm{image}(\mathscr{S}|_{S^\circ} \to \varphi_+^*\mathscr{W} \twoheadrightarrow \varphi_+^*\mathscr{W}'').$$

Then $\mathscr{S}'' \cong \mathscr{S}|_{S^\circ}/\mathscr{S}'$ and it projects isomorphically onto the L_{-,S° part of $\varphi_+^*\mathscr{W}''$. The inclusion $\mathscr{S}'' \subset \varphi_+^*\mathscr{W}''$ gives a morphism $S^\circ \to \mathbf{P}\mathscr{W}'' = S^\nu$ of schemes over C, and comparing with the diagram for $\mathscr{K}|_{S^\nu,\circ}$ shows that this is an inverse for ν. ■

Putting 4.4.11, 4.4.12, and 4.4.17 together yield the commutative diagram in:

4.4.18. Corollary. — *There is a commutative diagram of morphisms*

$$\begin{tikzcd} S^\nu \arrow[r, "\nu"'] \arrow[d, "\tilde\varphi_+"'] & S \arrow[d, "\varphi_-"] \\ C \arrow[r, "\phi_C"] & C \end{tikzcd}$$

which is equivariant for the action

$$\mathbf{G}_m \times \mathrm{U}_3(q) \cong (\mathbf{Aut}(L_+ \subset U, \beta_U) \cap \mathbf{Aut}(L_- \subset U, \beta_U)) \times \mathbf{Aut}(W, \beta_W).$$

Proof. — That the diagram is equivariant is because $\tilde\varphi_+$ and φ_- both arise from Cone Situations, so that the discussion of 4.2.28 applies. Since the unipotent part must act trivially to preserve L_+, the computation of 3.3.1 implies $\mathbf{Aut}(L_+ \subset U, \beta_U) \cong \mathbf{G}_m$. ■

4.4.19. Conductors. — Let

$$\mathrm{cond}_{\nu,S} := \mathscr{A}nn_{\mathscr{O}_S}(\nu_*\mathscr{O}_{S^\nu}/\mathscr{O}_S) \subset \mathscr{O}_S \quad \text{and} \quad \mathrm{cond}_{\nu,S^\nu} := \nu^{-1}\mathrm{cond}_{\nu,S}\cdot\mathscr{O}_{S^\nu} \subset \mathscr{O}_{S^\nu}$$

be the conductor ideals associated with the normalization $\nu\colon S^\nu \to S$, and let

$$D := \mathrm{V}(\mathrm{cond}_{\nu,S}) \subset S \quad \text{and} \quad D^\nu := \mathrm{V}(\mathrm{cond}_{\nu,S^\nu}) \subset S^\nu$$

be the conductor subschemes of S and S^ν, respectively. The conductor ideal of S is characterized as the largest ideal of $\mathscr{O}_S$ which is also an ideal of $\nu_*\mathscr{O}_{S^\nu}$, so there is a commutative diagram of exact sequences of sheaves on S:

$$\begin{tikzcd}
 & \operatorname{cond}_{\nu,S} \arrow[r, "\cong"'] \arrow[d, hook] & \nu_* \operatorname{cond}_{\nu,S^\nu} \arrow[d, hook] & & \\
0 \arrow[r] & \mathscr{O}_S \arrow[r, "\nu^\#"] \arrow[d] & \nu_*\mathscr{O}_{S^\nu} \arrow[r] \arrow[d] & \nu_*\mathscr{O}_{S^\nu}/\mathscr{O}_S \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathscr{O}_D \arrow[r, "\nu^\#"] & \nu_*\mathscr{O}_{D^\nu} \arrow[r] & \nu_*\mathscr{O}_{D^\nu}/\mathscr{O}_D \arrow[r] & 0\,.
\end{tikzcd}$$

The basic properties of the conductor subschemes are as follows:

4.4.20. Lemma. — *Let $D \subset S$ and $D^\nu \subset S^\nu$ be the conductor subschemes associated with the normalization morphism $\nu\colon S^\nu \to S$. Then*

(i) *there is a $\mathbf{G}_m \times \mathrm{U}_3(q)$-equivariant commutative diagram*

$$\begin{tikzcd}
D^\nu \arrow[r, "\nu"'] \arrow[d, "\tilde\varphi_+"'] & D \arrow[d, "\varphi_-"] \\
C \arrow[r, "\phi_C"] & C\,,
\end{tikzcd}$$

(ii) $D^{\mathbf{G}_m} = D_{\mathrm{red}} = C_+$ *and* $D^{\nu,\mathbf{G}_m} = D^\nu_{\mathrm{red}} = C^\nu_+$, *and*

(iii) *the morphisms $\varphi_-\colon D \to C$ and $\tilde\varphi_+\colon D^\nu \to C$ are finite.*

Proof. — The diagram of (i) exists and commutes because of 4.4.18; the action of $\mathbf{G}_m \times \mathrm{U}_3(q)$ on the surfaces restricts to an action on the conductors because $\nu\colon S^\nu \to S$ is equivariant, and so $\mathscr{A}nn_{\mathscr{O}_S}(\nu_*\mathscr{O}_{S^\nu}/\mathscr{O}_S)$ is an equivariant ideal.

For (ii), note that D_+ and D^ν_+ are supported on C_+ and C^ν_+, respectively, since $\nu\colon S^\nu \to S$ is an isomorphism away from these curves by 4.4.17. That $D_+^{\mathbf{G}_m} = C_+$ follows from the corresponding statement for S from 4.4.6; likewise, that $D_+^{\nu,\mathbf{G}_m} = C^\nu_+$ is because $S^\nu \to C$ is a $\mathbf{P}^1$-bundle and $\mathbf{G}_m$ acts along fibres with fixed locus $C^\nu_+ \cup C^\nu_-$. This now implies (iii) using properness of S and S^ν over C. ∎

The conductor ideal is determined in 4.4.23 via the identification $\operatorname{cond}_{\nu,S^\nu} \cong \omega_{S^\nu/S}$ from duality theory. This is aided by two auxiliary computations. The first involves the rank 2 subbundle $\mathscr{K} \subset V_{S^\nu}$ constructed in 4.4.15:

4.4.21. Lemma. — *There are short exact sequences on S^ν given by*

$$0 \to \mathscr{O}_{\tilde{\varphi}_+}(-2q) \otimes \wedge^3 \tilde{\mathscr{W}} \to \mathscr{O}_{\tilde{\varphi}_+}(-q) \otimes \wedge^2 \tilde{\mathscr{W}} \to \mathscr{K} \to 0, \text{ and}$$
$$0 \to \mathscr{K} \to \tilde{\mathscr{W}} \to \mathscr{O}_{\tilde{\varphi}_+}(q) \to 0.$$

In particular, $\det(\mathscr{K}) \cong \tilde{\varphi}_+^* \mathscr{O}_C(-1) \otimes \mathscr{O}_{\tilde{\varphi}_+}(-q-1) \otimes L_+$.

Proof. — The concatenation of the two sequences is the Koszul complex for the surjection $\beta_{\tilde{\mathscr{W}}} \colon \tilde{\mathscr{W}} \to \mathscr{O}_{\tilde{\varphi}_+}(q)$ from 4.4.15. Compute the determinant by using, for example, the second exact sequence:

$$\begin{aligned}\det(\mathscr{K}) \cong \det(\tilde{\mathscr{W}}) \otimes \mathscr{O}_{\tilde{\varphi}_+}(-q) &\cong \tilde{\varphi}_+^* \det(\mathscr{W}') \otimes \mathscr{O}_{\tilde{\varphi}_+}(-q-1) \\ &\cong \tilde{\varphi}_+^* \mathscr{O}_C(-1) \otimes \mathscr{O}_{\tilde{\varphi}_+}(-q-1) \otimes L_+\end{aligned}$$

upon using the identification $\mathscr{W}' = L_{+,C} \oplus \mathscr{O}_C(-1)$ from 4.4.14. ■

The second computation identifies the ν-pullback of the Plücker line bundle:

4.4.22. Lemma. — $\nu^* \mathscr{O}_S(1) = \tilde{\varphi}_+^* \mathscr{O}_C(1) \otimes \mathscr{O}_{\tilde{\varphi}_+}(q+1) \otimes L_+^\vee$.

Proof. — By its construction in 4.4.15, the pullback of the tautological subbundle $\mathscr{S}$ on S via ν is the bundle $\mathscr{K}$ on S^ν. Therefore,

$$\nu^* \mathscr{O}_S(1) \cong \nu^* \det(\mathscr{S})^\vee \cong \det(\mathscr{K})^\vee \cong \tilde{\varphi}_+^* \mathscr{O}_C(1) \otimes \mathscr{O}_{\tilde{\varphi}_+}(q+1) \otimes L_+^\vee,$$

upon using the determinant computation of 4.4.21. ■

4.4.23. Proposition. — *The conductor ideal of S^ν is isomorphic to*

$$\mathrm{cond}_{\nu,S^\nu} \cong \mathscr{O}_{\tilde{\varphi}_+}(-\delta-1) \otimes (L_+^{\otimes 2q-1} \otimes L_-) \quad \textit{where } \delta := 2q^2 - q - 2,$$

and the conductor subscheme D^ν is the δ-order neighbourhood of C_+^ν.

Proof. — Evaluation at $1 \in \nu_* \mathscr{O}_{S^\nu}$ yields the first isomorphism in

$$\nu_* \mathrm{cond}_{\nu,S^\nu} \cong \mathscr{H}om_{\mathscr{O}_S}(\nu_* \mathscr{O}_{S^\nu}, \mathscr{O}_S) \cong \nu_* \omega_{S^\nu/S}$$

and duality theory for $\nu \colon S^\nu \to S$ gives the second isomorphism; see [**Stacks**, 0FKW], for instance. As ν is affine, it follows that $\mathrm{cond}_{\nu,S^\nu}$ is isomorphic to the relative dualizing sheaf $\omega_{S^\nu/S} \cong \omega_{S^\nu} \otimes \nu^*\omega_S^\vee$. Since S^ν the projective bundle over C associated with $\mathscr{W}'' = \mathscr{T}_C(-1) \oplus L_{-,C}$, the relative Euler sequence gives

$$\omega_{S^\nu} \cong \omega_{S^\nu/C} \otimes \tilde{\varphi}_+^*\omega_C \cong \mathscr{O}_{\tilde{\varphi}_+}(-2) \otimes \tilde{\varphi}_+^*(\omega_C^{\otimes 2} \otimes \mathscr{O}_C(1)) \otimes L_-^\vee.$$

By 4.1.1, $\omega_S \cong \mathscr{O}_S(2q-3) \otimes (L_+ \otimes L_-)^{\vee,\otimes 2}$, so 4.4.22 gives

$$\begin{aligned}\nu^*\omega_S &\cong \nu^*\mathscr{O}_S(2q-3) \otimes (L_+ \otimes L_-)^{\vee,\otimes 2} \\ &\cong \tilde{\varphi}_+^*\mathscr{O}_C(2q-3) \otimes \mathscr{O}_{\tilde{\varphi}_+}\big((2q-3)(q+1)\big) \otimes L_+^{\vee,\otimes 2q-1} \otimes L_-^{\vee,\otimes 2}.\end{aligned}$$

Since C is a plane curve of degree $q+1$, $\omega_C^{\otimes 2} \otimes_{\mathscr{O}_C} \mathscr{O}_C(1) \cong \mathscr{O}_C(2q-3)$. Putting the computations together yields

$$\omega_{S^\nu/S} \cong \omega_{S^\nu} \otimes \nu^*\omega_S^\vee \cong \mathscr{O}_{\tilde{\varphi}_+}(-\delta-1) \otimes (L_+^{\otimes 2q-1} \otimes L_-).$$

That D^ν is the δ-order neighbourhood of C_+^ν now follows from 4.4.20(ii). ■

4.4.24. The sheaf $\mathscr{F}$. — By 4.4.20, the $\mathscr{O}_C$-modules given by

$$\mathscr{D} := \varphi_{-,*}\mathscr{O}_D \quad \text{and} \quad \mathscr{D}^\nu := \phi_{C,*}\tilde{\varphi}_{+,*}\mathscr{O}_{D^\nu}$$

are graded $\mathscr{O}_C$-algebras whose spectra over C yield the finite morphisms $\varphi_- \colon D \to C$ and $\phi_C \circ \tilde{\varphi}_+ \colon D^\nu \to C$. There is a sequence of graded coherent $\mathscr{O}_C$-modules

$$0 \to \mathscr{D} \xrightarrow{\nu^\#} \mathscr{D}^\nu \to \mathscr{F} \to 0$$

where the morphism $\nu^\#$ is induced by $\nu \colon D^\nu \to D$. The graded module $\mathscr{F}$ will play an important role in the computation of the invariants of S thanks to the following:

4.4.25. Lemma. — *The graded coherent $\mathscr{O}_C$-module*

$$\mathscr{F} := \mathscr{D}^\nu/\mathscr{D} \cong \varphi_{-,*}(\nu_*\mathscr{O}_{D^\nu}/\mathscr{O}_D) \cong \varphi_{-,*}(\nu_*\mathscr{O}_{S^\nu}/\mathscr{O}_S)$$

is locally free and fits into a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{O}_C & \longrightarrow & \phi_{C,*}\mathscr{O}_C & \longrightarrow & \mathscr{F} & \longrightarrow & \mathbf{R}^1\varphi_{-,*}\mathscr{O}_S & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\| & & & & \\ 0 & \longrightarrow & \mathscr{D} & \longrightarrow & \mathscr{D}^\nu & \longrightarrow & \mathscr{F} & \longrightarrow & 0 & & \end{array}$$

in which the rows are exact and the surjection $\mathscr{F} \to \mathbf{R}^1\varphi_{-,*}\mathscr{O}_S$ splits.

Proof. — The identifications of $\mathscr{F}$ come from affineness of the morphisms $D \to C$ and $D^\nu \to C$ from 4.4.20 together with the commutative diagram of 4.4.19. Pushing the rows of that diagram down to C yields the diagram in the statement; exactness of the first row comes from $\varphi_{-,*}\mathscr{O}_S \cong \tilde{\varphi}_{+,*}\mathscr{O}_{S^\nu} \cong \mathscr{O}_C$ by 4.3.10(ii), the commutative diagram of 4.4.18, and that $\mathbf{R}^1\tilde{\varphi}_{+,*}\mathscr{O}_{S^\nu} = 0$ as it is a projective bundle over C. The top row of the diagram shows that $\mathscr{F}$ is an extension of the locally free sheaves $\mathbf{R}^1\varphi_{-,*}\mathscr{O}_S$, which is locally free by 4.3.10(iii), by $\phi_{C,*}\mathscr{O}_C/\mathscr{O}_C$, which is locally free since C is regular so the morphism $\phi_{C,*} \colon C \to C$, which up to an automorphism is the q^2-Frobenius by 2.9.1, is flat by [Kun69, Theorem 2.1] or [Stacks, 0EC0].

To show that $\mathscr{F} \to \mathbf{R}^1\varphi_{-,*}\mathscr{O}_S$ splits, observe that $\mathscr{O}_C \hookrightarrow \mathscr{D}$ and $\phi_{C,*}\mathscr{O}_C \hookrightarrow \mathscr{D}^\nu$ are the inclusion of the constant functions which, by 4.4.20(ii), make up the degree 0 components. Therefore the positively graded components of $\mathscr{F}$ map isomorphically to $\mathbf{R}^1\varphi_{-,*}\mathscr{O}_S$, providing the desired splitting. ■

4.4.26. Lemma. — $\mathscr{O}_S(1)|_D = \varphi_-^*\mathscr{O}_C(1)|_D \otimes L_+^\vee$.

Proof. — By 4.2.6, $\varphi_- \colon S \setminus C_- \to C$ is induced by the line subbundle of $\mathscr{S}|_{S \setminus C_-}$ obtained by intersecting with $(L_- \oplus W)_{S \setminus C_-}$. Thus there is a short exact sequence

$$0 \to \varphi_-^*\mathscr{O}_C(-1)|_{S\setminus C_-} \to \mathscr{S}|_{S\setminus C_-} \to L_{+,S\setminus C_-} \to 0.$$

Taking determinants and restricting to D yields the desired identification. ■

The following shows that sheaf $\mathscr{F}$ is, in a sense, dual to the sheaf $\mathscr{D}$:

4.4.27. Proposition. — *There are canonical isomorphisms*

(i) $\nu_*\mathscr{O}_{S^\nu}/\mathscr{O}_S \cong \mathbf{R}\mathscr{H}om_{\mathscr{O}_S}(\mathscr{O}_D, \mathscr{O}_S)[1]$ *in the derived category of S, and*
(ii) $\mathscr{F} \cong \mathscr{D}^\vee \otimes \mathscr{O}_C(-q+1) \otimes L_+^{\otimes 2q-1} \otimes L_-^{\otimes 2}$ *as graded $\mathscr{O}_C$-modules.*

Proof. — Consider the conductor subscheme exact sequence on S:

$$0 \to \mathrm{cond}_{\nu,S} \to \mathscr{O}_S \to \mathscr{O}_D \to 0.$$

Identifying $\mathrm{cond}_{\nu,S}$ with $\nu_*\omega_{S^\nu/S}$ as in 4.4.23 and applying $\mathbf{R}\mathscr{H}om_{\mathscr{O}_S}(-,\mathscr{O}_S)$ to the corresponding triangle in the derived category yields a triangle

$$\mathbf{R}\mathscr{H}om_{\mathscr{O}_S}(\mathscr{O}_D,\mathscr{O}_S)\to\mathscr{O}_S\to\nu_*\mathscr{O}_{S^\nu}\xrightarrow{+1}$$

since $\mathbf{R}\mathscr{H}om_{\mathscr{O}_S}(\mathscr{O}_S,\mathscr{O}_S)=\mathscr{O}_S$, and $\mathbf{R}\mathscr{H}om_{\mathscr{O}_S}(\nu_*\omega_{S^\nu/S},\mathscr{O}_S)=\nu_*\mathscr{O}_{S^\nu}$ by duality for ν. The map $\mathscr{O}_S\to\nu_*\mathscr{O}_{S^\nu}$ is dual to evaluation at 1, and hence is the $\mathscr{O}_S$-module map determined by $1\mapsto 1$; in other words, this is the map $\nu^\#$, yielding the first statement.

Pushing forward (i) and applying relative duality for $\varphi_-\colon S\to C$ yields

$$\mathscr{F}\cong\mathbf{R}\varphi_{-,*}\mathbf{R}\mathscr{H}om_{\mathscr{O}_S}(\mathscr{O}_D,\mathscr{O}_S)[1]\cong\mathbf{R}\mathscr{H}om_{\mathscr{O}_C}(\mathbf{R}\varphi_{-,*}(\mathscr{O}_D\otimes\omega_{\varphi_-}),\mathscr{O}_C)[1]$$

in which $\omega_{\varphi_-}=(\omega_S\otimes\varphi_-^*\omega_C^\vee)[1]$. By 4.1.1 together with 4.4.26,

$$\mathscr{O}_D\otimes\omega_S\cong\varphi_-^*\mathscr{O}_C(2q-3)|_D\otimes L_+^{\vee,\otimes 2q-1}\otimes L_-^{\vee,\otimes 2}.$$

Combining with $\omega_C\cong\mathscr{O}_C(q-2)$ gives

$$\mathbf{R}\varphi_{-,*}(\mathscr{O}_D\otimes\omega_{\varphi_-})=(\mathbf{R}\varphi_{-,*}\mathscr{O}_D)\otimes\mathscr{O}_C(q-1)\otimes L_+^{\vee,\otimes 2q-1}\otimes L_-^{\vee,\otimes 2}[1].$$

Since $D\to C$ is of relative dimension 0, $\mathbf{R}\varphi_{-,*}\mathscr{O}_D=\varphi_{-,*}\mathscr{O}_D=\mathscr{D}$, yielding the isomorphism in (ii). All identifications are functorial, so the isomorphism respects the action of $\mathbf{G}_m$ from 4.4.18, and so it is an isomorphism of graded modules. ■

The following relates the cohomology of the structure sheaf of S with the cohomology of the sheaf $\mathscr{F}$ on C:

4.4.28. Proposition. — *The cohomology of $\mathscr{O}_S$ is given by*

$$\mathrm{H}^i(S,\mathscr{O}_S)\cong\begin{cases}\mathrm{H}^0(C,\mathscr{O}_C) & \text{if } i=0,\\ \mathrm{H}^0(C,\mathscr{F}) & \text{if } i=1, \text{ and}\\ \mathrm{H}^1(C,\mathscr{F})/\mathrm{H}^1(C,\mathscr{O}_C) & \text{if } i=2.\end{cases}$$

Proof. — Consider the cohomology sequence associated with the exact sequence

$$0\to\mathscr{O}_S\to\nu_*\mathscr{O}_{S^\nu}\to\nu_*\mathscr{O}_{S^\nu}/\mathscr{O}_S\to 0.$$

Note that $\varphi_{-,*}\mathcal{O}_S \cong \mathcal{O}_C$ by 4.3.10(ii), and, for each $i = 0, 1, 2$,

$$\mathrm{H}^i(S, \nu_*\mathcal{O}_{S^\nu}/\mathcal{O}_S) \cong \mathrm{H}^i(C, \mathcal{F}) \quad \text{and} \quad \mathrm{H}^i(S^\nu, \mathcal{O}_{S^\nu}) \cong \mathrm{H}^i(C, \mathcal{O}_C)$$

by 4.4.25 and the fact that $S^\nu \to C$ is a projective bundle. Thus the long exact sequence yields $\mathrm{H}^0(S, \mathcal{O}_S) \cong \mathrm{H}^0(S^\nu, \mathcal{O}_{S^\nu}) \cong \mathrm{H}^0(C, \mathcal{O}_C)$ and an exact sequence

$$0 \to \mathrm{H}^0(C, \mathcal{F}) \xrightarrow{a} \mathrm{H}^1(S, \mathcal{O}_S) \xrightarrow{b} \mathrm{H}^1(S, \nu_*\mathcal{O}_{S^\nu}) \to \mathrm{H}^1(C, \mathcal{F}) \to \mathrm{H}^2(S, \mathcal{O}_S) \to 0.$$

The result will follow upon verifying that $b \colon \mathrm{H}^1(S, \mathcal{O}_S) \to \mathrm{H}^1(S, \nu_*\mathcal{O}_{S^\nu})$ vanishes. Since $\varphi_{-,*}\mathcal{O}_S = \mathcal{O}_C$, the Leray spectral sequence gives a short exact sequence

$$0 \to \mathrm{H}^1(C, \mathcal{O}_C) \xrightarrow{c} \mathrm{H}^1(S, \mathcal{O}_S) \xrightarrow{d} \mathrm{H}^0(C, \mathbf{R}^1\varphi_{-,*}\mathcal{O}_S) \to 0.$$

Now 4.4.25 implies that the composite $d \circ a \colon \mathrm{H}^0(C, \mathcal{F}) \to \mathrm{H}^0(C, \mathbf{R}^1\varphi_{-,*}\mathcal{O}_S)$ is a surjection. So exactness of the long sequence means it remains to show that $b \circ c \colon \mathrm{H}^1(C, \mathcal{O}_C) \xrightarrow{c} \mathrm{H}^1(S, \nu_*\mathcal{O}_{S^\nu})$ vanishes. Pushing down to C along φ_- and applying 4.4.18 shows that this is

$$\phi_C \colon \mathrm{H}^1(C, \mathcal{O}_C) \to \mathrm{H}^1(C, \phi_{C,*}\mathcal{O}_C)$$

which, by 2.9.8, is the map induced by the q^2-power Frobenius up to an automorphism. Thus by 2.6.2, this is the zero map. ∎

4.5. Structure of the algebra $\mathcal{D}$

The computation of 4.4.28 reduces the problem of computing the cohomology of the structure sheaf of S to the problem of computing cohomology of the graded $\mathcal{O}_C$-module $\mathcal{F}$ from 4.4.24. This will be accomplished, at least in the case $q = p$ is a prime, in the following Section, see 4.6.14. The structure of the module $\mathcal{F}$ is accessed via its duality with the graded $\mathcal{O}_C$-algebra $\mathcal{D} = \varphi_{-,*}\mathcal{O}_D$ from 4.4.27. The object of this Section is to describe $\mathcal{D}$, the main result being 4.5.21; a summary of its consequences for $\mathcal{F}$ is given in 4.5.25.

4.5.1. Ambient affine bundles. — The coordinate ring $\mathscr{D}$ of $\varphi_- \colon D \to C$ can be expressed as a quotient of coordinate rings of affine spaces over C via the following commutative diagram of schemes:

$$\begin{tikzcd}
D \arrow[r, hook] \arrow[dr] & S^\circ \arrow[r, hook] \arrow[d] & \mathbf{B} \arrow[d, "\rho"] \arrow[dd, bend left, "\varphi"] \\
& T^\circ \arrow[r, hook] \arrow[dr] & \mathbf{A} \arrow[d, "\pi"] \\
& & C
\end{tikzcd}$$

in which $S^\circ := S \setminus C_-$ and T° are as in 4.2.7 associated with the Smooth Cone Situation (X, x_-); the scheme $\mathbf{A}$ is the fibre product $\mathbf{A}_1 \times_C \mathbf{A}_2$ where

$$\mathbf{A}_1 := \mathbf{P}\mathscr{V}_1 \setminus \mathbf{P}L_{-,C} \quad \text{and} \quad \mathbf{A}_2 := \mathbf{P}\mathscr{V}_2 \setminus \mathbf{P}(\mathscr{T}_{\mathbf{P}W}(-1)|_C)$$

with $\mathscr{V}_1$ and $\mathscr{V}_2$ as in 4.2.10; and, finally,

$$\mathbf{B} := \mathbf{P}\mathscr{V} \setminus \mathbf{P}(\mathscr{T}_{\pi_1}(-1,0))|_{\mathbf{A}}$$

where $\mathscr{V}$ is as in 4.2.21. That T° is contained in $\mathbf{A} \subset \mathbf{P}$ follows from the description of the points of $T \setminus T^\circ$ from 4.3.5 together with the fact from 4.2.3 that the set of lines in X through x_- coincide with the set of lines in X in $\mathbf{P}\,\mathrm{Fr}^*(L_-)^\perp$; compare with 4.2.10 for a description of the points in the boundary of the projective bundles. That S° is contained in $\mathbf{B}$ is as observed in 4.2.24.

Consider the sheaf of $\mathscr{O}_C$-algebras $\mathscr{A} := \pi_*\mathscr{O}_{\mathbf{A}}$ and $\mathscr{B} := \varphi_*\mathscr{O}_{\mathbf{B}}$. The first observation is that $\mathscr{A}$ and $\mathscr{B}$ are equivariant for a linear algebraic group:

4.5.2. Lemma. — *The morphisms $\mathbf{B} \to \mathbf{A} \to C$ are equivariant for the linear action of*

$$\mathbf{Aut}(L_- \subset U) \times \mathrm{U}_3(q) = \mathbf{Aut}(L_- \subset U) \times \mathbf{Aut}(W, \beta_W) \subset \mathbf{GL}(V).$$

The unipotent radical of $\mathbf{Aut}(L_- \subset U)$ acts trivially on $\mathbf{A}$.

Proof. — By their construction from the Subquotient Situation as in 4.2.10, the morphisms $\mathbf{P}\mathscr{V} \to \mathbf{P} \to C$ are equivariant for the action of $\mathbf{Aut}(L_- \subset U) \times \mathrm{U}_3(q)$ induced by its linear action on V. Since $L_{-,C} \subset \mathscr{V}_1$, $\mathscr{V}_2 \twoheadrightarrow L_{+,C}$, and $\mathbf{Aut}(L_- \subset U)$

acts via the diagonal action on $\mathbf{P} = \mathbf{P}\mathscr{V}_1 \times_C \mathbf{P}\mathscr{V}_2$, the action of its unipotent radical is trivial. By 4.2.23,

$$\mathbf{P}(\mathscr{T}_{\pi_1}(-1,0)) = \left\{ \big((y \in \ell) \mapsto (y_0 \in \ell_0)\big) \;\middle|\; \ell = \langle y_0, x_- \rangle \right\} \subset \mathbf{P}\mathscr{V}$$

and so is stable under the action $\mathbf{Aut}(L_- \subset U)$. Thus the action restricts to the complement $\mathbf{P}\mathscr{V}^\circ := \mathbf{P}\mathscr{V} \setminus \mathbf{P}(\mathscr{T}_{\pi_1}(-1,0))$. Restricting to $\mathbf{A}$ gives the result. ∎

To describe the equivariant structure on $\mathscr{A}$ and $\mathscr{B}$, choose an isomorphism

$$\mathbf{Aut}(L_- \subset U) \cong \left\{ \begin{pmatrix} \lambda_-^{-1} & \epsilon \\ 0 & \lambda_+ \end{pmatrix} \in \mathbf{GL}(L_- \oplus L_+) \right\}.$$

The maximal torus acts on $L_-^{\vee,\otimes a} \otimes L_+^{\otimes b}$ with weight $(a,b) \in \mathbf{Z}^2_{\geq 0}$ and equips both $\mathscr{A}$ and $\mathscr{B}$ with a bigrading, and the bigraded pieces are described in the next statement; the action of the unipotent radical is described in 4.5.6.

4.5.3. Lemma. — *The q-bic form β induces an equivariant isomorphism of bigraded $\mathscr{O}_C$-algebras*

$$\mathscr{A} \cong \mathrm{Sym}^*(\mathscr{O}_C(-1) \otimes L_-^\vee \oplus \Omega^1_{\mathbf{P}W}(1)|_C \otimes L_+).$$

The $\mathscr{O}_C$-algebra $\mathscr{B}$ is a filtered $\mathscr{A}$-algebra with increasing $\mathbf{Z}_{\geq 0}$-filtration $\mathrm{Fil}_\bullet \mathscr{B}$ whose associated graded pieces are equivariantly identified as

$$\mathrm{gr}_i \mathscr{B} := \mathrm{Fil}_i \mathscr{B} / \mathrm{Fil}_{i-1} \mathscr{B} \cong \mathscr{A} \otimes (L_-^\vee \otimes L_+)^{\otimes i} \quad \textit{for all } i \in \mathbf{Z}_{\geq 0}.$$

Proof. — By 4.4.1, β induces an orthogonal decomposition $V \cong U \oplus W$. This induces splittings $\mathscr{V}_1 \cong L_{-,C} \oplus \mathscr{O}_C(-1)$ and $\mathscr{V}_2 \cong L_{+,C} \oplus \mathscr{T}_{\mathbf{P}W}(-1)|_C$ of the short exact sequences from 4.2.10. Then A.1.10 gives an isomorphism

$$\mathbf{A} \cong \mathbf{A}(L_- \otimes \mathscr{O}_C(1)) \times_C \mathbf{A}(\mathscr{T}_{\mathbf{P}W}(-1)|_C \otimes L_+^\vee) \cong \mathbf{A}(L_- \otimes \mathscr{O}_C(1) \oplus \mathscr{T}_{\mathbf{P}W}(-1)|_C \otimes L_+^\vee),$$

whence the identification of $\pi_* \mathscr{O}_{\mathbf{A}}$.

As for $\mathbf{B}$, the computation A.1.9 together with the exact sequence from 4.2.23 gives an isomorphism of $\mathscr{O}_{\mathbf{A}}$-algebras

$$\rho_* \mathscr{O}_{\mathbf{B}} \cong \mathrm{colim}_n \, \mathrm{Sym}^n(\mathscr{V}^\vee(0,-1))|_{\mathbf{A}}.$$

The exact sequence for $\mathscr{V}$ induces a two step filtration

$$\mathscr{O}_T = \mathrm{Fil}_0(\mathscr{V}^\vee(0,-1)) \subset \mathrm{Fil}_1(\mathscr{V}^\vee(0,-1)) = \mathscr{V}^\vee(0,-1).$$

This induces an $n+1$ step filtration on $\mathrm{Sym}^n(\mathscr{V}^\vee(0,-1))$. The description of the transition maps from A.1.9 of the colimit show that they are compatible with the filtrations, so $\rho_*\mathscr{O}_{\mathbf{B}}$ inherits a filtration by $\mathbf{Z}_{\geq 0}$ and its graded pieces are

$$\mathrm{gr}_i\,\rho_*\mathscr{O}_{\mathbf{B}} = \Omega^1_{\pi_1}(1,-1)|_{\mathbf{A}}^{\otimes i} \cong \mathscr{O}_{\mathbf{A}} \otimes (L_-^\vee \otimes L_+)^{\otimes i} \quad \text{for all } i \in \mathbf{Z}_{\geq 0}$$

upon using that $\Omega^1_{\pi_1}(1,0)|_{\mathbf{A}} \cong L^\vee_{-,\mathbf{A}}$ and $\mathscr{O}_{\mathbf{P}}(0,-1)|_{\mathbf{A}} \cong L_{+,\mathbf{A}}$, see A.1.11. ■

The bigraded pieces of $\mathscr{A}$ and $\mathscr{B}$ can be described in terms of the sheaves appearing in 4.5.3. Throughout this Section, the sheaf $W_C^\vee(-1)$ is viewed as a filtered $\mathscr{O}_C$-module via the Euler sequence

$$0 \to \Omega^1_{\mathbf{P}W}|_C \to W_C^\vee(-1) \to \mathscr{O}_C \to 0.$$

This induces a $d+1$ step filtration on $\mathrm{Sym}^d(W_C^\vee(-1))$ with graded pieces

$$\mathrm{gr}_i\,\mathrm{Sym}^d(W_C^\vee(-1)) \cong \mathrm{Sym}^{d-i}(\Omega^1_{\mathbf{P}W}(1)|_C) \quad \text{for } 0 \leq i \leq d.$$

4.5.4. Lemma. — *The form β induces equivariant isomorphisms of filtered bundles*

$$\mathscr{B}_{(a,0)} \cong \mathscr{A}_{(a,0)} \cong \mathscr{O}_C(-a), \quad \mathscr{B}_{(0,b)} \cong \mathscr{A}_{(0,b)} \cong \mathrm{Sym}^b(\Omega^1_{\mathbf{P}W}(1)|_C), \quad \mathscr{B}_{(1,1)} \cong W_C^\vee(-1),$$

and $\mathscr{B}_{(a,b)} \cong \mathrm{Fil}_a\,\mathrm{Sym}^b(W_C^\vee(-1)) \otimes \mathscr{O}_C(b-a)$ *for all integers* $a, b \geq 0$.

Proof. — Apply A.4.8 with $\mathscr{W}_1 = \mathscr{O}_C(-1)$, $\mathscr{W}_2 = \mathscr{T}_{\mathbf{P}W}(-1)|_C$, $\mathscr{W} = W_C^\vee(-1)$, $L_1 = L_{-,C}$, and $L_2 = L_{+,C}$. ■

A basic, useful point about the algebra structure of $\mathscr{B}$ is:

4.5.5. Lemma. — *The multiplication map* $\mathscr{B}_{(a,b)} \otimes \mathscr{B}_{(c,0)} \to \mathscr{B}_{(a+c,b)}$ *is injective for all integers* $a, b, c \geq 0$*, and is an isomorphism if and only if* $a \geq b$.

Proof. — Injectivity is because $\mathscr{B}$ is locally a polynomial algebra and $\mathscr{B}_{(c,0)}$ is generated by a monomial. Surjectivity if and only if $a \geq b$ now follows from 4.5.4, since $\mathrm{Fil}_a\,\mathrm{Sym}^b(W_C^\vee(-1)) = \mathrm{Sym}^b(W_C^\vee(-1))$ if and only if $a \geq b$. ■

4.5.6. Unipotent automorphisms. — The unipotent radical of $\mathbf{Aut}(L_- \subset U)$ from 4.5.2 gives $\mathscr{B}$ an $\mathscr{A}$-linear $\mathbf{G}_a$-equivariant structure. This is equivalent to an $\mathscr{A}$-comodule structure

$$\mathscr{B} \to \mathscr{B} \otimes_{\mathbf{k}} \mathbf{k}[\epsilon] : z \mapsto \sum\nolimits_{j=0}^{\infty} \partial_j(z) \otimes \epsilon^j$$

where $\mathbf{G}_a := \operatorname{Spec}(\mathbf{k}[\epsilon])$, and the $\partial_j : \mathscr{B} \to \mathscr{B}$ are $\mathscr{A}$-linear maps such that a given local section z lies in the kernel of all but finitely many ∂_j, $\partial_0 = \mathrm{id}$, and

$$\partial_j \circ \partial_k = \binom{j+k}{j} \partial_{j+k} \quad \text{for integers } j, k \geq 0.$$

See [Jan03, I.7.3, I.7.8, and I.7.12] for details. The ∂_j in the setting at hand will now be described in several steps:

Step 1. The $\mathbf{G}_a$-equivariant structure is induced by a representation on V given by

$$\mathbf{Aut}_{\mathrm{uni}}(L_- \subset U) \cong \left\{ \begin{pmatrix} 1 & -\epsilon \\ 0 & 1 \end{pmatrix} \right\} \subset \mathbf{GL}(V).$$

It will be convenient to describe the dual representation, so let $U = \langle u_-, u_+ \rangle$ be a choice of basis compatible with this embedding into $\mathbf{GL}(V)$, and let $U^\vee = \langle u_-^\vee, u_+^\vee \rangle$ be the dual basis. Then the corresponding comodule $V^\vee \to V^\vee \otimes_{\mathbf{k}} \mathbf{k}[\epsilon]$ is given by $\partial_1(u_-^\vee) = u_+^\vee$, $\partial_1(W^\vee \oplus L_+^\vee) = 0$, and $\partial_j = 0$ for all $j \geq 2$. In particular, the operator ∂_1 restricts to an isomorphism $\partial_1 : L_-^\vee \to L_+^\vee$.

Step 2. Consider the filtration on $V_C^\vee$ determined by the short exact sequence from 4.2.10: this is the two step filtration with $\mathrm{Fil}_0 V_C^\vee = \mathscr{V}_2^\vee$ and $\mathrm{Fil}_1 V_C^\vee = V_C^\vee$. Since $L_{+,C}^\vee \subset \mathscr{V}_2^\vee$ and $\mathscr{V}_1^\vee \twoheadrightarrow L_{-,C}^\vee$, this implies that $\partial_1 : V_C^\vee \to V_C^\vee$ satisfies

$$\partial_1(\mathrm{Fil}_i V_C^\vee) \subseteq \mathrm{Fil}_{i-1} V_C^\vee \quad \text{for all } i \in \mathbf{Z}.$$

Pulling this up along $\pi : \mathbf{P} \to C$, twisting by the $\mathbf{G}_a$-invariant sheaf $\mathscr{O}_{\mathbf{P}}(0,-1)$, and using 4.2.21 to write

$$\mathscr{V}^\vee(0,-1) = \mathscr{H}(\Omega^1_{\pi_2} \hookrightarrow V_{\mathbf{P}}^\vee(0,-1) \twoheadrightarrow \mathscr{O}_{\mathbf{P}}(1,-1))$$

shows that ∂_1 passes through homology to yield an $\mathscr{O}_{\mathbf{P}}$-linear map $\partial_1 : \mathscr{V}^\vee(0,-1) \to \mathscr{V}^\vee(0,-1)$. The filtration on V_C induces the two step filtration $\mathscr{O}_{\mathbf{P}} \subset \mathscr{V}^\vee(0,-1)$

corresponding to the short exact sequence of 4.2.23, and so

$$\partial_1(\operatorname{Fil}_i \mathscr{V}^\vee(0,-1)) \subseteq \operatorname{Fil}_{i-1} \mathscr{V}^\vee(0,-1) \quad \text{for all } i \in \mathbf{Z}.$$

Step 3. Symmetric powers induce a $\mathbf{G}_a$-equivariant structure on $\operatorname{Sym}^n(\mathscr{V}^\vee(0,-1))$ for every $n \geq 0$. For each $0 \leq j \leq n$, the maps

$$\partial_j \colon \operatorname{Sym}^n(\mathscr{V}^\vee(0,-1)) \to \operatorname{Sym}^n(\mathscr{V}^\vee(0,-1))$$

giving the corresponding comodule structure are obtained by summing all possible j-fold tensor powers of ∂_1 on $\mathscr{V}^\vee(0,-1)^{\otimes n}$, and passing to the symmetric quotient. This is compatible with the convolution filtration on the n-th tensor product, so **Step 2** implies

$$\partial_j(\operatorname{Fil}_i \operatorname{Sym}^n(\mathscr{V}^\vee(0,-1))) \subseteq \operatorname{Fil}_{i-j} \operatorname{Sym}^n(\mathscr{V}^\vee(0,-1))$$

for all integers $i, j, n \geq 0$.

Step 4. As in the proof of 4.5.3, write

$$\rho_*\mathscr{O}_{\mathbf{P}\mathscr{V}^\circ} = \operatorname{colim}_n \operatorname{Sym}^n(\mathscr{V}^\vee(0,-1)).$$

By A.1.9, the transition maps in the colimit are multiplication by a generator of $\operatorname{Fil}_0 \mathscr{V}^\vee(0,-1)$. By **Step 2**, this is annihilated by ∂_1, so the description of the $\mathbf{G}_a$-equivariant structure on $\operatorname{Sym}^n(\mathscr{V}^\vee(0,-1))$ from **Step 3** shows that they pass to the colimit and induce a $\mathbf{G}_a$-equivariant structure on $\rho_*\mathscr{O}_{\mathbf{P}\mathscr{V}^\circ}$. The corresponding $\mathscr{O}_{\mathbf{P}}$-linear maps $\partial_j \colon \rho_*\mathscr{O}_{\mathbf{P}\mathscr{V}^\circ} \to \rho_*\mathscr{O}_{\mathbf{P}\mathscr{V}^\circ}$ also satisfy

$$\partial_j(\operatorname{Fil}_i \rho_*\mathscr{O}_{\mathbf{P}\mathscr{V}^\circ}) \subseteq \operatorname{Fil}_{i-j} \rho_*\mathscr{O}_{\mathbf{P}\mathscr{V}^\circ}$$

for all integers $i, j \geq 0$. Finally, since $\mathscr{B} = \pi_*(\rho_*\mathscr{O}_{\mathbf{P}\mathscr{V}^\circ}|_{\mathbf{A}})$, this gives:

4.5.7. Lemma. — *For each integer $j \geq 0$, the $\mathscr{A}$-module map $\partial_j \colon \mathscr{B} \to \mathscr{B}$ satisfies*

$$\partial_j(\operatorname{Fil}_i \mathscr{B}) \subseteq \operatorname{Fil}_{i-j} \mathscr{B} \quad \textit{for each integer } i \geq 0$$

and is of bidegree $(-j,-j)$. ■

4.5.8. Lemma. — *For each integer $i \geq 0$, the i-th associated graded map*

$$\mathrm{gr}_i \partial \colon \mathrm{gr}_i \mathscr{B} \to \mathrm{gr}_{i-1} \mathscr{B}$$

induced by $\partial := \partial_1$ is an isomorphism if $p \nmid i$ and is zero otherwise.

Proof. — As in the proof of 4.5.7, it suffices to show the analogous conclusion about the restriction to $\mathbf{A}$ of the i-th associated graded map of ∂ acting on $\mathrm{Sym}^n(\mathscr{V}^\vee(0,-1))$ with $n \geq i$. As in 4.5.3, the relative Euler sequence gives a canonical isomorphism

$$\Omega_{\pi_1}(1,-1)|_{\mathbf{A}} \cong (L_-^\vee \otimes L_+)_{\mathbf{A}}.$$

Thus the i-th associated graded map is identified with a map

$$\mathrm{gr}_i \partial \colon (L_-^\vee \otimes L_+)_{\mathbf{A}}^{\otimes i} \to (L_-^\vee \otimes L_+)_{\mathbf{A}}^{\otimes i-1}.$$

By **Step 2** of 4.5.6, $\mathrm{gr}_1 \partial$ is in this way identified with the map $L_-^\vee \otimes L_+ \to \mathbf{k}$ adjoint to the isomorphism $\partial \colon L_-^\vee \to L_+^\vee$ from **Step 1**. For $i \geq 1$, the description of the induced $\mathbf{G}_a$-equivariant structure on symmetric powers from **Step 3** shows that $\mathrm{gr}_i \partial$ may be identified with the map

$$i \cdot \mathrm{gr}_1 \partial \colon (L_-^\vee \otimes L_+)^{\otimes i} \to (L_-^\vee \otimes L_+)^{\otimes i-1}$$

where $\mathrm{gr}_1 \partial$ acts on, say, the first tensor factor. This is an isomorphism whenever $p \nmid i$ and zero otherwise. ■

4.5.9. Coordinate ring of T°. — Consider the coordinate ring of T° over C. First, by 4.2.28 and the computation of 3.3.1, $\pi_* \mathscr{O}_{T^\circ}$ remains equivariant for the subgroup

$$\left\{ \begin{pmatrix} \lambda^{-1} & \epsilon \\ 0 & \lambda^q \end{pmatrix} \in \mathbf{GL}_2(L_- \oplus L_+) \;\middle|\; \lambda \in \mathbf{G}_m, \epsilon \in \boldsymbol{\alpha}_q \right\} \cong \mathbf{Aut}(L_- \subset U, \beta_U).$$

The torus endows each of $\mathscr{A}$, $\mathscr{B}$, and $\pi_* \mathscr{O}_{T^\circ}$ with a $\mathbf{Z}_{\geq 0}$-grading; the degree d component of this grading is related to the previous bigrading by

$$\mathscr{A}_d = \bigoplus\nolimits_{a+bq=d} \mathscr{A}_{(a,b)} \quad \text{and} \quad \mathscr{B}_d = \bigoplus\nolimits_{a+bq=d} \mathscr{B}_{(a,b)}.$$

Next, by 4.2.13, $T^\circ \subset \mathbf{A}$ is the complete intersection cut out by the sections

$$v_1 := \mathrm{eu}_{\pi_2}^\vee \circ \beta^\vee \circ \mathrm{eu}_{\pi_1}^{(q)} \colon \mathcal{O}_{\mathbf{P}}(-q,-1) \to \mathcal{O}_{\mathbf{P}}$$
$$v_2 := \mathrm{eu}_{\pi_2}^{(q),\vee} \circ \beta \circ \mathrm{eu}_{\pi_1} \colon \mathcal{O}_{\mathbf{P}}(-1,-q) \to \mathcal{O}_{\mathbf{P}}$$

restricted to $\mathbf{A}$. By A.1.11, $\mathcal{O}_{\mathbf{P}}(-1,0)|_{\mathbf{A}} \cong \pi^*\mathcal{O}_C(-1)|_{\mathbf{A}}$ and $\mathcal{O}_{\mathbf{P}}(0,-1)|_{\mathbf{A}} \cong L_{+,\mathbf{A}}$, so pushing along π gives a presentation of $\pi_*\mathcal{O}_{T^\circ}$ as a graded $\mathscr{A}$-algebra:

$$\pi_*\mathcal{O}_{T^\circ} \cong \operatorname{coker}\big(v_1 \oplus v_2 \colon (\mathscr{A}(-q) \otimes L_+) \oplus (\mathscr{A}(-1) \otimes L_+^{\otimes q}) \to \mathscr{A}\big).$$

The components v_1 and v_2 of the presentation are describe as follows:

4.5.10. Lemma. — *The maps $v_1 \colon \mathscr{A}(-q) \otimes L_+ \to \mathscr{A}$ and $v_2 \colon \mathscr{A}(-1) \otimes L_+^{\otimes q} \to \mathscr{A}$ are the maps of graded $\mathscr{A}$-modules induced by*

$$v_1 = (\beta_U^\vee, \delta) \colon \mathcal{O}_C(-q) \otimes L_+ \to \mathcal{O}_C(-q) \otimes L_-^{\vee,\otimes q} \oplus \Omega^1_{\mathbf{P}W}(1)|_C \otimes L_+$$
$$v_2 = (\beta_W \circ \mathrm{eu}) \colon \mathcal{O}_C(-1) \otimes L_+^{\otimes q} \to \mathrm{Fr}^*(\Omega^1_{\mathbf{P}W}(1))|_C \otimes L_+^{\otimes q}$$

where $\delta \colon \mathcal{O}_C(-q) \to \Omega^1_{\mathbf{P}W}(1)|_C$ is the conormal map of $C \subset \mathbf{P}W$.

Proof. — By A.1.13, v_1 is the $\mathscr{A}$-module map induced by

$$\begin{aligned}\mathcal{O}_C(-q) \otimes L_+ &\to \mathrm{Fr}^*\big(L_- \otimes (\mathcal{O}_C(-1) \otimes L_-^\vee) \oplus \mathcal{O}_C(-1)\big) \otimes L_+ \\ &\to \big(L_+^\vee \otimes (\mathcal{O}_C(-1) \otimes L_-^\vee) \oplus \Omega_{\mathbf{P}W}(1)|_C\big) \otimes L_+\end{aligned}$$

where the first map is $\mathrm{Fr}^*(\mathrm{tr}_{L_-}^\vee, \mathrm{id}_{\mathcal{O}_C(-1)}) \otimes \mathrm{id}_{L_+}$ and the second is $\beta_U^\vee \oplus \beta_W^\vee$. The action of the second map uses the computation of 2.2.10 in the first component and that, since $\mathrm{type}(\beta_U) = \mathbf{N}_2$, $\beta_U^\vee \colon \mathrm{Fr}^*(L_-) \to L_+^\vee$ is an isomorphism in the second.

Likewise, v_2 is the $\mathscr{A}$-module map induced by

$$\mathcal{O}_C(-1) \otimes L_+^{\otimes q} \to \big(L_- \otimes (\mathcal{O}_C(-1) \otimes L_-^\vee) \oplus \mathcal{O}_C(-1)\big) \otimes L_+^{\otimes q} \to \mathrm{Fr}^*(\Omega_{\mathbf{P}W}(1))|_C \otimes L_+^{\otimes q}$$

where the first map is $(\mathrm{tr}_{L_-}^\vee, \mathrm{id}_{\mathcal{O}_C(-1)})$ and the second map is $\beta_U \oplus \beta_W$. This time, the map $\beta_U \colon L_- \to \mathrm{Fr}^*(L_+)^\vee$ is zero, whence the second summand vanishes. ■

4.5.11. Coordinate ring of S°. — By 4.2.27, S° is the hypersurface in $\mathbf{B} \times_{\mathbf{A}} T^\circ$ cut out by the restriction of the section

$$v_3 := u_3^{-1}\beta_{\mathscr{V}_T}(\mathrm{eu}_\rho^{(q),\vee}, \mathrm{eu}_\rho)\colon \mathscr{O}_\rho(-q) \otimes \rho^*\mathscr{O}_T(0,-1)|_{\mathbf{P}\mathscr{V}_T} \to \mathscr{O}_{\mathbf{P}\mathscr{V}_T}.$$

Set $\mathscr{B}_T := \varphi_*\mathscr{O}_{\mathbf{B}\times_{\mathbf{A}}T^\circ} \cong \mathscr{B} \otimes_{\mathscr{A}} \pi_*\mathscr{O}_{T^\circ}$. By A.1.11 together with the exact sequences in 4.2.23 and 4.2.10, $\mathscr{O}_\rho(-1)|_{\mathbf{B}} \cong \rho^*(\mathscr{O}_{\mathbf{P}}(0,-1)|_{\mathbf{A}}) \cong L_{+,\mathbf{B}}$. Thus the coordinate ring of S° over C admits a presentation as a graded $\mathscr{B}_T$-algebra

$$\varphi_*\mathscr{O}_{S^\circ} \cong \mathrm{coker}(v_3\colon \mathscr{B}_T \otimes L_+^{\otimes q+1} \to \mathscr{B}_T).$$

By 4.5.3, $\mathscr{B}_T$ carries an increasing filtration in which $\mathrm{Fil}_i\,\mathscr{B}_T$ consists of all local sections with degree at most i in the affine fibre coordinate of $\mathbf{P}\mathscr{V}_T^\circ := \mathbf{P}\mathscr{V}_T \setminus \mathbf{P}(\mathscr{T}_{\pi_1}(-1,0)|_T)$ over T. Since v_3 is of degree q over T, this gives the first statement in the following:

4.5.12. Lemma. — *For each $i \in \mathbf{Z}_{\geq 0}$, $v_3(\mathrm{Fil}_i\,\mathscr{B}_T \otimes L_+^{\otimes q+1}) \subseteq \mathrm{Fil}_{i+q}\,\mathscr{B}_T$. The induced map on associated graded pieces is the isomorphism*

$$\mathrm{gr}_i\,\mathscr{B}_T \otimes L_+^{\otimes q+1} \cong \pi_*\mathscr{O}_{T^\circ} \otimes (L_-^\vee \otimes L_+)^{\otimes i} \otimes L_+^{\otimes q+1} \to \pi_*\mathscr{O}_{T^\circ} \otimes (L_-^\vee \otimes L_+)^{\otimes i+q} \cong \mathrm{gr}_{i+q}\,\mathscr{B}_T$$

given by $\beta_U^\vee\colon L_+^{\otimes q+1} \to (L_-^\vee \otimes L_+)^{\otimes q}$.

Proof. — Begin by considering the pushfoward along ρ of v_3 restricted to $\mathbf{P}\mathscr{V}_T^\circ$. Identifying $\mathscr{O}_\rho(-1)|_{\mathbf{P}\mathscr{V}_T^\circ}$ with $\rho^*\mathscr{O}_T(0,-1)$, this gives a map

$$\rho_*\rho^*\mathscr{O}_T(0,-q-1) \to \rho_*\mathscr{O}_{\mathbf{P}\mathscr{V}_T^\circ} \cong \mathrm{colim}_n\,\mathrm{Sym}^n(\mathscr{V}_T^\vee(0,-1)).$$

By its definition and the description of the Euler section from A.1.12, this is the $\rho_*\mathscr{O}_{\mathbf{P}\mathscr{V}_T^\circ}$-module map induced by

$$\gamma\colon \mathscr{O}_T(0,-q-1) \xrightarrow{\mathrm{tr}_{\mathscr{V}_T}^\vee} \mathscr{V}_T \otimes \mathscr{V}_T^\vee \otimes \mathscr{O}_T(0,-q-1) \xrightarrow{\beta_{\mathscr{V}_T}} \mathrm{Fr}^*(\mathscr{V}_T^\vee(0,-1)) \otimes \mathscr{V}_T^\vee(0,-1)$$

followed by the multiplication map into $\mathrm{Sym}^{q+1}(\mathscr{V}_T^\vee(0,-1))$.

Next, consider the restriction of γ to T°. As explained in A.1.11, there are canonical isomorphisms $\mathscr{T}_{\pi_1}(-1,0)|_{T^\circ} \cong L_{-,T^\circ}$ and $\mathscr{O}_T(0,-1)|_{T^\circ} \cong L_{+,T^\circ}$. Via the short exact sequence 4.2.24, this gives a canonical isomorphism $(\mathscr{V}_T, \beta_{\mathscr{V}_T})|_{T^\circ} \cong (U_{T^\circ}, \beta_U)$ of q-bic forms over T°. So upon passing through these isomorphisms, γ restricts to the map

$$\gamma|_{T^\circ} \colon L_{+,T^\circ}^{\otimes q+1} \xrightarrow{\mathrm{tr}_U^\vee} U_{T^\circ} \otimes U_{T^\circ}^\vee \otimes L_{+,T^\circ}^{\otimes q+1} \xrightarrow{\beta_U} \mathrm{Fr}^*(U)_{T^\circ}^\vee \otimes U_{T^\circ}^\vee \otimes L_{+,T^\circ}^{\otimes q+1}$$

Finally, to consider the map on graded pieces, recall that the filtration on $\rho_*\mathscr{O}_{\mathbf{P}\mathscr{V}_T^\circ}$ is induced by the filtration on $\mathscr{V}_T^\vee(0,-1)$ given by the short exact sequence 4.2.24; in particular, the graded pieces of $\mathscr{V}_T^\vee(0,-1)$ are

$$\mathrm{gr}_0\,\mathscr{V}_T^\vee(0,-1) = \mathscr{O}_T \quad \text{and} \quad \mathrm{gr}_1\,\mathscr{V}_T^\vee(0,-1) = \Omega^1_{\pi_1}(1,-1)|_T.$$

Tensor functors applied to filtered vector bundles also carry filtrations. In the case at hand, the graded pieces are given by:

$$\begin{aligned}
\mathrm{gr}_i(\mathscr{E})^\vee &= \mathrm{gr}_{-i}(\mathscr{E}^\vee),\\
\mathrm{Fr}^*(\mathrm{gr}_i(\mathscr{E})) &= \mathrm{gr}_{iq}(\mathrm{Fr}^*(\mathscr{E})),\\
\mathrm{gr}_i(\mathscr{E}_1 \otimes \mathscr{E}_2) &= \bigoplus\nolimits_{a+b=i} \mathrm{gr}_a(\mathscr{E}_1) \otimes \mathrm{gr}_b(\mathscr{E}_2),
\end{aligned}$$

for filtered vector bundles $\mathscr{E}$, $\mathscr{E}_1$, and $\mathscr{E}_2$. Since $\mathrm{tr}_U^\vee$ preserves the filtration degree and β_U shifts the filtration degree by q, thanks to the first statement of the Lemma, the map induced by $\gamma|_{T^\circ}$ on associated graded pieces is

$$L_{+,T^\circ}^{\otimes q+1} \to (L_{+,T^\circ} \otimes L_{+,T^\circ}^\vee \oplus L_{-,T^\circ} \otimes L_{-,T^\circ}^\vee) \otimes L_{+,T^\circ}^{\otimes q+1} \to L_{-,T^\circ}^{\vee,\otimes q} \otimes L_{+,T^\circ}^\vee \otimes L_{+,T^\circ}^{\otimes q+1}$$

where the first map is $(\mathrm{tr}_{L_+}^\vee, \mathrm{tr}_{L_-}^\vee)$, and the second map is the isomorphism $\beta_U \colon L_+ \to L_-^{\vee,\otimes q}$ on the first summand and the zero map on the second summand. Identifying the target as $(L_-^\vee \otimes L_+)_{T^\circ}^{\otimes q}$ then implies the second statement of the Lemma. ∎

Hence the map v_3 becomes a strict morphism of filtered bundles upon shifting one of the filtrations by q. Recall that a morphism $f : \mathscr{E}_1 \to \mathscr{E}_2$ between two filtered modules is called *strict* if

$$f(\mathrm{Fil}_i\,\mathscr{E}_1) = f(\mathscr{E}_1) \cap \mathrm{Fil}_i\,\mathscr{E}_2 \quad \text{for all } i \in \mathbf{Z},$$

see [**Stacks**, 0123]. In the following statement, a *strict short exact sequence* of filtered modules is a short exact sequence in which all morphisms are strict.

4.5.13. Corollary. — *The sheaf $\varphi_*\mathscr{O}_{S^\circ}$ is a filtered graded $\pi_*\mathscr{O}_{T^\circ}$-algebra with*

$$\operatorname{gr}_i \varphi_*\mathscr{O}_{S^\circ} \cong \begin{cases} \pi_*\mathscr{O}_{T^\circ} \otimes (L_-^\vee \otimes L_+)^{\otimes i} & \text{if } 0 \leq i \leq q-1, \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

which fits into a strict short exact sequence of graded filtered $\pi_*\mathscr{O}_{T^\circ}$-modules

$$0 \to (\mathscr{B}_T \otimes L_+^{\otimes q+1}, \mathrm{Fil}_{\bullet-q}) \xrightarrow{\nu_3} (\mathscr{B}_T, \mathrm{Fil}_\bullet) \to (\varphi_*\mathscr{O}_{S^\circ}, \mathrm{Fil}_\bullet) \to 0.$$

Proof. — The first statement of 4.5.12 shows that the short exact sequence in question is one of filtered modules; the second statement there with [**Stacks**, 0127] shows that it is strict. To identify the graded pieces, since the sequence is strict, for each integer $i \geq 0$, there is a short exact sequence of graded pieces

$$0 \to \operatorname{gr}_{i-q} \mathscr{B}_T \otimes L_+ \to \operatorname{gr}_i \mathscr{B}_T \to \operatorname{gr}_i \varphi_*\mathscr{O}_{S^\circ} \to 0.$$

When $0 \leq i \leq q-1$, the first term is 0 and so

$$\operatorname{gr}_i \varphi_*\mathscr{O}_{S^\circ} \cong \operatorname{gr}_i \mathscr{B}_T \cong \pi_*\mathscr{O}_{T^\circ} \otimes (L_-^\vee \otimes L_+)^{\otimes i}.$$

When $i \geq q$, the second statement of 4.5.12 means that the first arrow in the sequence is an isomorphism, and so $\operatorname{gr}_i \varphi_*\mathscr{O}_{S^\circ} = 0$. ∎

4.5.14. Action of $\boldsymbol{\alpha}_q$. — By 4.2.28 together with the comments in 4.5.9, the quotient $\varphi_*\mathscr{O}_{S^\circ}$ of $\mathscr{B}$ remains equivariant for the action of the subgroup $\boldsymbol{\alpha}_q \subset \mathbf{G}_a$. Thus the $\mathscr{A}$-comodule structure on $\mathscr{B}$ from 4.5.14 induces a $\pi_*\mathscr{O}_{T^\circ}$-comodule structure

$$\varphi_*\mathscr{O}_{S^\circ} \to \varphi_*\mathscr{O}_{S^\circ} \otimes_{\mathbf{k}} \mathbf{k}[\epsilon]/(\epsilon^q) \colon z \mapsto \sum\nolimits_{j=0}^{q-1} \partial_j(z) \otimes \epsilon^j$$

where $\partial_j \colon \varphi_*\mathscr{O}_{S^\circ} \to \varphi_*\mathscr{O}_{S^\circ}$ are induced by the ∂_j on $\mathscr{B}$. By 4.5.7, the ∂_j satisfy

$$\partial_j(\mathrm{Fil}_i \varphi_*\mathscr{O}_{S^\circ}) \subseteq \varphi_*\mathscr{O}_{S^\circ} \quad \text{for each integer } i \geq 0,$$

and are of degree $-j(q+1)$. A neat summary of the structure provided by ∂_1 is:

4.5.15. Corollary. — *The map $\partial := \partial_1$ fits into a complex of $\pi_*\mathscr{O}_{T^\circ}$-modules*

$$0 \to \mathrm{Fil}_0 \varphi_*\mathscr{O}_{S^\circ} \to \varphi_*\mathscr{O}_{S^\circ} \xrightarrow{\partial} \varphi_*\mathscr{O}_{S^\circ} \to \operatorname{gr}_{q-1} \varphi_*\mathscr{O}_{S^\circ} \to .$$

This is exact when $q = p$.

Proof. — This follows from 4.5.8 together with 4.5.13. ∎

4.5.16. The equation v_1. — The next goal is to describe some of the graded pieces of $\mathscr{D}$ in terms of the sheaves appearing in 4.5.4. Recall by 4.4.23 that $\mathscr{D}$ is the quotient of $\varphi_*\mathscr{O}_{S^\circ}$ by the ideal of all sections of weight greater than $\delta = 2q^2 - q - 2$. Thus the task is to describe the low degree pieces of $\varphi_*\mathscr{O}_{S^\circ}$ and this demands a careful study of the equations v_1, v_2, and v_3 appearing in 4.5.9 and 4.5.11. The equation v_1 is the simplest: by its description from 4.5.10, it is locally of the form $t^q + \delta$, where t is a local fibre coordinate of $\mathbf{A}_1$ over C and δ is a linear form on $\mathbf{A}_2$ over C. Set

$$\mathscr{B}' := \mathrm{coker}(v_1 \colon \mathscr{B} \otimes \mathscr{O}_C(-q) \otimes L_+ \to \mathscr{B}).$$

Using v_1 to locally eliminate t^q makes it possible to identify the components of $\mathscr{B}'$ of weight at most $q^2 - 1$:

4.5.17. Lemma. — *The form β induces equivariant isomorphisms of filtered sheaves*

$$\mathscr{B}'_a \cong \mathscr{O}_C(-a), \quad \mathscr{B}'_q \cong \Omega^1_{\mathbf{P}W}(1)|_C, \quad \mathscr{B}'_{q+1} \cong W_C^\vee(-1),$$

and $\mathscr{B}'_{bq+a} \cong \mathrm{Sym}^b(W_C^\vee(-1)) \otimes \mathscr{O}_C(b-a)$ *for all* $0 \leq b \leq a \leq q-1$.

Proof. — Since v_1 is of degree q, $\mathscr{B}'_a = \mathscr{B}_a \cong \mathscr{O}_C(-a)$ for all $0 \leq a \leq q-1$, the second identification is due to 4.5.4. To describe the higher degree components, observe that by 4.5.10, the composite

$$\mathscr{O}_C(-q) \otimes L_+ \xrightarrow{v_1} \mathscr{O}_C(-q) \otimes L_-^{\vee,\otimes q} \oplus \Omega^1_{\mathbf{P}W}(1)|_C \otimes L_+ \xrightarrow{\mathrm{pr}_1} \mathscr{O}_C(-q) \otimes L_-^{\vee,\otimes q}$$

is the isomorphism induced by β_U. Since the middle term is $\mathscr{B}_q$ by 4.5.4, this implies that there is an isomorphism of $\mathscr{O}_C$-modules

$$\mathscr{B}'_q = \mathrm{coker}(v_1 \colon \mathscr{O}_C(-q) \otimes L_+ \to \mathscr{B}_q) \cong \Omega^1_{\mathbf{P}W}(1)|_C.$$

Let $0 \leq b \leq a \leq q-1$. Consider the composition of

$$\xi \colon \mathscr{B}_{(b-1)q+a} \otimes \mathscr{O}_C(-q) \otimes L_+ \xrightarrow{v_1} \mathscr{B}_{bq+a} \twoheadrightarrow \mathscr{B}_{(bq+a,0)} \oplus \mathscr{B}_{((b-1)q+a,1)} \oplus \cdots \oplus \mathscr{B}_{(q+a,b-1)}$$

the map ν_1 together with the projection of $\mathscr{B}_{bq+a} = \bigoplus_{i=0}^{b} \mathscr{B}_{((b-i)q+a,i)}$ onto the first b pieces of its bigraded decomposition. I claim that ξ is an isomorphism. Combined with 4.5.4, this will complete the proof as it gives the isomorphism in

$$\mathscr{B}'_{bq+a} = \operatorname{coker}(\nu_1 \colon \mathscr{B}_{(b-1)q+a} \otimes \mathscr{O}_C(-q) \otimes L_+ \to \mathscr{B}_{bq+a}) \cong \mathscr{B}_{(a,b)}.$$

Since both the projection and ν_1 are compatible with the natural filtrations, this is gives an isomorphism of filtered $\mathscr{O}_C$-modules.

To see that ξ is an isomorphism, consider the bigraded decomposition

$$\mathscr{B}_{(b-1)q+a} = \mathscr{B}_{((b-1)q+a,0)} \oplus \mathscr{B}_{((b-2)q+a,1)} \oplus \cdots \oplus \mathscr{B}_{(a,b-1)}.$$

Let $0 \leq i \leq b-1$. Restricting ξ to the component $\mathscr{B}_{((b-1-i)q+a,i)} \otimes \mathscr{O}_C(-q) \otimes L_+$ gives a map

$$\xi_i \colon \mathscr{B}_{((b-1-i)q+a,i)} \otimes \mathscr{O}_C(-q) \otimes L_+ \to \mathscr{B}_{((b-i)q+a,i)} \oplus \mathscr{B}_{((b-1-i)q+a,i+1)}$$

where, by the computation of 4.5.10, the map to the first summand is multiplication by a generator of $\mathscr{B}_{(q,0)}$, and the map to the second summand is multiplication by the subbundle $\delta \colon \mathscr{O}_C(-q) \otimes L_+ \to \mathscr{B}_{(0,1)}$. Since $(b-1-i)q+a \geq i$ for each $0 \leq i \leq b-1$, 4.5.5 shows that post-composing ξ_i with projection onto the second factor yields an isomorphism. With the ordering of the bigraded decompositions of $\mathscr{B}_{(b-1)q+a}$ and $\mathscr{B}_{bq+a}$ above, ξ is upper triangular with isomorphisms on the diagonal, and so is itself an isomorphism. ∎

The identification $\mathscr{B}'_{q+1} \cong W_C^\vee(-1)$ of 4.5.17 shows that the subalgebra generated in degree $q+1$ is strikingly simple. Higher weight pieces of $\mathscr{B}'$ can be partially accessed by comparing with this subalgebra. In particular, the pieces $\mathscr{B}'_{dq+q-1}$ with $q \leq d \leq 2q-2$ turn out to be rather simple and useful. In the following, let

$$\iota \colon \operatorname{Sym}^*(\mathscr{B}'_{q+1}) \to \mathscr{B}'$$

be the inclusion of the subalgebra generated in degree $q+1$.

4.5.18. Lemma. — *For each $q \leq d \leq 2q-2$, the multiplication map factors as*

$$\mathrm{mult} \colon \mathscr{B}'_{dq+q-1} \otimes \mathscr{B}'_{d-q+1} \xrightarrow{\mu'_d} \operatorname{Sym}^d(\mathscr{B}'_{q+1}) \xrightarrow{\iota} \mathscr{B}'_{d(q+1)}$$

where μ'_d is a strict morphism of filtered bundles, and

$$\operatorname{coker}(\mu'_d) \cong \operatorname{Sym}^d(\mathscr{B}'_{q+1})/(\operatorname{Sym}^{d-1}(\mathscr{B}'_{q+1})(-q-1) + \operatorname{Fil}_{q-1}\operatorname{Sym}^d(\mathscr{B}'_{q+1})).$$

Proof. — To show that the multiplication map factors on $\mathscr{B}'$ as claimed, it suffices to show that the multiplication map up on $\mathscr{B}$ factors as

$$\mathrm{mult}\colon \mathscr{B}_{dq+q-1} \otimes \mathscr{B}_{d-q+1} \xrightarrow{\mu_d} \operatorname{Sym}^d(\mathscr{B}_{q+1}) \to \mathscr{B}_{d(q+1)}.$$

Multiplication respects the bigrading of $\mathscr{B}$. Noting that $q \leq d \leq 2q-2$, it decomposes into a sum of maps

$$(0 \to \mathscr{B}_{(d-q,d+1)}) \oplus \bigoplus_{i=0}^{d}\Big(\mathscr{B}_{(iq+q-1,d-i)} \otimes \mathscr{B}_{(d-q+1,0)} \to \mathscr{B}_{(iq+d,d-i)}\Big).$$

Since $\mathscr{B}_{q+1} = \mathscr{B}_{(q+1,0)} \oplus \mathscr{B}_{(1,1)}$ by 4.5.4 and 4.5.9, the targets of the latter sum add up to $\operatorname{Sym}^d(\mathscr{B}_{q+1})$. Set μ_d to be the latter sum of the nonzero maps. Since multiplication is strict, so is μ_d. Since $\mathscr{B}'$ carries the quotient filtration induced from $\mathscr{B}$, this implies that the induced map μ'_d is also strict, see [**Stacks**, 0124].

By the construction of μ_d, its cokernel is identified as

$$\begin{aligned}
\operatorname{coker}(\mu_d) &= \bigoplus_{i=0}^{d}\operatorname{coker}\Big(\mathscr{B}_{(iq+q-1,d-i)} \otimes \mathscr{B}_{(d-q+1,0)} \to \mathscr{B}_{(iq+d,d-i)}\Big) \\
&= \operatorname{coker}(\mathscr{B}_{(q-1,d)} \otimes \mathscr{B}_{(d-q+1,0)} \to \mathscr{B}_{(d,d)}) \\
&\cong \operatorname{Sym}^d(\mathscr{B}'_{q+1})/\operatorname{Fil}_{q-1}\operatorname{Sym}^d(\mathscr{B}'_{q+1})
\end{aligned}$$

where the second equality is due to 4.5.5 since $2q-1 > d$, and the third isomorphism is due to 4.5.4. To relate this with $\operatorname{coker}(\mu'_d)$, since the equation v_1 is of degree q,

$$\begin{aligned}
\mathscr{B}'_{dq+q-1} \otimes \mathscr{B}'_{d-q+1} &= \operatorname{coker}\Big(\mathscr{B}_{(d-1)q+q-1} \otimes \mathscr{O}_C(-q) \otimes L_+ \xrightarrow{v_1} \mathscr{B}_{dq+q-1}\Big) \otimes \mathscr{B}_{d-q+1}, \\
\operatorname{Sym}^d(\mathscr{B}'_{q+1}) &= \operatorname{coker}\Big(\operatorname{Sym}^{d-1}(\mathscr{B}_{q+1}) \otimes \mathscr{B}_1 \otimes \mathscr{O}_C(-q) \otimes L_+ \xrightarrow{v_1} \operatorname{Sym}^d(\mathscr{B}_{q+1})\Big).
\end{aligned}$$

Since v_1 is a map of algebras, the presentations above fit into an exact commutative diagram with the maps μ_{d-1}, μ_d, and μ'_d to yield a short exact sequence

$$0 \to \operatorname{coker}(\mu_{d-1}) \otimes \mathscr{B}_1 \otimes \mathscr{O}_C(-q) \otimes L_+ \xrightarrow{v_1} \operatorname{coker}(\mu_d) \to \operatorname{coker}(\mu'_d) \to 0.$$

Since v_1 respects filtrations, this gives the desired description of $\operatorname{coker}(\mu'_d)$. ∎

The next goal is to descend this structure along the quotient $\mathscr{B}' \to \mathscr{D}$ to identify $\mathscr{D}_{dq+q-1}$ for $q \leq d \leq 2q-3$. This is achieved in 4.5.21. Begin with a simple linear algebra fact. In the following, for a vector bundle $\mathscr{E}$ and an integer $d \geq q$, write

$$(\mathrm{Sym}^d \otimes \mathrm{Fr}^*)(\mathscr{E}) := \mathrm{Sym}^d(\mathscr{E}) \otimes \mathrm{Fr}^*(\mathscr{E}),$$
$$(\mathrm{Sym}^d / \mathrm{Sym}^{d-q} \otimes \mathrm{Fr}^*)(\mathscr{E}) := \mathrm{Sym}^d(\mathscr{E})/(\mathrm{Sym}^{d-q}(\mathscr{E}) \otimes \mathrm{Fr}^*(\mathscr{E}))$$

where $\mathrm{Sym}^{d-q} \otimes \mathrm{Fr}^*$ includes into Sym^d via multiplication.

4.5.19. Lemma. — *Let V be a finite dimensional vector space with a two step filtration $\mathrm{Fil}_0 V \subseteq \mathrm{Fil}_1 V = V$. If $\mathrm{gr}_1 V$ is one-dimensional, then, for all integers $d \geq q$, the map*

$$(\mathrm{Sym}^{d-q} \otimes \mathrm{Fr}^*)(V) \to \mathrm{Sym}^d(V)/\mathrm{Fil}_{q-1} \mathrm{Sym}^d(V)$$

induced by multiplication is surjective, and it induces a canonical isomorphism

$$\mathrm{Fil}_{q-1} \mathrm{Sym}^d(V)/\mathrm{Fil}_{q-1}(\mathrm{Sym}^{d-q}(V) \otimes \mathrm{Fr}^*(V)) \cong (\mathrm{Sym}^d / \mathrm{Sym}^{d-q} \otimes \mathrm{Fr}^*)(V).$$

Proof. — Choose a basis $\mathrm{Fil}_0 V = \langle v_1, \ldots, v_n \rangle$ and extend it to a basis of V by taking $w \in V$ mapping to a basis of $\mathrm{gr}_1 V$. Then $\mathrm{Sym}^d(V)$ has a basis given by the monomials of degree d in the $v_1, \ldots, v_n, w$, and the $(q-1)$-st piece of the induced filtration is

$$\mathrm{Fil}_{q-1} \mathrm{Sym}^d(V) = \langle v_1^{i_1} \cdots v_n^{i_n} w^j \mid i_1 + \cdots i_n + j = d \text{ and } j \leq q-1 \rangle.$$

The quotient therefore has a basis of the form

$$\mathrm{Sym}^d(V)/\mathrm{Fil}_{q-1} \mathrm{Sym}^d(V) = \langle v_1^{i_1} \cdots v_n^{i_n} w^j \mid i_1 + \cdots + i_n + j = d \text{ and } j \geq q \rangle.$$

This shows that every element of $\mathrm{Sym}^d(V)/\mathrm{Fil}_{q-1} \mathrm{Sym}^d(V)$ can be written as a product of w^q with an element of $\mathrm{Sym}^{d-q}(V)$, so the multiplication map is surjective. Since

$$\mathrm{Fil}_{q-1}(\mathrm{Sym}^{d-q} \otimes \mathrm{Fr}^*)(V) = \ker\big((\mathrm{Sym}^{d-q} \otimes \mathrm{Fr}^*)(V) \to \mathrm{Sym}^d(V)/\mathrm{Fil}_{q-1} \mathrm{Sym}^d(V)\big)$$

the second statement now follows from the Five Lemma. ■

Since the ideal of $\mathscr{D}$ in $\mathscr{B}'$ contains the generators v_2 and v_3 in degrees q^2 and $q(q+1)$, see 4.5.9 and 4.5.11, the map $\mathrm{Sym}^*(\mathscr{B}'_{q+1})$ onto the subalgebra of $\mathscr{D}$ generated by $\mathscr{D}_{q+1}$ is not injective. The following uses the filtration to eliminate the equation v_3 via 4.5.12 to give a simple relationship between $\mathscr{D}_{dq+q-1}$ and $\mathrm{Sym}^d(\mathscr{B}'_{q+1})$ for $q \leq d \leq 2q-3$:

4.5.20. Lemma. — *Let $q \leq d \leq 2q-3$ and let*

$$\mathscr{D}^d_{q+1} := \mathrm{image}\big(\mathrm{Sym}^d(\mathscr{B}'_{q+1}) \to \mathscr{D}_{d(q+1)}\big).$$

Then the following hold:

(i) *Multiplication $\mathscr{D}_{dq+q-1} \otimes \mathscr{D}_{d-q+1} \to \mathscr{D}_{d(q+1)}$ is an isomorphism onto $\mathscr{D}^d_{q+1}$.*

(ii) *The natural map $\mathrm{Fil}_{q-1}\,\mathrm{Sym}^d(\mathscr{B}'_{q+1}) \to \mathscr{D}^d_{q+1}$ is strict surjective and its kernel is*

$$\mathrm{image}\big(v_2 \colon \mathrm{Sym}^{d-q}(\mathscr{B}'_{q+1}) \otimes L_+^{\otimes q}(-q-1) \to \mathrm{Fil}_{q-1}\,\mathrm{Sym}^d(\mathscr{B}'_{q+1})\big).$$

(iii) *The surjection $\mathrm{Fil}_{q-1}\,\mathrm{Sym}^d(\mathscr{B}'_{q+1}) \to \mathscr{D}^d_{q+1}$ induces a strict surjection*

$$\mathscr{D}^d_{q+1} \to (\mathrm{Sym}^d \,/\, \mathrm{Sym}^{d-q} \otimes \mathrm{Fr}^*)(\mathscr{B}'_{q+1}).$$

Proof. — The factorization of the multiplication map on $\mathscr{B}'$ from 4.5.18 implies that the multiplication map $\mathscr{D}_{dq+q-1} \otimes \mathscr{D}_{d-q+1} \to \mathscr{D}_{d(q+1)}$ factors through $\mathscr{D}^d_{q+1}$. Thus there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Fil}_{q-1}(\mathscr{B}'_{dq+q-1} \otimes \mathscr{B}'_{d-q+1}) \arrow[r, "\cong", "\mathrm{Fil}_{q-1}\mu'_d"'] \arrow[d, two heads] & \mathrm{Fil}_{q-1}\,\mathrm{Sym}^d(\mathscr{B}'_{q+1}) \arrow[d, two heads] \\
\mathscr{D}_{dq+q-1} \otimes \mathscr{D}_{d-q+1} \arrow[r, hook, "\mathrm{mult}"] & \mathscr{D}^d_{q+1}
\end{tikzcd}$$

in which

- the vertical maps are induced by the quotient map $\mathscr{B}' \to \mathscr{D}$ and are strict surjective since the filtration on $\mathscr{D}$ has q steps by 4.5.13;
- the top map is an isomorphism by the computation of $\mathrm{coker}(\mu'_d)$ in 4.5.18; and
- the multiplication map below is injective since S is integral: it is irreducible by 4.4.17 and reduced as it is Cohen–Macaulay and generically smooth, see 4.4.5.

Commutativity of the diagram implies that the multiplication map on the bottom is an isomorphism, establishing (i). This also shows the first statement of (ii).

To compute the kernel in (ii), use the horizontal isomorphisms and the identifications $\mathscr{B}'_{d-q+1} \cong \mathscr{D}_{d-q+1} \cong \mathscr{O}_C(-d+q-1)$ from 4.5.17 to obtain

$$\ker\big(\operatorname{Fil}_{q-1}\operatorname{Sym}^d(\mathscr{B}'_{q+1}) \to \mathscr{D}^d_{q+1}\big) \cong \ker\big(\operatorname{Fil}_{q-1}\mathscr{B}'_{dq+q-1} \to \mathscr{D}_{dq+q-1}\big) \otimes \mathscr{O}_C(-d+q-1).$$

Since the kernel of $\mathscr{B}' \to \mathscr{D}$ is, in low degrees, the ideal generated by v_2 and v_3, and the latter lies strictly in $\operatorname{Fil}_q \mathscr{B}'$ by 4.5.12, it follows that

$$\ker\big(\operatorname{Fil}_{q-1}\mathscr{B}'_{dq+q-1} \to \mathscr{D}_{dq+q-1}\big) = \operatorname{image}\big(\mathscr{B}'_{(d-q)q+q-1} \otimes L_{+,C}^{\otimes q}(-1) \xrightarrow{v_2} \operatorname{Fil}_{q-1}\mathscr{B}'_{dq+q-1}\big)$$

where $\operatorname{Fil}_{q-1}\mathscr{B}'_{(d-q)q+q-1} = \mathscr{B}'_{(d-q)q+q-1}$ since $(d-q)q+q-1 < q(q+1)$. Putting these together with the isomorphism

$$\mathscr{B}_{(d-q)q+q-1} \cong \operatorname{Sym}^{d-q}(\mathscr{B}'_{q+1}) \otimes \mathscr{O}_C(d-2q+1)$$

from 4.5.17 now identifies the kernel in (ii).

Finally, 4.5.10 shows that v_2 factors through $\operatorname{Fr}^*(\mathscr{B}'_q) \subset \mathscr{B}'_{q^2}$, and so the morphism giving the kernel in (ii) factors as

$$\begin{tikzcd}
\operatorname{Sym}^{d-q}(\mathscr{B}'_{q+1}) \otimes L_+^{\otimes q}(-q-1) \arrow[r, "v_2"'] \arrow[d, hook, "\beta_W \circ \mathrm{eu}_{\mathbf{P}W}"'] & \operatorname{Fil}_{q-1}\operatorname{Sym}^d(\mathscr{B}'_{q+1}) \\
\operatorname{Sym}^{d-q}(\mathscr{B}'_{q+1}) \otimes \operatorname{Fr}^*(\mathscr{B}'_q \otimes \mathscr{B}'_1) \arrow[r, equal] & \operatorname{Fil}_{q-1}(\operatorname{Sym}^{d-q} \otimes \operatorname{Fr}^*)(\mathscr{B}'_{q+1}) . \arrow[u, hook]
\end{tikzcd}$$

In other words, the kernel of the surjection $\operatorname{Fil}_{q-1}\operatorname{Sym}^d(\mathscr{B}'_{q+1}) \to \mathscr{D}^d_{q+1}$ is contained in the subbundle $\operatorname{Fil}_{q-1}(\operatorname{Sym}^{d-q} \otimes \operatorname{Fr}^*)(\mathscr{B}'_{q+1})$. Thus there is an induced surjection

$$\mathscr{D}^d_{q+1} \to \operatorname{Fil}_{q-1}\operatorname{Sym}^d(\mathscr{B}'_{q+1}) / \operatorname{Fil}_{q-1}(\operatorname{Sym}^{d-q} \otimes \operatorname{Fr}^*)(\mathscr{B}'_{q+1}).$$

This is strict since both sheaves carry the quotient filtration from $\operatorname{Fil}_{q-1}\operatorname{Sym}^d(\mathscr{B}'_{q+1})$, see [**Stacks**, 0124]. Since the quotient on the right is canonically isomorphic to $(\operatorname{Sym}^d / \operatorname{Sym}^{d-q} \otimes \operatorname{Fr}^*)(\mathscr{B}'_{q+1})$ by 4.5.19, this completes the proof of (iii). ■

The main result regarding the structure of the pieces of $\mathscr{D}$ is the following:

4.5.21. Proposition. — *There are equivariant isomorphisms of filtered bundles*

$$\mathscr{D}_a \cong \mathscr{O}_C(-a), \quad \mathscr{D}_q \cong \Omega^1_{\mathbf{P}W}(1)|_C, \quad \mathscr{D}_{q+1} \cong W_C^\vee(-1),$$

and $\mathscr{D}_{bq+a} \cong \mathrm{Sym}^b(W_C^\vee(-1)) \otimes \mathscr{D}_{a-b}$ *for each* $0 \leq b \leq a \leq q-1$. *There are equivariant strict surjections of filtered bundles*

$$\bar{\mu}_d \colon \mathscr{D}_{dq+q-1} \otimes \mathscr{D}_{d-q+1} \to (\mathrm{Sym}^d / \mathrm{Sym}^{d-q} \otimes \mathrm{Fr}^*)(W_C^\vee(-1))$$

for each $q \leq d \leq 2q-3$, *with* $\ker(\bar{\mu}_d) \cong \mathrm{Sym}^{d-q}(W_C^\vee(-1)) \otimes \mathscr{O}_C(-2q+1)$.

Proof. — By construction, $\mathscr{D}$ is a quotient of $\mathscr{B}'$ by an ideal generated in degrees at least q^2, so $\mathscr{D}_d = \mathscr{B}'_d$ for all $0 \leq d \leq q^2-1$ and the first series of statements follows from 4.5.17.

For the latter statements, let $q \leq d \leq 2q-3$ and let $\bar{\mu}_d$ be the composition

$$\bar{\mu}_d \colon \mathscr{D}_{dq+q-1} \otimes \mathscr{D}_{d-q+1} \cong \mathscr{D}^d_{q+1} \twoheadrightarrow (\mathrm{Sym}^d / \mathrm{Sym}^{d-q} \otimes \mathrm{Fr}^*)(\mathscr{B}'_{q+1})$$

of the isomorphism from 4.5.20(i) with the surjection of 4.5.20(iii). Thus this is strict and there is a commutative square

$$\begin{tikzcd}
\mathrm{Fil}_{q-1}\mathrm{Sym}^d(\mathscr{B}'_{q+1}) \arrow[r, two heads] \arrow[d, equal] & \mathscr{D}_{dq+q-1} \otimes \mathscr{D}_{d-q+1} \arrow[d, two heads, "\bar{\mu}_d"] \\
\mathrm{Fil}_{q-1}\mathrm{Sym}^d(\mathscr{B}'_{q+1}) \arrow[r, two heads] & (\mathrm{Sym}^d / \mathrm{Sym}^{d-q} \otimes \mathrm{Fr}^*)(\mathscr{B}'_{q+1}) .
\end{tikzcd}$$

The kernels of the maps in the diagram fit into a short exact sequence

$$0 \to \mathrm{Sym}^{d-q}(\mathscr{B}'_{q+1}) \otimes \mathscr{O}_C(-q-1) \xrightarrow{\nu_2} \mathrm{Fil}_{q-1}(\mathrm{Sym}^{d-q} \otimes \mathrm{Fr}^*)(\mathscr{B}'_{q+1}) \to \ker(\bar{\mu}_d) \to 0$$

where the first term is identified by 4.5.20(ii). Since

$$\mathrm{Fil}_{q-1}(\mathrm{Sym}^{d-q} \otimes \mathrm{Fr}^*)(\mathscr{B}'_{q+1}) = \mathrm{Sym}^{d-q}(\mathscr{B}'_{q+1}) \otimes \mathrm{Fr}^*\,\Omega^1_{\mathbf{P}W}|_C,$$

as is explained in the proof of 4.5.12, the map ν_2 can be identified using 4.5.10 as the map $\beta_W \circ \mathrm{eu}_{\mathbf{P}W} \colon \mathscr{O}_C(-q-1) \to \mathrm{Fr}^*\,\Omega^1_{\mathbf{P}W}|_C$. It follows from 2.2.10 and the diagram of 2.2.8 that the cokernel of this is identified via β with $\mathscr{T}_C(-q-1) \cong \mathscr{O}_C(-2q+1)$. This shows that $\ker(\bar{\mu}_d) \cong \mathrm{Sym}^{d-q}(\mathscr{B}'_{q+1}) \otimes \mathscr{O}_C(-2q+1)$. The statement follows upon identifying $\mathscr{B}'_{q+1} \cong W_C^\vee(-1)$ as in 4.5.17. ∎

4.5.22. The algebra $\mathscr{D}^\nu$. — Consider the short exact sequence of graded $\mathscr{O}_C$-modules from 4.4.24:

$$0 \to \mathscr{D} \xrightarrow{\nu^\#} \mathscr{D}^\nu \to \mathscr{F} \to 0.$$

The algebra $\mathscr{D}^\nu$ and low degree pieces of the map $\nu^\#$ are simple to describe and are given in the next two statements. These descriptions will be used to explicitly compute the global sections of low degree pieces of $\mathscr{F}$ in 4.6.11.

4.5.23. Lemma. — *$\mathscr{D}^\nu \cong \phi_{C,*}\big(\bigoplus_{i=0}^{\delta}(\mathscr{T}_C(-1) \otimes L_-^\vee)^{\otimes i}\big)$ as graded $\mathscr{O}_C$-modules.*

Proof. — Recall from 4.4.15 that $S^\nu = \mathbf{P}(L_{-,C} \oplus \mathscr{T}_C(-1))$ over C, and that C_+^ν is the subbundle $\mathbf{P}(\mathscr{T}_C(-1))$. Thus by 4.4.23, D^ν is the δ-order neighbourhood of the zero section in the affine bundle over C given by

$$S^\nu \setminus C_-^\nu = \mathbf{P}(L_{-,C} \oplus \mathscr{T}_C(-1)) \setminus \mathbf{P}L_{-,C} \cong \mathbf{A}(\Omega_C^1(1) \otimes L_-),$$

using the identification of A.1.10. Whence $\tilde{\varphi}_{+,*}\mathscr{O}_{D^\nu} \cong \bigoplus_{i=0}^{\delta}(\mathscr{T}_C(-1) \otimes L_-^\vee)^{\otimes i}$ and the result follows from 4.4.20(i). ■

By 4.5.21 and 4.5.23, the degree 1 component of $\nu^\#$ is an $\mathscr{O}_C$-module map

$$\nu_1^\# \colon \mathscr{O}_C(-1) \to \phi_{C,*}(\mathscr{T}_C(-1)).$$

Its adjoint is a map between line bundles on C; the following identifies that map:

4.5.24. Lemma. — *The adjoint to the map $\nu_1^\# \colon \mathscr{O}_C(-1) \to \phi_{C,*}(\mathscr{T}_C(-1))$ fits into a commutative diagram*

$$\begin{tikzcd} \phi_C^*(\mathscr{O}_C(-1)) \arrow[r] \arrow[d, "\cong"] & \mathscr{T}_C(-1) \\ \mathrm{Fr}^*(\mathscr{N}_{C/\mathbf{P}W}(-1))^\vee \arrow[r, "\phi_C"] & \mathscr{T}_C^{\mathrm{e}} \arrow[u] \end{tikzcd}$$

where $\phi_C \colon \mathrm{Fr}^*(\mathscr{N}_{C/\mathbf{P}W}(-1))^\vee \to \mathscr{T}_C^{\mathrm{e}}$ is the map induced by $\beta_W^{-1} \circ \delta^{(q)}$ as in 2.2.14.

Proof. — Write $S^{\nu,\circ} := S^\nu \setminus C_-^\nu$ and consider the diagram

$$\begin{tikzcd} S^{\nu,\circ} \arrow[r, "\nu"'] \arrow[d, "\tilde\varphi_+"'] & S^\circ \arrow[r, "\psi"'] \arrow[d, "\varphi_-"] & \mathbf{A}_1 \arrow[dl, "\pi_1"] \\ C \arrow[r, "\phi_C"] & C & \end{tikzcd}$$

where notation is as in 4.5.1 and the square commutes by 4.4.18. Since the degree 1 generator of $\mathscr{D}$ is the fibre coordinate of $\mathbf{A}_1 = \mathbf{P}\mathscr{V}_1 \setminus \mathbf{P}L_{-,C}$ over C, the desired map $\phi_C^*(\mathscr{O}_C(-1)) \to \mathscr{T}_C(-1)$ arises as a map

$$\nu^*\psi^*(\mathscr{O}_{\mathbf{P}\mathscr{V}_1}(-1)|_{\mathbf{A}_1}) \to \mathscr{O}_{\tilde\varphi_+}(-1)|_{S^{\nu,\circ}}$$

upon identifying $\mathscr{O}_{\mathbf{P}\mathscr{V}_1}(-1)|_{\mathbf{A}_1} \cong \pi_1^*\mathscr{O}_C(-1)$ and $\mathscr{O}_{\tilde\varphi_+}(-1)|_{S^{\nu,\circ}} \cong \tilde\varphi_+^*(\mathscr{T}_C(-1))$ via the relative Euler sequences, as in A.1.11. Note both identifications are induced by the linear projection $V \to W$.

To identify this map, note that the discussion of 4.2.12 implies

$$\mathbf{P}\mathscr{V}_1 = \{\, (y \mapsto y_0) \mid y_0 \in C, y \in \langle y_0, x_- \rangle \,\}$$

and 4.2.6 shows that the map $\psi([\ell]) = \ell \cap \mathbf{P}\,\mathrm{Fr}^*(L_-)^\perp =: y$ for a line $\ell \subset X$ not passing through x_-. Since the pullback under ν of the tautological subbundle on S is the bundle $\mathscr{K}$ by its construction in 4.4.15, it follows that

$$\nu^*\psi^*(\mathscr{O}_{\mathbf{P}\mathscr{V}_1}(-1)|_{\mathbf{A}_1}) = \ker(\mathscr{K} \subset V_{S^\nu} \twoheadrightarrow (V/\mathrm{Fr}^*(L_-)^\perp)_{S^\nu})|_{S^{\nu,\circ}}.$$

Post-composing with the restriction to $S^{\nu,\circ}$ of the natural map $\mathscr{K} \subset \tilde{\mathscr{W}} \twoheadrightarrow \mathscr{O}_{\tilde\varphi_+}(-1)$ arising from its construction, see again 4.4.15, gives the desired map relating the fibre coordinates of the two affine bundles. Since

$$\mathscr{K} \subset \tilde{\mathscr{W}} \subset \tilde\varphi_+^*\mathscr{W} = U_{S^\nu} \oplus \tilde\varphi_+^*\mathscr{T}_C^{\mathrm{e}},$$

linear projection $V \to W$ induces the following commutative diagram on S^ν:

$$\begin{tikzcd} \nu^*\psi^*(\mathscr{O}_{\mathbf{P}\mathscr{V}_1}(-1)|_{\mathbf{A}_1}) \arrow[r] \arrow[d, "\cong"'] & \mathscr{K}|_{S^{\nu,\circ}} \arrow[r] \arrow[d] & \mathscr{O}_{\tilde\varphi_+}(-1)|_{S^{\nu,\circ}} \arrow[d, "\cong"] \\ \tilde\varphi_+^*\phi_C^*(\mathscr{O}_C(-1))|_{S^{\nu,\circ}} \arrow[r, "\phi_C"] & \tilde\varphi_+^*\mathscr{T}_C^{\mathrm{e}}|_{S^{\nu,\circ}} \arrow[r, two heads] & \tilde\varphi_+^*(\mathscr{T}_C(-1))|_{S^{\nu,\circ}} \end{tikzcd}$$

where the left-most and right-most vertical maps are the identifications of the tautological bundles arising from the Euler sequence, see A.1.11. The commutative diagram of schemes above implies that the bottom-left map gives the line subbundle of $\mathscr{T}_C^{\mathrm{e}} \subset W_C$ inducing the map $\phi_C \colon C \to C$; by 3.5.2 this is the sheaf map ϕ_C. ■

The following summarizes the consequences on the structure of $\mathscr{F}$.

4.5.25. Duality between $\mathscr{D}$ and $\mathscr{F}$. — By 4.4.27, there is a canonical isomorphism

$$\mathscr{F} \cong \mathscr{D}^\vee \otimes \mathscr{O}_C(-q+1) \otimes L_+^{\otimes 2q-1} \otimes L_-^{\otimes 2}.$$

This identifies graded components as $\mathscr{F}_i \cong \mathscr{D}_{\delta-i}^\vee \otimes \mathscr{O}_C(-q+1)$ for each $0 \leq i \leq \delta = 2q^2-q-2$. In particular, $\mathscr{F}_{dq+q-1} \cong \mathscr{D}^\vee_{(2q-3-d)q+q-1} \otimes \mathscr{O}_C(-q+1)$ for $0 \leq d \leq 2q-3$ and 4.5.21 gives canonical short exact sequences

$$0 \to \mathrm{Div}^{2q-3-d}_{\mathrm{red}}(W) \otimes \mathscr{O}_C \to \mathscr{F}_{dq+q-1} \to \mathrm{Div}^{q-3-d}(W) \otimes \mathscr{O}_C(q-1) \to 0$$

where for each $j \in \mathbf{Z}$, set $\mathrm{Div}^j(W) := 0$ if $j < 0$, and

$$\mathrm{Div}^j_{\mathrm{red}}(W) := \ker(\mathrm{Div}^j(W) \to \mathrm{Div}^{j-q}(W) \otimes \mathrm{Fr}^*(W))$$

where the map is dual to the multiplication map on symmetric powers. Note that $\mathrm{Div}^j_{\mathrm{red}}(W) = \mathrm{Div}^j(W)$ for $j < q$.

The q step increasing filtration of $\mathscr{D}$ induces one on $\mathscr{F}$ with

$$\mathrm{Fil}_i \mathscr{F} := (\mathscr{D}/\mathrm{Fil}_{q-1-i} \mathscr{D})^\vee \otimes \mathscr{O}_C(-q+1) \otimes L_+^{\otimes 2q-1} \otimes L_-^{\otimes 2}$$

for $0 \leq i \leq q-1$ so that $\mathrm{gr}_i(\mathscr{F}) \cong \mathrm{gr}_{q-1-i}(\mathscr{D})^\vee \otimes \mathscr{O}_C(-q+1) \otimes L_+^{\otimes 2q-1} \otimes L_-^{\otimes 2}$. Finally, the maps $\partial_j \colon \mathscr{D} \to \mathscr{D}$, with $0 \leq j \leq q-1$, from 4.5.14 giving the $\boldsymbol{\alpha}_q$-action dually yield maps $\partial_j \colon \mathscr{F} \to \mathscr{F}$. The diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Fil}_{q-1-i+j} \mathscr{D} \arrow[r] \arrow[d, "\partial_j"] & \mathscr{D} \arrow[r] \arrow[d, "\partial_j"] & \mathscr{D}/\mathrm{Fil}_{q-1-i+j} \mathscr{D} \arrow[r] \arrow[d, "\partial_j"] & 0 \\
0 \arrow[r] & \mathrm{Fil}_{q-1-i} \mathscr{D} \arrow[r] & \mathscr{D} \arrow[r] & \mathscr{D}/\mathrm{Fil}_{q-1-i} \mathscr{D} \arrow[r] & 0
\end{tikzcd}$$

implies that $\partial_j(\mathrm{Fil}_i \mathscr{F}) \subseteq \mathrm{Fil}_{i-j} \mathscr{F}$ and that ∂_j is of degree $-j(q+1)$.

4.6. Cohomology of $\mathscr{F}$

This Section computes the cohomology of the sheaf $\mathscr{F}$ introduced in 4.4.24 in the case $q = p$ is a prime; see 4.6.14 for the final result. This is done by identifying each graded component $\mathrm{H}^0(C, \mathscr{F}_i)$ as a representation $\mathrm{SU}_3(p)$ using the structure results from 4.5.25, and proceeds in three steps: First, many global sections are constructed for the components $\mathscr{F}_{ap+p-1}$ by using their explicit structure from 4.5.25. Second, the action of $\boldsymbol{\alpha}_p$ on $\mathscr{F}$ gives maps

$$\mathscr{F}_{ap+p-1} \xrightarrow{\partial} \mathscr{F}_{ap+p-1-(p+1)} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \begin{cases} \mathscr{F}_{p-1-a} & \text{if } 0 \leq a \leq p-2, \text{ and} \\ \mathscr{F}_{(a-p+1)p} & \text{if } p-1 \leq a \leq 3p-3. \end{cases}$$

Each of the maps in the sequence are injective on global sections, see 4.6.5, and so this gives a lower bound on the space of sections of each component, see 4.6.6. Third, a corresponding upper bound is determined for those rightmost sheaves, see 4.6.7. The matching bounds give equality throughout, completing the computation.

4.6.1. Notation. — Throughout this Section, *assume that $q = p$ is prime*. In particular, Fr denotes the p-power Frobenius. Let $\mathrm{SU}_3(p) := \mathrm{U}_3(p) \cap \mathbf{SL}(W)$ be the special unitary group associated with the q-bic form (W, β_W). As in 4.5.25, for each integer b, write

$$\mathrm{Div}^b_{\mathrm{red}}(W) := \ker(\mathrm{Div}^b(W) \to \mathrm{Div}^{b-p}(W) \otimes \mathrm{Fr}^*(W))$$

with the convention that $\mathrm{Div}^b(W) = 0$ when $b < 0$. For small b, this is the simple $\mathrm{SU}_3(p)$ representation obtained by restricting simple representations of $\mathbf{SL}(W) = \mathbf{SL}_3$:

$$\mathrm{Div}^b_{\mathrm{red}}(W) = \begin{cases} L(0,b) & \text{for } 0 \leq b \leq p-1, \text{ and} \\ L(b-p+1, 2p-2-b) & \text{for } p \leq b \leq 2p-3. \end{cases}$$

See B.3.3 for the notation on the $\mathbf{SL}_3$-representations and B.3.6 in regards to restriction of representations to $\mathrm{SU}_3(p)$.

The following determines the global sections of symmetric powers of the tangent bundle of $\mathbf{P}W$. The restriction $q = p$ makes it possible to use the Borel–Weil–Bott Theorem in the cases at hand.

4.6.2. Lemma. — *Let $0 \leq b \leq p-1$ and let $a \leq 0$ be integers. Then*

$$\mathrm{H}^0(C, \mathrm{Sym}^b(\mathcal{T}_{\mathbf{P}W}(-1))(a)|_C) = \begin{cases} \mathrm{Sym}^b(W) & \text{if } a = 0, \text{ and} \\ 0 & \text{if } a < 0, \end{cases}$$

as representations of $\mathrm{SU}_3(p)$.

Proof. — The restriction sequence

$$0 \to \mathrm{Sym}^b(\mathcal{T}_{\mathbf{P}W}(-1))(a-p-1) \to \mathrm{Sym}^b(\mathcal{T}_{\mathbf{P}W}(-1))(a) \to \mathrm{Sym}^b(\mathcal{T}_{\mathbf{P}W}(-1))(a)|_C \to 0$$

implies it suffices to show that $\mathrm{H}^0(\mathbf{P}W, \mathrm{Sym}^b(\mathcal{T}_{\mathbf{P}W}(-1))) = \mathrm{Sym}^b(W)$ and

$$\mathrm{H}^0(\mathbf{P}W, \mathrm{Sym}^b(\mathcal{T}_{\mathbf{P}W}(-1))(a)) = \mathrm{H}^1(\mathbf{P}W, \mathrm{Sym}^b(\mathcal{T}_{\mathbf{P}W}(-1))(a-p)) = 0 \quad \text{when } a < 0.$$

The identification of global sections follows the Euler sequence; since $0 \leq b \leq p-1$, the vanishing follows from the Borel–Weil–Bott Theorem *á la* Griffith, see B.2.3. ■

The sections coming from the 0-th filtered piece of $\mathcal{F}$ can now be determined.

4.6.3. Lemma. — *As a representation of* $\mathbf{Aut}(L_- \subset U) \times \mathrm{SU}_3(p)$,

$$\mathrm{H}^0(C, \mathrm{Fil}_0 \mathcal{F}) \cong \bigoplus\nolimits_{i=0}^{p-2} \mathrm{Div}^{p-2-i}(W) \otimes L_+^{\otimes i}.$$

Proof. — The duality between $\mathcal{D}$ and $\mathcal{F}$ as in 4.5.25 yields an identification

$$\mathrm{Fil}_0 \mathcal{F} \cong \mathrm{gr}_{p-1}(\mathcal{D})^\vee \otimes \mathcal{O}_C(-p+1) \otimes L_+^{\otimes 2p-1} \otimes L_-^{\otimes 2}.$$

By 4.5.13 and 4.4.23, the right hand side is

$$\big((\pi_* \mathcal{O}_{T^\circ})^\vee \otimes \mathcal{O}_C(-p+1) \otimes L_+^{\otimes p} \otimes L_-^{\otimes p+1}\big)_{\geq 0} = \bigoplus\nolimits_{d=0}^{p^2-p-1} (\pi_* \mathcal{O}_{T^\circ})^\vee_{p^2-p-1-d} \otimes \mathcal{O}_C(-p+1).$$

Since $p^2 - p - 1$ is strictly less than the degree p^2 of the equation v_2 from 4.5.10, the graded pieces of $\pi_* \mathcal{O}_{T^\circ}$ appearing above coincide with the graded pieces of $\mathcal{B}'$

as given in 4.5.17. Thus this gives a canonical isomorphism

$$\mathrm{Fil}_0 \mathscr{F} \cong \bigoplus\nolimits_{i=0}^{p-2} \bigoplus\nolimits_{j=0}^{p-1} \mathrm{Div}^{p-2-i}(\mathscr{T}_{\mathbf{P}W}(-1))(-j)|_C \otimes L_+^{\otimes i} \otimes L_-^{\vee,\otimes j}.$$

Since all divided powers have exponent less than p, they coincide with symmetric powers. Then 4.6.2 applies to show that the summands with $j = 0$ have the required spaces of sections, and all other summands have no sections. ■

Let $\partial : \mathscr{F} \to \mathscr{F}$ be the operator induced by the action of $\boldsymbol{\alpha}_p$, as explained in 4.5.14 and 4.5.25. Its kernel is easily determined:

4.6.4. Lemma. — $\ker(\partial : \mathscr{F} \to \mathscr{F}) = \bigoplus_{i=0}^{p} \mathscr{F}_i \oplus \bigoplus_{i=p+1}^{\delta} \mathrm{Fil}_0 \mathscr{F}_i$.

Proof. — Since ∂ is of degree $-p-1$, each of $\mathscr{F}_i$ for $0 \leq i \leq p$ must be annihilated by ∂. For $p+1 \leq i \leq \delta$, the duality of 4.5.25 applied to 4.5.15 gives an exact sequence

$$0 \to \mathrm{Fil}_0 \mathscr{F}_i \to \mathscr{F}_i \xrightarrow{\partial} \mathscr{F}_{i-p-1} \to \mathrm{gr}_{p-1}(\mathscr{F}_{i-p-1}) \to 0$$

and this proves the statement. ■

4.6.5. Corollary. — *$\partial : \mathscr{F}_i \to \mathscr{F}_{i-p-1}$ is injective on global sections except when*

- *$\mathscr{F}_i$ is on top, so that $0 \leq i \leq p-1$, or*
- *$\mathscr{F}_i$ is on the left, so that $i = jp$ for $0 \leq j \leq p-2$.*

Proof. — This follows from 4.6.4 and 4.6.3. ■

The following gives a lower bound on the spaces of sections of the $\mathscr{F}_i$ by using the explicit description of the $\mathscr{F}_{ap+p-1}$ from 4.5.25, and successively applying the operators ∂ to propagate these sections to other components:

4.6.6. Lemma. — *There are inclusions of $\mathrm{SU}_3(p)$-representations*

$$\mathrm{Div}^{2p-3-a}_{\mathrm{red}}(W) \subseteq \mathrm{H}^0(C, \mathscr{F}_{ap+p-1-b(p+1)})$$

for $0 \leq a \leq 2p-3$ and $0 \leq b \leq \min(a, p-1)$.

Proof. — The sheaves $\mathscr{F}_{ap+p-1}$ are identified in 4.5.25 and taking global sections gives the stated inclusion for $b=0$. The remaining statements for $b>0$ now following from the injectivity statements on global sections for ∂ from 4.6.5. ■

4.6.7. Upper bounds. — The statement of 4.6.6 might be thought of as a lower bound on the groups $\mathrm{H}^0(C,\mathscr{F}_i)$. It remains to give a matching upper bound. The injectivity statements for $\partial^\vee$ from 4.6.5 means it suffices to give a bound when

- $\mathscr{F}_i$ is on top, so that $0 \leq i \leq p-1$, and
- $\mathscr{F}_i$ is on the left, so that $i = jp$ for $0 \leq j \leq 2p-2$.

The cases $0 \leq i \leq p$ are dealt with an explicit cohomology computation, see 4.6.10 and 4.6.11; the remaining cases then follow from this explicit calculation by further analyzing the action of ∂ on global sections, see 4.6.13.

4.6.8. — Let $0 \leq i \leq p$ and consider the defining short exact sequence from 4.4.24:

$$0 \to \mathscr{D}_i \to \mathscr{D}_i^\nu \to \mathscr{F}_i \to 0.$$

Using the identification of the low degree pieces of $\mathscr{D}$ from 4.5.21, of $\mathscr{D}^\nu$ from 4.5.23, and taking the long exact sequence in cohomology shows that, as $\mathrm{SU}_3(p)$-representations,

$$\mathrm{H}^0(C,\mathscr{F}_i) \cong \begin{cases} \ker\big(\mathrm{H}^1(C,\mathscr{O}_C(-i)) \to \mathrm{H}^1(C,\phi_{C,*}(\mathscr{T}_C(-1)^{\otimes i}))\big) & \text{if } 0 \leq i \leq p-1, \\ \ker\big(\mathrm{H}^1(C,\Omega^1_{\mathbf{P}W}(1)|_C) \to \mathrm{H}^1(C,\phi_{C,*}(\mathscr{T}_C(-1)^{\otimes p}))\big) & \text{if } i = p. \end{cases}$$

Then 4.5.24 identifies the map $\mathscr{D}_i \to \phi_{C,*}\mathscr{D}_i^\nu$, when $0 \leq i \leq p-1$, as

$$\phi_{C,*}(\phi_C^i) \circ \phi_C^\#\colon \mathscr{O}_C(-i) \to \phi_{C,*}(\mathscr{O}_C(-iq^2)) \to \phi_{C,*}(\mathscr{T}_C(-1)^{\otimes i}).$$

In the case $i = p$, the composite $\mathscr{D}_1^{\otimes p} \hookrightarrow \mathscr{D}_p \to \phi_{C,*}\mathscr{D}_p^\nu$ is also given by this map. Its action on $\mathrm{H}^1(C,\mathscr{O}_C(-i))$ may be computed explicitly as follows: Let

$$f \in \mathrm{H}^0(\mathbf{P}W,\mathscr{O}_{\mathbf{P}W}(p+1)) \quad \text{and} \quad \tilde{\phi}_C \in \mathrm{H}^0(\mathbf{P}W,\mathscr{O}_{\mathbf{P}^2}(p^2-p+1))$$

be an equation for C and any lift of $\phi_C \in \mathrm{H}^0(C, \mathcal{O}_C(p^2-p+1))$, respectively. Then there is a commutative diagram sheaves on $\mathbf{P}W$ with exact rows given by

$$\begin{tikzcd}
\mathcal{O}_{\mathbf{P}W}(-i-p-1) \arrow[r, hook, "f"'] \arrow[d, "\phi_C^\#"] & \mathcal{O}_{\mathbf{P}W}(-i) \arrow[r, two heads] \arrow[dd, "\phi_C^\#"] & \mathcal{O}_C(-i) \arrow[dd, "\phi_C^\#"] \\
\phi_{C,*}\mathcal{O}_{\mathbf{P}W}(-(i+p+1)p^2) \arrow[d, "f^{p^2-1}"] & & \\
\phi_{C,*}\mathcal{O}_{\mathbf{P}W}(-ip^2-p-1) \arrow[r, hook, "\phi_{C,*}(f)"] \arrow[d, "\phi_{C,*}(\tilde\phi_C^i)"] & \phi_{C,*}\mathcal{O}_{\mathbf{P}W}(-ip^2) \arrow[r, two heads] \arrow[d, "\phi_{C,*}(\tilde\phi_C^i)"] & \phi_{C,*}\mathcal{O}_C(-ip^2) \arrow[d, "\phi_{C,*}(\phi_C^i)"] \\
\phi_{C,*}\mathcal{O}_{\mathbf{P}W}(-i(p-1)-p-1) \arrow[r, hook] & \phi_{C,*}\mathcal{O}_{\mathbf{P}W}(-i(p-1)) \arrow[r, two heads] & \phi_{C,*}(\mathcal{T}_C(-1)^{\otimes i}).
\end{tikzcd}$$

Consider the long exact sequence in cohomology associated with the top row of the diagram. The following identifies the resulting spaces as representations for $\mathrm{SU}_3(p)$ using some of the notation and computations from Appendix B. See, in particular, B.1.1 and B.1.4 for the notations $L(a,b)$ and $\Delta(a,b)$.

4.6.9. Lemma. — *The* $\mathrm{SU}_3(p)$ *representations* $\mathrm{H}^1(C, \mathcal{O}_C(-i))$ *for* $0 \leq i \leq p$ *are:*

(i) *If* $0 \leq i \leq 1$, *then* $\mathrm{H}^1(C, \mathcal{O}_C(-i)) \cong \mathrm{Div}^{p+i-2}(W)$ *is simple.*

(ii) *If* $2 \leq i \leq p$, *then there is a short exact sequence*

$$0 \to \mathrm{Div}^{p+i-2}_{\mathrm{red}}(W) \to \mathrm{H}^1(C, \mathcal{O}_C(-i)) \to \Delta(1, i-2) \to 0.$$

(iii) *If* $i \neq p$, *then the quotient* $\Delta(1, i-2) = L(1, i-2)$ *is simple.*

(iv) *If* $i = p$, *then* $\Delta(1, p-2) \cong \mathrm{H}^0(C, \Omega^1_{\mathbf{P}W}(1)|_C)$ *and a short exact sequence*

$$0 \to \mathrm{Div}^{p-3}(W) \to \mathrm{H}^0(C, \Omega^1_{\mathbf{P}W}(1)|_C) \to L(1, p-2) \to 0.$$

Proof. — The cohomology sequence associated with

$$0 \to \mathcal{O}_{\mathbf{P}W}(-i-p-1) \xrightarrow{f} \mathcal{O}_{\mathbf{P}W}(-i) \to \mathcal{O}_C(-i) \to 0$$

shows that $\mathrm{H}^1(C, \mathcal{O}_C(-i)) \cong \ker(\mathrm{Div}^{i+p-2}(W) \xrightarrow{f} \mathrm{Div}^{i-3}(W))$. When $0 \leq i \leq 1$, B.3.3(i) shows that $\mathrm{Div}^{p+i-2}(W)$ is the simple representation $L(0, p+i-2)$, proving (i). Suppose from now on that $2 \leq i \leq p$. Then the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Div}^{i+p-2}_{\mathrm{red}}(W) \arrow[r] & \mathrm{Div}^{i+p-2}(W) \arrow[r] \arrow[d, "f"'] & \mathrm{Fr}^*(W) \otimes \mathrm{Div}^{i-2}(W) \arrow[r] \arrow[d, "f"] & 0 \\ & & \mathrm{Div}^{i-3}(W) \arrow[r, equal] & \mathrm{Div}^{i-3}(W) & \end{tikzcd}$$

is commutative with exact top row, and **B.3.7** gives the identification

$$\ker\left(\mathrm{Fr}^*(W) \otimes \mathrm{Div}^{i-2}(W) \xrightarrow{f} \mathrm{Div}^{i-3}(W)\right) \cong \Delta(1, i-2).$$

Taking kernels of the vertical maps in the diagram yields the exact sequence in (ii):

$$0 \to \mathrm{Div}^{i+p-2}_{\mathrm{red}}(W) \to \mathrm{H}^1(C, \mathscr{O}_C(-i)) \to \Delta(1, i-2) \to 0.$$

If $2 \leq i \leq p-1$, then $\Delta(1, i-2) = L(1, i-2)$ is simple by **B.3.4**(i), proving (iii). If $i = p$, then the Euler sequence yields an $\mathrm{SU}_3(p)$-equivariant identification

$$\mathrm{H}^1(C, \Omega^1_{\mathbf{P}W}(1)|_C) \cong \ker\left(W^\vee \otimes \mathrm{H}^1(C, \mathscr{O}_C) \to \mathrm{H}^1(C, \mathscr{O}_C(-1))\right) \cong \Delta(1, p-2)$$

and **B.3.4**(ii) gives the short exact sequence finishing the proof of (iv). ∎

The following shows that $\mathrm{H}^1(C, \mathscr{D}) \to \mathrm{H}^1(C, \mathscr{D}^\nu)$ is nonzero in low degree components by using the explicit computation in the cohomology of $\mathbf{P}W$ using the diagram of **4.6.8** and the identification of $\nu^\#$ from **4.5.24**.

4.6.10. Lemma. — *The map* $\phi_{C,*}(\phi_C^i)\colon \mathrm{H}^1(C, \mathscr{O}_C(-i)) \to \mathrm{H}^1(C, \phi_{C,*}(\mathscr{T}_C(-1)^{\otimes i}))$ *is nonzero for each* $2 \leq i \leq p$.

Proof. — Choose coordinates $(x : y : z)$ on $\mathbf{P}W = \mathbf{P}^2$ so that $f = x^p y + x y^p - z^{p+1}$. A lift $\tilde{\phi}_C$ of ϕ_C is computed in **3.5.4** and is given by

$$\tilde{\phi}_C := \frac{x^{p^2} y - x y^{p^2}}{x^p y + x y^p} z = \left(x^{p(p-1)} - x^{(p-1)(p-1)} y^{p-1} + \cdots - y^{p(p-1)}\right) z.$$

View the cohomology groups of $\mathbf{P}^2$ as a module over its homogeneous coordinate ring as explained in [**Stacks**, 01XT], and consider a class

$$\xi := \frac{1}{xyz} \frac{1}{x^{i+p-2}} \in \mathrm{H}^2(\mathbf{P}^2, \mathscr{O}_{\mathbf{P}^2}(-i-p-1)).$$

Such a class acts on homogeneous polynomials by contraction, see [**Stacks**, 01XV]. In particular, $\xi \cdot f = 0$ since f does not contain a pure power of x, so ξ represents a class in $\mathrm{H}^1(C, \mathcal{O}_C(-i))$. I claim that $\phi_{C,*}(\phi_C^i)(\xi) \neq 0$. Indeed, since f is a Hermitian q-bic equation, $\phi_C = \mathrm{Fr}^2$ is the q^2-power Frobenius by **2.9.8**, and the diagram of **4.6.8** shows that $\phi_{C,*}(\phi_C^i)(\xi)$ is represented by the product

$$\xi^{p^2} \cdot (f^{p^2-1}\tilde{\phi}_C^i) = \Big(\frac{1}{xyz}\frac{1}{x^{d+p-2}}\Big)^{p^2} \cdot \Big((x^p y + x y^p - z^{p+1})^{p^2-1}\Big(\frac{x^{p^2}y - xy^{p^2}}{x^p y + x y^p}\Big)^i z^i\Big).$$

To see this is nonzero, consider the coefficient of $z^{(p+1)(p-2)+i}$ in $f^{p^2-1}\tilde{\phi}_C^i$. Since $0 < i < p+1$, this is the coefficient of z^i in $\tilde{\phi}_C^i$ multiplied by the coefficient of $z^{(p+1)(p-2)}$ in f^{p^2-1}. The latter is $-(x^p y + x y^p)^{p^2-p+1}$, as found by writing

$$f^{p^2-1} = \big((x^p y + x y^p)^p - z^{p(p+1)}\big)^{p-1}\big((x^p y + x y^p) - z^{p+1}\big)^{p-1}.$$

Therefore $\xi^{p^2} \cdot (f^{p^2-1}\tilde{\phi}_C^i)$ has a summand given by

$$\begin{aligned}\frac{1}{x^{(i+p-1)p^2}y^{p^2}z^{p+2-i}} &\cdot \Big(-(x^p y + x y^p)^{p^2-p+1}\Big(\frac{x^{p^2}y - xy^{p^2}}{x^p y + x y^p}\Big)^i\Big) \\ &= \frac{-1}{x^{(i+p-2)p^2+p-1}y^{p-1}z^{p+2-i}} \cdot \Big((x^{p-1} + y^{p-1})^{p^2-p+1-i}(x^{p^2-1} - y^{p^2-1})^i\Big).\end{aligned}$$

Since all monomials in y involve at least y^{p-1}, the only potentially nonzero contribution is the pure power of x, so this is equal to

$$\frac{-1}{x^{(i+p-1)p^2+p-1}y^{p-1}z^{p+2-i}} \cdot x^{(p-1)(p^2-p+1-i)+(p^2-1)i} = \frac{-1}{x^{(i-1)p}y^{p-1}z^{p+2-i}}.$$

This is nonzero in $\mathrm{H}^2(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(-(i+1)(p+1)))$ if $2 \leq i \leq p$, so $\phi_{C,*}(\phi_C^i)(\xi) \neq 0$. ∎

The following completely identifies the global sections of the low degree graded components of $\mathscr{F}$:

4.6.11. Proposition. — *There are* $\mathrm{SU}_3(p)$*-equivariant isomorphisms*

$$\mathrm{H}^0(C, \mathscr{F}_i) \cong \begin{cases} \mathrm{Div}^{p+i-2}_{\mathrm{red}}(W) & \text{if } 0 \leq i \leq p-1, \text{ and} \\ \mathrm{Div}^{p-3}(W) & \text{if } i = p. \end{cases}$$

Proof. — Fix $0 \le i \le p$ and let L denote the $\mathrm{SU}_3(p)$-representation appearing on the right hand side of the statement. Consider the short exact sequence

$$0 \to \mathrm{H}^0(C, \mathscr{F}_i) \to \mathrm{H}^1(C, \mathscr{D}_i) \xrightarrow{\phi} \mathrm{H}^1(C, \phi_{C,*}(\mathscr{D}_i^\nu)) \to 0.$$

Applying 4.6.6 with $a = b = p - i - 1$ when $i \neq p$ and $a = b + 1 = p$ when $i = p$ shows that $L \subseteq \mathrm{H}^0(C, \mathscr{F}_i) \subseteq \mathrm{H}^1(C, \mathscr{D}_i)$. When $0 \le i \le 1$, 4.5.21 with 4.6.9(i) already show that $\mathrm{H}^1(C, \mathscr{D}_i) = L$, proving the Proposition in that case. For $2 \le i \le p$, I claim that the map ϕ is nonzero and that $\mathrm{H}^1(C, \mathscr{D}_i)/L$ is a simple $\mathrm{SU}_3(p)$-representation. This implies the Proposition upon comparing with the short exact sequence above.

Consider the claim for $2 \le i \le p-1$. Then $\mathscr{D}_i \cong \mathscr{O}_C(-i)$ by 4.5.21, $\phi = \phi_{C,*}(\phi_C^i)$ by 4.5.24 which is nonzero by 4.6.10, and $\mathrm{H}^1(C, \mathscr{D}_i)/L \cong L(1, i-2)$ is simple by 4.6.9(iii). When $i = p$, then $\mathscr{D}_p \cong \Omega^1_{\mathbf{P}W}(1)|_C$ by 4.5.21. The subbundle given by $\mathscr{D}_1^{\otimes p} = \mathscr{O}_C(-p)$ maps to $\phi_{C,*}\mathscr{D}_p^\nu$ via $\phi_{C,*}(\phi_C^p)$ by 4.5.24. Thus there is a diagram in cohomology

$$\begin{tikzcd} \mathrm{H}^1(C, \mathscr{D}_p) \arrow[r, "\phi"] & \mathrm{H}^1(C, \phi_{C,*}\mathscr{D}_p^\nu) \\ \mathrm{H}^1(C, \mathscr{O}_C(-p)) \arrow[u, two heads] \arrow[ur, "\phi_{C,*}(\phi_C^p)"'] & \end{tikzcd}$$

where the vertical arrow is surjective since $\mathscr{D}_p/\mathscr{D}_1^{\otimes p} \cong \mathscr{O}_C(p-1)$. Since $\phi_{C,*}(\phi_C^p) \neq 0$ by 4.6.10, $\phi \neq 0$. Finally, 4.6.9(iv) identifies the quotient $\mathrm{H}^0(C, \mathscr{D}_p)/L$ with the simple module $L(1, p-2)$, completing the proof of the claim in the case $i = p$. ■

4.6.12. Corollary. — *For each $0 \le i \le 2p-3$, there are $\mathrm{SU}_3(p)$-equivariant isomorphisms*

$$\mathrm{H}^0(C, \mathscr{F}_{ip+p-1}) \cong \mathrm{Div}^{2p-3-i}_{\mathrm{red}}(W).$$

Proof. — When $p-2 \le i \le 2p-3$, $\mathscr{F}_{ip+p-1} \cong \mathrm{Div}^{2p-3-i}_{\mathrm{red}}(W) \otimes \mathscr{O}_C$ by the sequence in 4.5.25, yielding the conclusion in this case. When $0 \le i \le p-3$, 4.6.6, 4.6.5, and 4.6.11 together give a sequence of inclusions

$$\mathrm{Div}^{2p-3-i}_{\mathrm{red}}(W) \subseteq \mathrm{H}^0(C, \mathscr{F}_{ip+p-1}) \xhookrightarrow{\partial^i} \mathrm{H}^0(C, \mathscr{F}_{p-1-i}) = \mathrm{Div}^{2p-3-i}_{\mathrm{red}}(W).$$

Therefore equality holds throughout. ■

4.6.13. Proposition. — *There are* $\mathrm{SU}_3(p)$*-equivariant isomorphisms*

$$\mathrm{H}^0(C,\mathscr{F}_{ip}) \cong \begin{cases} \mathrm{Div}^{p-2-i}(W) & \text{if } 0 \leq i \leq p-2, \text{ and} \\ 0 & \text{if } p-1 \leq i \leq 2p-2. \end{cases}$$

Proof. — The cases $0 \leq i \leq 1$ are handled by 4.6.11. So assume that $2 \leq i \leq 2p-2$. Dualizing as in 4.4.27 together with 4.5.15 yields an exact sequence

$$0 \to \mathrm{Fil}_0 \mathscr{F}_{ip} \to \mathscr{F}_{ip} \xrightarrow{\partial} \mathscr{F}_{(i-2)p+p-1} \to \mathrm{gr}_{p-1} \mathscr{F}_{(i-2)p+p-1} \to 0.$$

Since $\mathrm{H}^0(C,\mathrm{Fil}_0 \mathscr{F}_{ip}) = \mathrm{Div}^{p-2-i}(W)$ by 4.6.3, with the convention that negative divided powers are zero as in 4.6.1, it suffices to show that ∂ vanishes on global sections. The exact sequence means that this is equivalent to injectivity of $\mathscr{F}_{(i-2)p+p-1} \to \mathrm{gr}_{p-1} \mathscr{F}_{(i-2)p+p-1}$ on global sections.

Consider the composite

$$\mathrm{Div}^{2p-1-i}_{\mathrm{red}}(W) \otimes \mathscr{O}_C \subset \mathscr{F}_{(i-2)p+p-1} \twoheadrightarrow \mathrm{gr}_{p-1} \mathscr{F}_{(i-2)p+p-1}$$

where the first map is the inclusion of the subbundle from 4.5.25. The first map is an isomorphism on global sections by 4.6.12, and $\mathrm{Div}^{2p-1-i}_{\mathrm{red}}(W)$ is a simple $\mathrm{SU}_3(p)$ representation by 4.6.1, so it suffices to see that the composite is a nonzero map of sheaves. Applying the duality of 4.5.25, this is equivalent to the assertion that

$$\bar{\mu}_{2p-1-i} \colon \mathrm{Fil}_0 \mathscr{D}_{(2p-1-i)p+p-1} \to (\mathrm{Sym}^{2p-1-i} / \mathrm{Sym}^{p-1-i} \otimes \mathrm{Fr}^*)(W_C^\vee(-1)) \otimes \mathscr{O}_C(p-i)$$

is nonzero. This follows from the final statement of 4.5.21: $\bar{\mu}_{2p-1-i}$ is a strict surjection of filtered bundles on all of $\mathscr{D}_{(2p-1-i)p+p-1}$, so it gives a surjection

$$\mathrm{Fil}_0 \bar{\mu}_{2p-1-i} \colon \mathrm{Fil}_0 \mathscr{D}_{(2p-1-i)p+p-1} \to (\mathrm{Sym}^{2p-1-i}(\Omega^1_{\mathbf{P}W}) / \mathrm{Sym}^{p-1-i}(\Omega^1_{\mathbf{P}W}))(p-i)|_C$$

between 0-th filtered pieces; the target is nonzero, so $\bar{\mu}_{2p-1-i}$ is nonzero. ■

4.6.14. Theorem. — *There is an isomorphism of* $\mathbf{Aut}(L_- \subset U, \beta_U) \times \mathrm{SU}_3(p)$*-representations* $\mathrm{H}^0(C,\mathscr{F}) \cong \Lambda_1 \oplus \Lambda_2$ *where*

$$\Lambda_1 := \bigoplus_{i=0}^{p-2} \mathrm{Div}^{p-2-i}(W) \otimes \mathrm{Sym}^{p-1}(U) \otimes L_-^{\vee,\otimes p-1} \otimes L_+^{\otimes i}, \text{ and}$$

$$\Lambda_2 := \bigoplus_{i=0}^{p-2} \mathrm{Div}^{p+i-1}_{\mathrm{red}}(W) \otimes \mathrm{Sym}^{p-2-i}(U) \otimes L_-^{\vee,\otimes p-1}.$$

Proof. — Begin by identifying each $\mathrm{H}^0(C,\mathscr{F}_i)$ as a representation for $\mathrm{SU}_3(p)$. The claim is that the inclusions from 4.6.6 are equalities: that

$$\mathrm{H}^0(C,\mathscr{F}_{ap+p-1-b(p+1)}) = \mathrm{Div}^{2p-3-a}_{\mathrm{red}}(W)$$

if $0 \leq a \leq 2p-3$ and $0 \leq b \leq \min(a,p-1)$, and that the group vanishes otherwise. Choose $0 \leq a \leq 3p-3$. Starting from $\mathrm{H}^0(C,\mathscr{F}_{ap+p-1})$ and successively applying ∂ a total of $\min(a,p-1)$ times produces, thanks to 4.6.5, a chain of inclusions

$$\mathrm{H}^0(C,\mathscr{F}_{ap+p-1}) \subseteq \cdots \subseteq \begin{cases} \mathrm{H}^0(C,\mathscr{F}_{p-1-a}) & \text{if } 0 \leq a \leq p-2, \text{ and} \\ \mathrm{H}^0(C,\mathscr{F}_{(a-p+1)p}) & \text{if } p-1 \leq a \leq 3p-3. \end{cases}$$

By convention, set $\mathrm{H}^0(C,\mathscr{F}_i) = 0$ whenever $i > \delta$. The spaces on the left are given by 4.6.12 whereas the spaces on the right are given by 4.6.11 and 4.6.13, and for each fixed a, the lower and upper bounds match. Therefore equality holds throughout. The $\mathbf{Aut}(L_- \subset U, \beta_U)$ side of the representation is obtained by matching weights and using 4.6.5 to identify the action of the unipotent radical. ■

4.6.15. Corollary. — $\dim_{\mathbf{k}} \mathrm{H}^0(C,\mathscr{F}) = (p^2+1)\binom{p}{2} + \binom{p}{3}$.

Proof. — Consider summing the cohomology groups in 4.6.14 column-wise, summing over residue classes modulo p:

$$\begin{aligned}
\dim_{\mathbf{k}} \mathrm{H}^0(C,\mathscr{F}) &= \sum\nolimits_{b=0}^{p-1} \sum\nolimits_{a=0}^{b+p-2} \dim_{\mathbf{k}} \mathrm{H}^0(C,\mathscr{F}_{(b+p-2-a)p+b}) \\
&= \sum\nolimits_{b=0}^{p-1} \sum\nolimits_{a=0}^{b+p-2} \dim_{\mathbf{k}} \mathrm{Div}^a_{\mathrm{red}}(W) \\
&= \sum\nolimits_{b=0}^{p-1} \sum\nolimits_{a=0}^{b+p-2} \big(\dim_{\mathbf{k}} \mathrm{Div}^a(W) - \dim_{\mathbf{k}}(\mathrm{Fr}^*(W) \otimes \mathrm{Div}^{a-p}(W)) \big).
\end{aligned}$$

Since W is a 3-dimensional vector space, $\mathrm{Div}^a(W)$ is $\binom{a+2}{2}$ dimensional for all $a \geq 0$, so using standard binomial coefficient identities gives

$$\begin{aligned}
\dim_{\mathbf{k}} \mathrm{H}^0(C,\mathscr{F}) &= \sum\nolimits_{b=0}^{p-1} \sum\nolimits_{a=0}^{b+p-2} \binom{a+2}{2} - 3 \sum\nolimits_{b=2}^{p-1} \sum\nolimits_{a=0}^{b-2} \binom{a+2}{2} \\
&= \sum\nolimits_{b=0}^{p-1} \binom{b+p+1}{3} - 3 \sum\nolimits_{b=2}^{p-1} \binom{b+1}{3} = \binom{2p+1}{4} - 4\binom{p+1}{4}.
\end{aligned}$$

It can now be directly verified that $\binom{2p+1}{4} - 4\binom{p+1}{4} = (p^2+1)\binom{p}{2} + \binom{p}{3}$. ■

28	36	42	46	51	48	55	42
24	36	39	42	56	72	60	46
18	30	36	36	51	73	75	48
10	18	24	28	36	51	56	48
6	10	15	21	28	36	42	46
3	6	10	15	24	36	39	42
1	3	6	10	18	30	36	36
0	1	3	6	10	18	24	28
0	0	1	3	6	10	15	21
0	0	0	1	3	6	10	15
0	0	0	0	1	3	6	10
0	0	0	0	0	1	3	6
0	0	0	0	0	0	1	3
0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	

36	45	52	58	60	61	71	60	52
31	45	45	58	75	60	77	90	57
21	31	36	45	58	57	63	71	60
15	21	28	36	45	52	58	60	61
11	18	21	31	45	45	58	75	60
6	11	15	21	31	36	45	58	57
3	6	10	15	21	28	36	45	52
1	3	6	11	18	21	31	45	45
0	1	3	6	11	15	21	31	36
0	0	1	3	6	10	15	21	28
0	0	0	1	3	6	11	18	21
0	0	0	0	1	3	6	11	15
0	0	0	0	0	1	3	6	10
0	0	0	0	0	0	1	3	6
0	0	0	0	0	0	0	1	3
0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	

FIGURE 1. The dimensions of the $\mathrm{H}^0(C, \mathscr{F}_i)$ are displayed with $q = 8$ on the left, and $q = 9$ on the right. The numbers are arranged so that the first row displays the dimensions of $\mathrm{H}^0(C, \mathscr{F}_i)$ for $0 \leq i \leq q-1$. These were obtained from a computer calculation.

Together with the computations in 4.4, this gives the coherent cohomology of $\mathscr{O}_S$ when $q = p$:

4.6.16. Theorem. — *If $q = p$, the dimensions of the cohomology groups of $\mathscr{O}_S$ are*

$$\dim_{\mathbf{k}} \mathrm{H}^i(S, \mathscr{O}_S) = \begin{cases} 1 & \text{if } i = 0, \\ (p^2+1)\binom{p}{2} + \binom{p}{3} & \text{if } i = 1, \text{ and} \\ \frac{1}{2}(5p^3 - 5p^2 + 3p - 6)\binom{p+1}{3} & \text{if } i = 2. \end{cases}$$

Proof. — By 4.4.28, $\mathrm{H}^0(S, \mathscr{O}_S) \cong \mathrm{H}^0(C, \mathscr{O}_C)$ and $\mathrm{H}^1(S, \mathscr{O}_S) \cong \mathrm{H}^0(C, \mathscr{F})$. Combined with 4.6.15, this gives the first two numbers. The dimension of $\mathrm{H}^2(S, \mathscr{O}_S)$ can now be obtained from the Euler characteristic computation of 4.1.3. ∎

This Section closes with some remarks on extending this computation for q not necessarily prime.

4.6.17. Remarks toward general q. — The assumption that $q = p$ was used in at least three ways:

(i) to apply the Borel–Weil–Bott Theorem in 4.6.2 and to identify divided powers with symmetric powers in 4.6.3;

(ii) to reduce the action of $\boldsymbol{\alpha}_q$ to the action of the single operator $\partial := \partial_1$ so that its action on associated graded modules is understood via 4.5.15; and

(iii) to show in 4.6.9 that the $\mathrm{SU}_3(q)$ representations appearing are either simple or have very short composition series.

In any case, part of the difficulty to extending the computation past the prime case is that the formula 4.6.15 does not hold for general q. A computer computation shows that

$$\dim_{\mathbf{k}} \mathrm{H}^0(C, \mathscr{F}) = \begin{cases} 106 \\ 2096 \\ 3231 \end{cases} \quad \text{whereas} \quad (q^2+1)\binom{q}{2} + \binom{q}{3} = \begin{cases} 106 & \text{if } q = 4, \\ 1876 & \text{if } q = 8, \text{ and} \\ 3036 & \text{if } q = 9. \end{cases}$$

The dimensions of $\mathrm{H}^0(C, \mathscr{F}_i)$ in the cases $q = 8$ and $q = 9$ are given in Figure 1. Certain general features from the prime case hold true—for instance, the action of $\boldsymbol{\alpha}_q$ still relates graded components which differ in weight by $q-1$—but there seem to jumps in certain entries, likely due to jumps in cohomology of homogeneous bundles on $\mathbf{P}W$.

4.7. Smooth q-bic threefolds

Smooth q-bic threefolds and their Fano schemes of lines have extraordinarily rich geometry, and it is here that the analogy with cubics starts to become quite striking. One of the main results of this Section is an analogue of a theorem of Clemens and Griffiths from [CG72, Theorem 11.19]: the intermediate Jacobian of a q-bic threefold X is purely inseparably isogenous to the Albanese variety of its Fano surface S of lines, see 4.7.21.

Much of the Section is devoted to the fascinating geometry of the surface S. To start, 4.7.3 shows that S does not lift to $\mathrm{W}_2(\mathbf{k})$ whenever $q > 2$, and so this surface is truly a positive characteristic phenomenon. In terms of invariants: its ℓ-adic Betti numbers are computed in 4.7.7, and its structure sheaf cohomology is computed in when $q = p$ in 4.7.27 and 4.7.28. Finally, some special divisors on S are constructed and studied through 4.7.10–4.7.15.

4.7.1. Generalities. — Specializing the general facts about smooth q-bic hypersurfaces to the case of threefolds gives the following:

(i) by 2.3.1, X admits a linear action by the finite unitary group $\mathrm{U}_5(q)$;

(ii) by 2.6.4, the middle ℓ-adic cohomology group $\mathrm{H}^3_{\mathrm{\acute{e}t}}(X, \mathbf{Q}_\ell)$ of X is an irreducible representation for $\mathrm{U}_5(q)$ of dimension $q(q-1)(q^2+1)$;

(iii) by 2.7.16, the Fano scheme S of lines is a smooth irreducible surface of general type with $\omega_S \cong \mathcal{O}_S(2q-3)$, and moreover, by 4.1.2, $\mathcal{T}_S \cong \mathcal{S}^\vee \otimes \mathcal{O}_S(1-q)$; and

(iv) by 2.7.19, the Plücker degree of S is $\deg(\mathcal{O}_S(1)) = (q+1)^2(q^2+1)$.

4.7.2. Liftings. — Let $\mathrm{W}_2(\mathbf{k})$ denote the second Witt vectors of the ground field $\mathbf{k}$. Given a scheme Y over $\mathbf{k}$, a *lift to* $\mathrm{W}_2(\mathbf{k})$ is a scheme $\mathcal{Y}$, flat over $\mathrm{W}_2(\mathbf{k})$, such that $\mathcal{Y} \otimes_{\mathrm{W}_2(\mathbf{k})} \mathbf{k} \cong Y$. The next result shows that when $q > 2$, the surface S does not lift to $\mathrm{W}_2(\mathbf{k})$, let alone characteristic 0; compare with the comments preceding 4.1.2. Note, in contrast, that when $q = 2$, S lifts to characteristic 0 because the canonical map

$$\mathrm{Fr}^*(W) \otimes W \hookrightarrow \mathrm{Sym}^3(W)$$

is an isomorphism for every 2-dimensional vector space W, and this means that the equations of S in $\mathbf{G}(2, V)$ lift.

4.7.3. Proposition. — *If $q > 2$, then S does not lift to* $\mathrm{W}_2(\mathbf{k})$.

Proof. — The sheaf of differentials of S is identified by 4.1.2 as

$$\Omega^1_S \cong \mathcal{S} \otimes \mathcal{O}_S(q-1) \cong \mathcal{S}^\vee \otimes \mathcal{O}_S(q-2)$$

where the second isomorphism arises since $\mathcal{S}$ is of rank 2 so $\mathcal{S} \cong \mathcal{S}^\vee \otimes \wedge^2 \mathcal{S}$. A nonzero global section of $\mathcal{S}^\vee$ induces a nonzero morphism $\mathcal{O}_S(q-2) \to \Omega^1_S$. Thus, when $q > 2$, Ω^1_S contains the ample line bundle $\mathcal{O}_S(q-2)$, and this implies S cannot lift to $\mathrm{W}_2(\mathbf{k})$ by Langer's analogue of Nakano–Akizuki–Kodaira Vanishing, see [Lan16, Proposition 4.1], arising from work on the Bogomolov–Miyaoka–Yau Inequality in positive characteristic. ■

4.7.4. Topology of the Fano scheme. — The next few paragraphs discuss the topology of the Fano surface S. Throughout, $\ell \neq p$ is a prime different from the characteristic. Its ℓ-adic cohomology groups are computed in 4.7.7 by relating S with certain Deligne–Lusztig varieties of type ${}^2\mathrm{A}_5$. Then in 4.7.9, the first cohomology group $\mathrm{H}^1_{\text{ét}}(S, \mathbf{Q}_\ell)$ is related to the middle cohomology $\mathrm{H}^3_{\text{ét}}(X, \mathbf{Q}_\ell)$ of the q-bic threefold via the Fano correspondence.

The following specializes the constructions from 2.9.8–2.9.16 to the case of lines on q-bic threefolds:

4.7.5. Proposition. — *There exists a commutative diagram*

$$\begin{tikzcd}
\tilde{X}^1 \arrow[r, "\phi\times\mathrm{id}"'] \arrow[d] & \tilde{S} \arrow[r, "\mathrm{id}\times\phi"'] \arrow[d] & \tilde{X}^1 \arrow[d] \\
X^1 \arrow[r, dashed, "\mathrm{cyc}^1_\phi"] & S \arrow[r, dashed, "{[\mathrm{id}\cap\phi]}"] & X^1
\end{tikzcd}$$

in which

(i) $\tilde{X}^1 \to X^1$ *is the blowup along the Hermitian points of* X^1,

(ii) $\tilde{S} \to S$ *is the blowup at the points corresponding to Hermitian lines in* X, *and*

(iii) $\phi \times \mathrm{id}$ *and* $\mathrm{id} \times \phi$ *are finite purely inseparable of degree* q^2.

Proof. — This is 2.9.16 upon noting that $S = S_{\mathrm{cyc}}$ by 2.9.15. ■

When X is the Fermat q-bic threefold, this diagram directly relates S with a compactified Deligne–Lusztig variety of type ${}^2\mathrm{A}_5$. This connection may be used to determine the zeta function of S:

4.7.6. Proposition. — *Let X be the Fermat q-bic threefold. Then the zeta function of its Fano scheme S of lines over $\mathbf{F}_{q^2}$ is given by*

$$Z(S;t) = \frac{(1+qt)^{b_1}(1+q^3t)^{b_3}}{(1-t)(1-q^2t)^{b_2}(1-q^4t)}$$

where $b_1 = b_3 = q(q-1)(q^2+1)$ *and* $b_2 = (q^4-q^3+1)(q^2+1)$.

Proof. — First, by 4.7.5(ii) and 2.9.14, $\tilde{S} \to S$ is a blowup along the $(q^3+1)(q^5+1)$ points corresponding to Hermitian lines in X; these are precisely the $\mathbf{F}_{q^2}$-points of S by 1.2.19(ii). Since $Z(\mathbf{P}^1;t)^{-1} = (1-t)(1-q^2t)$, this gives the first equality in

$$Z(S;t) = (1-q^2t)^{(q^3+1)(q^5+1)} Z(\tilde{S};t) = (1-q^2t)^{(q^3+1)(q^5+1)} Z(\tilde{X}^1;t).$$

The second equality follows from 4.7.5(iii), which shows that $\tilde{S}$ and $\tilde{X}^1$ are related by finite purely inseparable morphisms. Finally, by 2.9.12,

$$X^1 = \left\{ \begin{array}{c} x_0^{q+1} + x_1^{q+1} + x_2^{q+1} + x_3^{q+1} + x_4^{q+1} = 0 \\ x_0^{q^3+1} + x_1^{q^3+1} + x_2^{q^3+1} + x_3^{q^3+1} + x_4^{q^3+1} = 0 \end{array} \right\} \subset \mathbf{P}^4$$

and $\tilde{X}^1 \to X^1$ is the blowup along the $\mathbf{F}_{q^2}$-points of X^1. Therefore $\tilde{X}^1$ is the Deligne–Lusztig variety for $\mathrm{U}_5(q)$, denoted by $\overline{X}(s_1s_2)$ in [Rodoo, Théorème 8.1], and the result follows from the computation given in [Rodoo, Théorème 8.2]. ■

4.7.7. Corollary. — *For any smooth q-bic threefold X, the ℓ-adic Betti numbers of its surface S of lines are given by*

$$\dim_{\mathbf{Q}_\ell} \mathrm{H}^i_{\text{ét}}(S,\mathbf{Q}_\ell) = \begin{cases} 1 & \text{if } i=0 \text{ or } i=4, \\ q(q-1)(q^2+1) & \text{if } i=1 \text{ or } i=3, \text{ and} \\ (q^4-q^3+1)(q^2+1) & \text{if } i=2. \end{cases}$$

Proof. — This follows from 4.7.6 since X is isomorphic to the Fermat by 2.2.4. ■

4.7.8. — Consider the Fano correspondence, as in 2.8: let $\mathbf{L} := \mathbf{P}\mathscr{S}$ be the projective bundle associated with the universal subbundle on S and let

$$\begin{tikzcd} & \mathbf{L} \arrow[dl, "\mathrm{pr}_S"'] \arrow[dr, "\mathrm{pr}_X"] & \\ S & & X \end{tikzcd}$$

be the incidence correspondence. By 2.8.2 and 2.8.4, the morphism $\mathrm{pr}_X \colon \mathbf{L} \to X$ is generically finite of degree $q(q+1)$, inseparable of degree q, and its positive dimensional fibres are smooth q-bic curves lying over the Hermitian points of X.

As in 2.8.5, view $\mathbf{L}$ as a degree -1 correspondence from S to X, so that $\mathbf{L}$ acts on Chow rings and ℓ-adic cohomology. The results established so far imply that the cohomological action identifies the middle cohomology of X with the first cohomology of S:

4.7.9. Proposition. — *The morphism*

$$\mathbf{L}^* \colon \mathrm{H}^3_{\text{ét}}(X, \mathbf{Q}_\ell) \to \mathrm{H}^1_{\text{ét}}(S, \mathbf{Q}_\ell)$$

is an isomorphism of $\mathrm{U}_5(q)$*-representations.*

Proof. — The action of $\mathbf{L}$ is $\mathrm{U}_5(q)$-equivariant and injective by 2.8.7. Comparing 4.7.1(ii) and 4.7.7 shows that the source and target spaces are of the same dimension, whence the map is an isomorphism. ∎

4.7.10. Incidence divisors. — The next few paragraphs study two families of divisors on S, each defined by certain incidence conditions:

- the curve C_ℓ of lines incident with a fixed line $\ell \subset X$, and
- the curve C_x of lines incident with a fixed Hermitian point $x \in X$.

The former set of curves was constructed in 2.8.8. The latter set of curves was constructed in 4.2.3, since Smooth Cone Situations of X are precisely given by Hermitian points of X, see 2.4.8. In particular, the C_x are smooth q-bic curves.

When $\ell \subset X$ is a Hermitian line, the curve C_ℓ is made up of the curves C_x, with x ranging over the Hermitian points of X contained in ℓ:

4.7.11. Lemma. — *Let $\ell \subset X$ be a Hermitian line. Then*

$$C_\ell = \sum\nolimits_{x \in \ell \cap X_{\mathrm{Herm}}} C_x$$

as divisors on S. The sum ranges over the Hermitian points of X contained in ℓ.

Proof. — Let D be the sum appearing on the right. It is clear from definitions and symmetry that C_ℓ contains D to some positive multiplicity a. Consider the Plücker degree of the two divisors. On the one hand, by 2.8.9 and 4.7.1(iv),

$$\deg(\mathcal{O}_S(1)|_{C_\ell}) = (q+1)^{-1} \deg(\mathcal{O}_S(1)) = (q+1)(q^2+1)$$

On the other hand, 4.2.3 implies that $\mathscr{O}_S(1)|_{C_x}$ is the usual polarization of C_x as a plane curve. Since ℓ contains q^2+1 Hermitian points, this shows that D is the same degree as C_ℓ and so the two divisors coincide. ∎

The curves parameterizing lines through Hermitian points only intersect if they lie on a common Hermitian line, in which case they intersect transversely:

4.7.12. Lemma. — *Let $x, y \in X$ be distinct Hermitian points. Then, as 0-cycles on S,*

$$[C_x]\cdot[C_y] = \begin{cases} [\ell] & \text{if } x \text{ and } y \text{ lie on a Hermitian line } \ell \subset X, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. — Let $\ell := \langle x, y\rangle$ be the unique line in $\mathbf{P}V$ passing through both x and y. Then the 3-planes $\mathbf{G}(2,V;x)$ and $\mathbf{G}(2,V;y)$ in $\mathbf{G}(2,V)$ parameterizing lines in $\mathbf{P}V$ through x and y, respectively, intersect at the unique point $[\ell]$. Since these 3-planes are linearly embedded into $\mathbf{G}(2,V)$, as explained in the proof of 4.2.3, they intersect with multiplicity 1 at $[\ell]$, and so

$$[C_x]\cdot[C_y] = \big([\mathbf{G}(2,V;x)]\cdot[\mathbf{G}(2,V;y)]\big)\big|_S = \begin{cases} [\ell] & \text{if } \ell \subset X, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

It remains to observe that, by definition, any line with two distinct Hermitian points is a Hermitian line, see 1.2.1. ∎

The divisor C_ℓ, with $\ell \subset X$ Hermitian, intersects each of its irreducible components in a 0-cycle of degree 1:

4.7.13. Lemma. — *Let $\ell \subset X$ be a Hermitian line and $x \in \ell$ a Hermitian point. Then*

$$\mathscr{O}_S(C_\ell)|_{C_x} \cong \mathscr{O}_{C_x}([\ell]).$$

In particular, $\deg([C_\ell]\cdot[C_x]) = 1$.

Proof. — By 4.7.11, C_ℓ can be written as

$$C_\ell = C_x + \sum\nolimits_{y \in \ell \cap X_{\mathrm{Herm}} \setminus \{x\}} C_y$$

where the second sum ranges over the q^2 Hermitian points of X in ℓ different from x. Applying 4.7.12 then gives the first equality in

$$\mathcal{O}_S(C_\ell)|_{C_x} \cong \mathcal{N}_{C_x/S} \otimes \mathcal{O}_{C_x}(q^2[\ell]) \cong \mathcal{O}_{C_x}(-q+1) \otimes \mathcal{O}_{C_x}(q^2[\ell]) \cong \mathcal{O}_{C_x}([\ell]).$$

By 4.2.3(iii), $\mathcal{N}_{C_x/S} \cong \mathcal{O}_{C_x}(-q+1)$. Since $[\ell]$ is a Hermitian point of C_x, it follows from 3.5.2 that $\mathcal{O}_{C_x}(1) \cong \mathcal{O}_{C_x}((q+1)[\ell])$. This gives the latter two isomorphisms above, from which the result follows. ■

In general, whenever $\ell \subset X$ contains at least one Hermitian point of X, the curve C_ℓ contains C_x as an irreducible component, and the residual component intersects C_x only at the point $[\ell]$. The following establishes this, and a little more:

4.7.14. Proposition. — *Let $\ell \subset X$ be a line containing a Hermitian point x. Then there exists an effective divisor C'_ℓ such that*

(i) *$C_\ell = C'_\ell + C_x$ as divisors on S,*

(ii) *C'_ℓ does not contain C_x as an irreducible component, and*

(iii) *for every Hermitian point y of X, as 0-cycles on S,*

$$[C'_\ell] \cdot [C_y] = \begin{cases} q^2[\ell] & \text{if } x = y, \text{ and} \\ 0 & \text{if } x \neq y. \end{cases}$$

Proof. — If ℓ is itself Hermitian, then (i) and (ii) follow from 4.7.11, and (iii) follows from 4.7.12 and the computation in the proof of 4.7.13. So for the remainder of the proof, assume that x is the unique Hermitian point of X contained in ℓ.

By its construction from 2.8.8, C_ℓ is the locus in S of lines that are incident with ℓ. Since $x \in \ell$, it follows that C_x is an irreducible component of C_ℓ; also, C_y is not contained in C_ℓ for any Hermitian point $y \neq x$. This already shows that the divisor $C'_\ell := C_\ell - C_x$ is effective, proving (i).

Choose a Hermitian point y different from x such that $\langle x, y\rangle \subset X$. On the one hand, $[C'_\ell]\cdot[C_y]$ is an effective 0-cycle on S. On the other hand,

$$\begin{aligned}\deg([C'_\ell]\cdot[C_y]) &= \deg([C_\ell]\cdot[C_y]) - \deg([C_x]\cdot[C_y]) \\ &= \deg([C_{\langle x,y\rangle}]\cdot[C_y]) - \deg([C_x]\cdot[C_y]) = 0\end{aligned}$$

where the first equality follows from (i), the second from the fact **2.8.9** that the curves C_ℓ and $C_{\langle x,y\rangle}$ are algebraically equivalent, and the third from the computations of **4.7.12** and **4.7.13**. Therefore $[C'_\ell]\cdot[C_y] = 0$ and, by **4.7.12**, C'_ℓ does not contain C_x as an irreducible component, proving (ii). In fact, this argument applied to any Hermitian point $y \neq x$ proves the second case of (iii).

It remains to consider the 0-cycle $[C'_\ell]\cdot[C_x]$. By (ii), this is an effective 0-cycle. Since the only line that is incident with ℓ and passes through x is ℓ itself, it is supported on $[\ell]$. In other words, there is a nonnegative integer a such that

$$[C'_\ell]\cdot[C_x] = a[\ell] \in \mathrm{CH}_0(S).$$

The multiplicity is now determined as above: choosing a Hermitian line $\ell_0 \subset X$ containing x, it follows that

$$a = \deg([C'_\ell]\cdot[C_x]) = \deg([C'_{\ell_0}]\cdot[C_x]) = \sum\nolimits_{y \in \ell_0 \cap X_{\mathrm{Herm}} \setminus \{x\}} \deg([C_y]\cdot[C_x]) = q^2.$$

by (i), **2.8.9**, **4.7.11**, and **4.7.12**. ■

4.7.15. Remark. — When ℓ contains a single Hermitian point, further properties of the divisor C'_ℓ might be understood by relating it to the discriminant divisor D_1 associated with the q-bic curve fibration $X' \to \mathbf{P}W$ obtained by linear projection of $\mathbf{P}V$ away from ℓ: see **2.5.7**. The same methods may be used in the case ℓ does not contain any Hermitian points to study the divisor C_ℓ.

The next few paragraphs relate a certain abelian variety $\mathbf{Ab}^2_X$ attached to the smooth q-bic threefold X, referred to as the *intermediate Jacobian of* X, with the Albanese variety of S: see **4.7.21**. This is done by using the Fano correspondence from **4.7.8** to relate codimension 2 cycles of X with divisors of S, and by using the computations of 4.7.10–4.7.14. The definitions below follow [Bea77, §3.2] and [Mur85, §§1.5–1.8].

4.7.16. Algebraic representatives. — Let Y a smooth projective variety over an algebraically closed field $\mathbf{k}$ and let A be an abelian variety over $\mathbf{k}$. Given an integer $0 \leq k \leq \dim Y$, a homomorphism of groups

$$\phi : \mathrm{CH}^k(Y)_{\mathrm{alg}} \to A(\mathbf{k})$$

is said to be *regular* if for every pointed smooth projective variety (T, t_0) over $\mathbf{k}$, and every cycle class $Z \in \mathrm{CH}^k(T \times Y)$, the map

$$T(\mathbf{k}) \to \mathrm{CH}^k(Y)_{\mathrm{alg}} \to A(\mathbf{k}), \qquad t \mapsto \phi(Z_t - Z_{t_0})$$

is induced by a morphism $T \to A$ of varieties over $\mathbf{k}$. A regular homomorphism

$$\phi_Y^k : \mathrm{CH}^k(Y)_{\mathrm{alg}} \to \mathbf{Ab}_Y^k(\mathbf{k})$$

that is initial is called an *algebraic representative* for codimension k cycles in Y.

4.7.17. Theorem. — *Let Y be a smooth projective variety of dimension d over $\mathbf{k}$. Then an algebraic representative for codimension k cycles exists when*

(i) $k = d$, *and* $\mathbf{Ab}_Y^0 = \mathbf{Alb}_Y$,
(ii) $k = 1$, *and* $\mathbf{Ab}_Y^1 = \mathbf{Pic}_{Y,\mathrm{red}}^0$, *and*
(iii) $k = 2$, *and* $2 \dim \mathbf{Ab}_Y^2 \leq \dim_{\mathbf{Q}_\ell} \mathrm{H}^3_{\mathrm{\acute{e}t}}(Y, \mathbf{Q}_\ell)$.

Proof. — For (i) and (ii), see [Mur85, §1.4]. For (iii), see [Mur83] or [Mur85, Theorem 1.9], along with a correction by [Kah21]. ■

Returning to the situation of a smooth q-bic threefold X, its algebraic representative $\mathbf{Ab}_X^2$ in codimension 2 from 4.7.17(iii) is referred to as its *intermediate Jacobian*. The next results relate the intermediate Jacobian of X with the Albanese variety of S:

4.7.18. Lemma. — *There exists a commutative diagram of abelian groups*

$$\begin{tikzcd} \mathrm{CH}^2(S)_{\mathrm{alg}} \arrow[r, "\mathbf{L}_*"'] \arrow[d, "\phi_S^2"] & \mathrm{CH}^2(X)_{\mathrm{alg}} \arrow[r, "\mathbf{L}^*"'] \arrow[d, "\phi_X^2"] & \mathrm{CH}^1(S)_{\mathrm{alg}} \arrow[d, "\phi_S^1"] \\ \mathbf{Alb}_S(\mathbf{k}) \arrow[r] & \mathbf{Ab}_X^2(\mathbf{k}) \arrow[r] & \mathbf{Pic}_S^0(\mathbf{k}) \end{tikzcd}$$

and hence morphisms of abelian varieties

$$\mathbf{Alb}_S \xrightarrow{\mathbf{L}_*} \mathbf{Ab}_X^2 \xrightarrow{\mathbf{L}^*} \mathbf{Pic}_{S,\mathrm{red}}^0.$$

Proof. — The action of the Fano correspondence gives the top row of maps, see 2.8.5. The vertical maps are the universal regular homomorphisms recognizing the Albanese variety of S, the intermediate Jacobian of X, and the Picard scheme of S as, respectively, the algebraic representatives for algebraically trivial 0-cycles of S, 1-cycles of X, and 1-cycles of S; see 4.7.17. The morphisms of the group schemes arise from the corresponding universal property of each scheme. ■

Fix a Hermitian line $\ell_0 \subset X$ and consider the Albanese morphism $\mathrm{alb}_S \colon S \to \mathbf{Alb}_S$ centred at $[\ell_0]$. Composing this with the morphisms of abelian schemes from 4.7.18 yields a morphism $S \to \mathbf{Pic}_S^0$. Its action on $\mathbf{k}$-points is easily understood:

4.7.19. Lemma. — *The morphism $S \to \mathbf{Pic}_S^0$ acts on $\mathbf{k}$-points by*

$$[\ell] \mapsto \mathcal{O}_S(C_\ell - C_{\ell_0}).$$

Proof. — The Albanese morphism on $\mathbf{k}$-points factorizes as

$$\phi_S^2 \circ \mathrm{alb}_S(\mathbf{k}) \colon S(\mathbf{k}) \to \mathrm{CH}^2(S)_{\mathrm{alg}} \to \mathbf{Alb}_S(\mathbf{k})$$

where the first map is $[\ell] \mapsto [\ell] - [\ell_0]$, and the second map is the universal regular homomorphism from 4.7.17(i). The commutative diagram of 4.7.18 then shows that $S \to \mathbf{Pic}_S^0$ factors through the map $S(\mathbf{k}) \to \mathrm{CH}^1(S)_{\mathrm{alg}}$ given by

$$[\ell] \mapsto \mathbf{L}^*\mathbf{L}_*([\ell] - [\ell_0]) = [C_\ell] - [C_{\ell_0}],$$

where the action of the correspondence is as in 2.8.5 and 2.8.8. Composing with the map $\phi_S^1 \colon \mathrm{CH}^1(S)_{\mathrm{alg}} \to \mathbf{Pic}_S^0(\mathbf{k})$ gives the result. ■

Let C_{ℓ_0} be the divisor in S paramterizing lines in X incident with the fixed Hermitian line ℓ_0, as in 4.7.10. By 4.7.11, its normalization $C \to C_{\ell_0}$ is given by

$$C = \coprod\nolimits_{x \in \ell_0 \cap X_{\mathrm{Herm}}} C_x,$$

the 1-dimensional scheme obtained as the disjoint union over the smooth q-bic curves C_x parameterizing lines through the Hermitian points x of X contained in ℓ_0, and the morphism to C_{ℓ_0} is the disjoint union of the canonical inclusions. Let $\nu\colon C \to C_{\ell_0} \hookrightarrow S$ be the composite of the normalization morphism $C \to C_{\ell_0}$ with the closed immersion $C_{\ell_0} \hookrightarrow S$. Let

$$\mathbf{Jac}_C := \prod\nolimits_{x \in \ell_0 \cap X_{\mathrm{Herm}}} \mathbf{Jac}_{C_x}$$

denote the product of the Jacobian varieties of the connected components of C. On the one hand, viewing the Jacobian as the Picard variety of gives a map $\nu^*\colon \mathbf{Pic}^0_S \to \mathbf{Jac}_C$. On the other hand, viewing $[\ell_0] \in C_x$ as a base point for each connected component of C and taking the product of the corresponding Albanese morphisms gives a map $\mathrm{alb}_C\colon C \to \mathbf{Jac}_C$. The universal property of the Albanese then gives a commutative diagram

$$\begin{tikzcd} C \arrow[r, "\nu"'] \arrow[d, "\mathrm{alb}_C"'] & S \arrow[d, "\mathrm{alb}_S"] \\ \mathbf{Jac}_C \arrow[r, "\nu_*"] & \mathbf{Alb}_S \end{tikzcd}$$

of morphisms of schemes over $\mathbf{k}$. Composing these morphisms to and from $\mathbf{Jac}_C$ with those of 4.7.18 gives an endomorphism of $\mathbf{Jac}_C$, and it is determined as follows:

4.7.20. Proposition. — *The composite morphism*

$$\Phi\colon \mathbf{Jac}_C \xrightarrow{\nu_*} \mathbf{Alb}_S \xrightarrow{\mathbf{L}_*} \mathbf{Ab}^2_X \xrightarrow{\mathbf{L}^*} \mathbf{Pic}^0_{S,\mathrm{red}} \xrightarrow{\nu^*} \mathbf{Jac}_C$$

is given by multiplcation by q^2.

Proof. — Consider a $\mathbf{k}$-point of a connected component C_x of C and identify it with a $\mathbf{k}$-point $[\ell]$ of its image in S. Then by 4.7.19, its image under $\Phi \circ \mathrm{alb}_C\colon C \to \mathbf{Jac}_C$ is

$$\Phi(\mathrm{alb}_C([\ell])) = \nu^*\mathcal{O}_S(C_\ell - C_{\ell_0}).$$

Consider the restriction of $\mathcal{O}_S(C_\ell - C_{\ell_0})$ to each component C_y of C, where $y \in \ell_0 \cap X_{\mathrm{Herm}}$. Being a point of C_x, the line ℓ contains the Hermitian point x. So by 4.7.14, there is some effective divisor C'_ℓ such that $C_\ell = C'_\ell + C_x$ and,

$$[C'_\ell]\cdot[C_y] = \begin{cases} q^2[\ell] & \text{if } x = y, \text{ and} \\ 0 & \text{if } x \neq y, \end{cases}$$

in $\mathrm{CH}_0(S)$. Thus writing

$$C_\ell - C_{\ell_0} = C'_\ell - \sum\nolimits_{z \in \ell_0 \cap X_{\mathrm{Herm}} \setminus \{x\}} C_z$$

it follows from 4.7.12 and 4.7.13 that

$$\mathcal{O}_S(C_\ell - C_{\ell_0})|_{C_y} \cong \begin{cases} \mathcal{O}_{C_x}(q^2([\ell]-[\ell_0])) & \text{if } x = y, \text{ and} \\ \mathcal{O}_{C_y} & \text{if } x \neq y. \end{cases}$$

Therefore $\Phi(\mathrm{alb}_C([\ell])) = q^2 \mathrm{alb}_C([\ell])$. Since the image of C under alb_C generates $\mathbf{Jac}_C$, Φ is given by multiplication by q^2 on all of $\mathbf{Jac}_C$. ■

Putting everything together shows that each of the abelian varieties in question are related to one another via purely inseparable isogenies:

4.7.21. Theorem. — *Each of the morphisms of abelian varieties*

$$\nu_* \colon \mathbf{Jac}_C \to \mathbf{Alb}_S, \quad \mathbf{L}_* \colon \mathbf{Alb}_S \to \mathbf{Ab}^2_X, \quad \mathbf{L}^* \colon \mathbf{Ab}^2_X \to \mathbf{Pic}^0_{S,\mathrm{red}}, \quad \nu^* \colon \mathbf{Pic}^0_{S,\mathrm{red}} \to \mathbf{Jac}_C$$

is a purely inseparable p-power isogeny.

Proof. — The map Φ in 4.7.20 is multiplication by q^2, so its kernel is finite. This implies that the image of the partial composites from $\mathbf{Jac}_C$ to each of $\mathbf{Alb}_S$, $\mathbf{Ab}^2_X$, and $\mathbf{Pic}^0_{S,\mathrm{red}}$ are abelian subvarieties of dimension

$$\dim \mathbf{Jac}_C = q(q-1)(q^2+1)/2.$$

By 4.7.17(iii) together with 4.7.1(ii),

$$2\dim \mathbf{Ab}^2_X \leq \dim_{\mathbf{Q}_\ell} \mathrm{H}^3_{\text{ét}}(X, \mathbf{Q}_\ell) = q(q-1)(q^2+1)$$

so equality holds throughout. Since the Albanese and Picard varieties have dimension $\dim_{\mathbf{Q}_\ell} \mathrm{H}^1_{\text{ét}}(S, \mathbf{Q}_\ell)$, comparing with 4.7.7 shows that that each of $\mathbf{Ab}^2_X$, $\mathbf{Alb}_S$, $\mathbf{Pic}^0_{S,\mathrm{red}}$, and $\mathbf{Jac}_C$ are abelian varieties of dimension $q(q-1)(q^2+1)/2$. It now follows that all the maps factoring Φ are p-power isogenies.

It remains to see that all the maps occurring are purely inseparable. For this, recall from 2.6.5 that smooth q-bic curves are supersingular. Thus $\mathbf{Jac}_C$, being the product of Jacobians of smooth q-bic curves, is itself supersingular. Whence multiplication by q^2 is purely inseparable. The result now follows from the factorization 4.7.20. ■

That $\mathbf{L}_* \colon \mathbf{Alb}_S \to \mathbf{Ab}^2_X$ is purely inseparable is analogous to the fact that for a cubic threefold, its intermediate Jacobian is isomorphic to the Albanese of its surface of lines, see [CG72, Theorem 11.19]. The abelian variety $\mathbf{Jac}_C$ is a degenerate analogue of a Prym variety for the covering $C \to D$, where D is the discriminant curve to projection of X from ℓ_0, see 2.5.7 and 2.5.6(i). Thus the statement that $\nu_* \colon \mathbf{Jac}_C \to \mathbf{Alb}_S$ is an analogue of Mumford's identification between the Albanese of the Fano surface for a cubic with a Prym variety, see [CG72, Appendix C]. The statement regarding $\nu^* \circ \mathbf{L}^* \colon \mathbf{Ab}^2_X \to \mathbf{Jac}_C$ is an analogue of Murre's identification between the group of algebraically trivial 1-cycles on a cubic and the Prym, see [Mur72, Theorem 10.8].

4.7.22. Coherent cohomology of the Fano scheme. — The next goal is to compute the cohomology of $\mathscr{O}_S$, at least when $q = p$ is prime: see 4.7.27 and 4.7.28 for the result. The computation proceeds in three steps:

First, carefully degenerate S to the singular surface S_0 of lines in type $\mathbf{N}_2 \oplus \mathbf{1}^{\oplus 3}$ and relate the cohomology of S with that of S_0: see 4.7.23 and 4.7.24.

Second, explicitly show that certain cohomology classes in $\mathrm{H}^1(S_0, \mathscr{O}_{S_0})$ do not lift to classes in $\mathrm{H}^1(S, \mathscr{O}_S)$: see 4.7.25.

Third, apply uppersemicontinuity of cohomology and use the computation of $\mathrm{H}^1(S_0, \mathscr{O}_{S_0})$ from 4.6.16 to conclude.

4.7.23. — Choose cone points $x_-, x_+ \in X$ such that $\langle x_-, x_+ \rangle \not\subset X$ and let $\pi \colon \mathscr{X} \to \mathbf{A}^1$ be the associated family of q-bic threefolds in $\mathbf{P}V \times \mathbf{A}^1$ as constructed in 4.3.12, so that π is smooth away from $0 \in \mathbf{A}^1$ and $X_0 := \pi^{-1}(0)$ is of type $\mathbf{N}_2 \oplus \mathbf{1}^{\oplus 3}$. Let $\mathscr{S} \to \mathbf{A}^1$ be the relative Fano scheme of lines of π, and let

$$\begin{tikzcd} & \tilde{\mathscr{S}} \arrow[dl] \arrow[dr] & \\ \mathscr{S} \arrow[dr] & & \mathscr{T} \arrow[dl] \\ & C_{\mathbf{A}^1} & \end{tikzcd}$$

be the diagram resulting from the family of Smooth Cone Situations $(\mathscr{X}, x_-)$ as in 4.3.15 and 4.3.16. Let $\varphi\colon S \to C$ and $\varphi_0\colon S_0 \to C$ be the fibres of the morphism $\mathscr{S} \to C_{\mathbf{A}^1}$ over the points 1 and 0 of $\mathbf{A}^1$, respectively. Then the sheaves $\mathbf{R}^1\varphi_*\mathscr{O}_S$ and $\mathbf{R}^1\varphi_{0,*}\mathscr{O}_{S_0}$ are locally free $\mathscr{O}_C$-modules which carry a q-step filtration by 4.3.10(iii); they also admit gradings by $\mathbf{Z}/(q^2-1)\mathbf{Z}$ and $\mathbf{Z}$, respectively, see 4.3.17. Moreover, by the proof of 4.4.25, $\mathbf{R}^1\varphi_{0,*}\mathscr{O}_{S_0}$ consists of the positively graded parts of the sheaf $\mathscr{F}$ associated with φ_0 as in 4.4.24.

With this notation, the following refines the relationship between $\mathbf{R}^1\varphi_*\mathscr{O}_S$ and $\mathbf{R}^1\varphi_{0,*}\mathscr{O}_{S_0}$ given in 4.3.17:

4.7.24. Lemma. — *In the setting of 4.7.23, there are short exact sequences of filtered* $\mathrm{U}_3(q)$*-equivariant locally free* $\mathscr{O}_C$*-modules*

$$0 \to \mathscr{F}_\alpha \to (\mathbf{R}^1\varphi_*\mathscr{O}_S)_\alpha \to \mathscr{F}_{\alpha+q^2-1} \to 0$$

for each $\alpha = 1, 2, \ldots, q^2-1$.

Proof. — By 4.3.17, the sheaves $(\mathbf{R}^1\varphi_*\mathscr{O}_{\mathscr{S}})_\alpha$ carry a second filtration $\mathrm{Fil}_\bullet$ such that

$$\mathrm{gr}_i^{\mathrm{Fil}}(\mathbf{R}^1\varphi_*\mathscr{O}_S)_\alpha \cong (\mathbf{R}^1\varphi_{0,*}\mathscr{O}_{S_0})_{\alpha+i(q^2-1)} \quad \text{for each } i \in \mathbf{Z}$$

as graded $\mathscr{O}_C$-modules. Identify $\mathbf{R}^1\varphi_{0,*}\mathscr{O}_{S_0}$ with the positively graded components of $\mathscr{F}$ via the proof of 4.4.25. Since, by 4.5.25, the weights appearing in $\mathscr{F}_{>0}$ lie in $[1, 2q^2-q-2]$, so the filtration above has at most 2 steps, yielding the desired short exact sequences; they are equivariant for $\mathrm{U}_3(q)$ by 4.3.16(iv). ∎

Consider the indices $\alpha = iq$ with $1 \leq i \leq q-2$. In this case, 4.5.25 combined with 4.5.21 identifies the quotient bundle as

$$\mathscr{F}_{q^2+iq-1} \cong \mathrm{Div}^{q-2-i}(W) \otimes \mathscr{O}_C.$$

When $q = p$, this makes it easy to show its corresponding exact sequence is not split:

4.7.25. Lemma. — *If $q = p$, then for each $1 \leq i \leq p-2$, the sequence*

$$0 \to \mathscr{F}_{ip} \to (\mathbf{R}^1\varphi_*\mathscr{O}_S)_{ip} \to \mathrm{Div}^{p-2-i}(W) \otimes \mathscr{O}_C \to 0$$

is not split and the induced map on global sections yields an isomorphism

$$\mathrm{H}^0(C, \mathscr{F}_{ip}) \cong \mathrm{H}^0(C, (\mathbf{R}^1\varphi_*\mathscr{O}_S)_{ip}).$$

Proof. — By 4.7.24, the sequence is of $\mathrm{U}_3(q)$-equivariant filtered $\mathscr{O}_C$-modules, in which the filtration is as from 4.2.27(iii). The result will follow upon showing

(i) $\mathrm{gr}_{p-1}(\mathrm{Div}^{p-2-i}(W) \otimes \mathscr{O}_C) \neq 0$, and

(ii) $\mathrm{H}^0(C, \mathrm{gr}_{p-1}(\mathbf{R}^1\varphi_*\mathscr{O}_S)_{ip}) = 0$.

Indeed, there is a commutative square

$$\begin{tikzcd}
(\mathbf{R}^1\varphi_*\mathscr{O}_S)_{ip} \arrow[r] \arrow[d] & \mathrm{Div}^{p-2-i}(W) \otimes \mathscr{O}_C \arrow[d] \\
\mathrm{gr}_{p-1}(\mathbf{R}^1\varphi_*\mathscr{O}_S)_{ip} \arrow[r] & \mathrm{gr}_{p-1}(\mathrm{Div}^{p-2-i}(W) \otimes \mathscr{O}_C)
\end{tikzcd}$$

in which the vertical maps are the quotient maps. The nonvanishing from (i) means that the right map is nonzero, so there a nonzero global section of the form

$$s \colon \mathscr{O}_C \hookrightarrow \mathrm{Div}^{p-2-i}(W) \otimes \mathscr{O}_C \twoheadrightarrow \mathrm{gr}_{p-1}(\mathrm{Div}^{p-2-i}(W) \otimes \mathscr{O}_C).$$

If the sequence of the statement were split, then s would lift to a global section of $\mathrm{gr}_{p-1}(\mathbf{R}^1\varphi_*\mathscr{O}_S)_{ip}$; this cannot happen given (ii). Thus given (i) and (ii), the sequence of the statement is not split, and so the boundary map

$$\delta \colon \mathrm{H}^0(C, \mathrm{Div}^{p-2-i}(W) \otimes \mathscr{O}_C) \to \mathrm{H}^1(C, \mathscr{F}_{ip})$$

is nonzero. But δ is $\mathrm{U}_3(q)$-equivariant by 4.7.24 and $\mathrm{Div}^{p-2-i}(W)$ is simple by B.3.3(i). Therefore δ is injective, giving the second statement of the Lemma.

It remains to verify (i) and (ii). The nonvanishing of (i) follows from 4.5.25:

$$\begin{aligned}\mathrm{gr}_{p-1}(\mathrm{Div}^{p-2-i}(W)\otimes\mathcal{O}_C) &\cong \mathrm{gr}_{p-1}(\mathcal{F}_{p^2+ip-1}) \\ &\cong \mathrm{gr}_0(\mathcal{D}_{(p-i-2)p+p-1})^\vee\otimes\mathcal{O}_C(-q+1)\neq 0.\end{aligned}$$

The vanishing (ii) follows from the identification of $\mathrm{gr}_{p-1}(\mathbf{R}^1\varphi_*\mathcal{O}_S)_{ip}$ from 4.7.26 below together with the Borel–Weil–Bott computation of 4.6.2. ■

The following identifies the final graded pieces of the weight iq component of $\mathbf{R}^1\varphi_*\mathcal{O}_S$ with respect to the filtration of 4.3.10(iii):

4.7.26. Lemma. — *For each $1\leq i\leq q-2$, there is an isomorphism of $\mathcal{O}_C$-modules*

$$\mathrm{gr}_{q-1}(\mathbf{R}^1\varphi_*\mathcal{O}_S)_{iq}\cong\mathrm{Div}^{2q-2-i}(\mathcal{T}_{\mathbf{P}W}(-1)|_C)(-1).$$

Proof. — Taking $i=q-1$ in 4.3.10(iii) shows

$$\mathrm{gr}_{q-1}(\mathbf{R}^1\varphi_*\mathcal{O}_S)=\mathbf{R}^1\pi_*(\mathrm{gr}_{q-1}(\rho_*\mathcal{O}_{\tilde{S}}))=\mathbf{R}^1\pi_*(\mathcal{O}_T(1,-q)\otimes\pi^*\mathcal{O}_C(-1)\otimes L_-).$$

By 4.2.15(iii), $\mathcal{O}_T(1,-q)$ is resolved by a complex $[\mathcal{E}^{-2}\to\mathcal{E}^{-1}]$ of $\mathcal{O}_{\mathbf{P}}$-modules with

$$\mathcal{E}^{-2}=\mathcal{O}_{\mathbf{P}}(-q+1,-2q-1)\otimes\pi^*\mathcal{O}_C(-1)\oplus\mathcal{O}_{\mathbf{P}}(-q,-2q)\otimes L_+,\text{ and}$$
$$\mathcal{E}^{-1}=\mathcal{O}_{\mathbf{P}}(0,-2q)\oplus\mathcal{O}_{\mathbf{P}}(-q+1,-q-1)\oplus\mathcal{O}_{\mathbf{P}}(-q+1,-2q)\otimes\pi^*\mathcal{O}_C(-1)\otimes L_+.$$

Recall from 4.2.10 that $\mathbf{P}=\mathbf{P}\mathcal{V}_1\times_C\mathbf{P}\mathcal{V}_2$ with $\mathcal{V}_1\cong L_{-,C}\oplus\mathcal{O}_C(-1)$ and $\mathcal{V}_2\cong\mathcal{T}_{\mathbf{P}W}(-1)|_C\oplus L_{+,C}$. The relative dualizing sheaves of the individual factors are

$$\omega_{\mathbf{P}\mathcal{V}_1/C}\cong\mathcal{O}_{\pi_1}(-2)\otimes\pi_1^*\mathcal{O}_C(1)\otimes L_-^\vee\quad\text{and}\quad\omega_{\mathbf{P}\mathcal{V}_2/C}\cong\mathcal{O}_{\pi_2}(-3)\otimes\pi_2^*\mathcal{O}_C(-1)\otimes L_+^\vee$$

so $\omega_{\mathbf{P}/C}\cong\mathcal{O}_{\mathbf{P}}(-2,-3)\otimes L_-^\vee\otimes L_+^\vee$. The resolution provides a spectral sequence computing $\mathbf{R}^1\pi_*\mathcal{O}_T(1,-q)$ with E_1 page given by

$$\begin{matrix} E_1^{-2,3}\xrightarrow{d_1}E_1^{-1,3}\xrightarrow{d_1}E_1^{0,3} \\ E_1^{-2,2}\xrightarrow{d_1}E_1^{-1,2}\xrightarrow{d_1}E_1^{0,2}\end{matrix}\;=\;\begin{matrix}\mathbf{R}^3\pi_*\mathcal{E}^{-2}\xrightarrow{\phi}\mathbf{R}^3\pi_*\mathcal{E}^{-1} & 0 \\ 0\qquad\mathbf{R}^2\pi_*\mathcal{E}^{-1} & \xrightarrow{\wedge^2\phi^\vee}\mathbf{R}^2\pi_*\mathcal{O}_{\mathbf{P}}(1,-q)\end{matrix}$$

and with all other terms vanishing. Observe that since $\boldsymbol{\mu}_{q^2-1}$ acts through linear automorphisms of $\mathbf{P}$ over C, the differentials of the spectral sequence are compatible with the induced $\mathbf{Z}/(q^2-1)\mathbf{Z}$ gradings on each term.

Let $1 \leq i \leq q-2$. To identify the weight component

$$(\mathbf{R}^1\varphi_*\mathcal{O}_S)_{iq} \cong (\mathbf{R}^1\pi_*(\mathcal{O}_T(1,-q)\otimes \pi^*\mathcal{O}_C(-1)\otimes L_-))_{iq},$$

consider the corresponding weight component of the spectral sequence. Since the weights of L_- and L_+ are -1 and q, respectively, a direct computation shows that

$$\begin{aligned}
(\mathbf{R}^3\pi_*(\mathcal{E}^{-2}\otimes \pi^*\mathcal{O}_C(-1)\otimes L_-))_{iq} &\cong \mathrm{Div}^{2q-2-i}(\mathcal{T}_{\mathbf{P}W}(-1)|_C)(-1)\otimes L_-^{\otimes q}\otimes L_+^{\otimes i+1},\\
(\mathbf{R}^3\pi_*(\mathcal{E}^{-1}\otimes \pi^*\mathcal{O}_C(-1)\otimes L_-))_{iq} &\cong 0,\\
(\mathbf{R}^2\pi_*(\mathcal{E}^{-1}\otimes \pi^*\mathcal{O}_C(-1)\otimes L_-))_{iq} &\cong \mathrm{Div}^{q-2-i}(\mathcal{T}_{\mathbf{P}W}(-1)|_C)\otimes L_-\otimes L_+^{\otimes q+i},\\
(\mathbf{R}^2\pi_*(\mathcal{O}_{\mathbf{P}}(1,-q)\otimes \pi^*\mathcal{O}_C(-1)\otimes L_-))_{iq} &\cong \mathrm{Div}^{q-2-i}(\mathcal{T}_{\mathbf{P}W}(-1)|_C)\otimes L_+^{\otimes i}.
\end{aligned}$$

The differential $\wedge^2\phi^\vee$ between the latter two sheaves is given by $u_1v'_{21}+u_2v'_{22}$. By the computations of the components of v' from **4.2.18**, v'_{21} involves a q-power of the Euler section of $\mathbf{P}\mathcal{V}_2$. Since the divided power of $\mathcal{T}_{\mathbf{P}W}(-1)|_C$ is always strictly less than q, this acts by zero. The remaining component $u_2v'_{22}$ then acts as the isomorphism which is identity on $\mathrm{Div}^{q-2-i}(\mathcal{T}_{\mathbf{P}W}(-1)|_C)$ and the isomorphism $L_-\cong L_+^{\vee,\otimes q}$ provided by β. This shows that, at least for computing the weight iq component of $\mathbf{R}^2\varphi_*\mathcal{O}_S$, the spectral sequence degenerates on this page and that

$$(\mathbf{R}^2\varphi_*\mathcal{O}_S)_{iq} \cong (\mathbf{R}^3\pi_*(\mathcal{E}^{-2}\otimes \pi^*\mathcal{O}_C(-1)\otimes L_-))_{iq} \cong \mathrm{Div}^{2q-2-i}(\mathcal{T}_{\mathbf{P}W}(-1)|_C)(-1)$$

for each $1 \leq i \leq q-2$. ■

Putting everything together yields a computation of the cohomology of $\mathcal{O}_S$:

4.7.27. Theorem. — *Let X be a smooth q-bic threefold and let S be its Fano surface of lines. If $p=q$, then*

$$\dim_{\mathbf{k}} \mathrm{H}^1(S,\mathcal{O}_S) = \frac{1}{2}\dim_{\mathbf{Q}_\ell}\mathrm{H}^1_{\text{ét}}(S,\mathbf{Q}_\ell) = \frac{1}{2}p(p-1)(p^2+1).$$

In particular, the Picard scheme $\mathbf{Pic}_S$ of S is smooth.

Proof. — Since $\mathrm{H}^1(S,\mathcal{O}_S)$ is canonically the tangent space to $\mathbf{Pic}_S$ at the identity, and, for any prime $\ell \neq p$, $\mathrm{H}^1_{\text{ét}}(S,\mathbf{Z}_\ell)$ is the ℓ-adic Tate module of $\mathbf{Pic}_S$, there is always an inequality

$$\dim_{\mathbf{k}} \mathrm{H}^1(S,\mathcal{O}_S) = \dim_{\mathbf{k}} \mathcal{T}_{\mathbf{Pic}_S,[\mathcal{O}_S]} \geq \dim \mathbf{Pic}_S = \frac{1}{2}\operatorname{rank}_{\mathbf{Z}_\ell} \mathrm{T}_\ell \mathbf{Pic}_S = \frac{1}{2}\dim_{\mathbf{Q}_\ell} \mathrm{H}^1_{\text{ét}}(S,\mathbf{Q}_\ell).$$

By the étale cohomology computation for S in 4.7.7, the dimension of $\mathrm{H}^1(S,\mathcal{O}_S)$ is always at least $q(q-1)(q^2+1)/2$ with no assumption on q.

Assume $q=p$ is prime. The corresponding upper bound follows by semicontinuity of cohomology, see [Stacks, 0BDN], for the flat family $\mathcal{S} \to \mathbf{A}^1$ produced in 4.3.16, the cohomology computation for the singular surface S_0 in 4.6.14, and the non-splitting result of 4.7.25. In more detail, and slightly more directly, the Leray spectral sequence for $\varphi\colon S \to C$ yields a short exact sequence

$$0 \to \mathrm{H}^1(C,\varphi_*\mathcal{O}_S) \to \mathrm{H}^1(S,\mathcal{O}_S) \to \mathrm{H}^0(C,\mathbf{R}^1\varphi_*\mathcal{O}_S) \to 0.$$

By 4.3.10(ii), $\varphi_*\mathcal{O}_S = \mathcal{O}_C$ so the first term has dimension $p(p-1)/2$. For the second term, consider the $\mathbf{Z}/(p^2-1)\mathbf{Z}$ weight decomposition induced by the action of $\boldsymbol{\mu}_{p^2-1}$. Taking global sections of the short exact sequences from 4.7.24 yields inequalities

$$\dim_{\mathbf{k}} \mathrm{H}^0(C,(\mathbf{R}^1\varphi_*\mathcal{O}_S)_\alpha) \leq \dim_{\mathbf{k}} \mathrm{H}^0(C,\mathcal{F}_\alpha) + \dim_{\mathbf{k}} \mathrm{H}^0(C,\mathcal{F}_{\alpha+p^2-1})$$

for each $\alpha = 1,2,\ldots,q^2-1$. When $\alpha = ip$ with $1 \leq i \leq p-2$, 4.7.25 refines this to an equality

$$\dim_{\mathbf{k}} \mathrm{H}^0(C,(\mathbf{R}^1\varphi_*\mathcal{O}_S)_\alpha) = \dim_{\mathbf{k}} \mathrm{H}^0(C,\mathcal{F}_{ip}).$$

Summing the inequalities over α gives the inequality

$$\dim_{\mathbf{k}} \mathrm{H}^0(C,\mathbf{R}^1\varphi_*\mathcal{O}_S) \leq \dim_{\mathbf{k}} \mathrm{H}^0(C,\mathcal{F}) - \dim_{\mathbf{k}} \mathrm{H}^0(C,\mathcal{F}_0) - \sum_{i=1}^{p-2} \dim_{\mathbf{k}} \mathrm{H}^0(C,\mathcal{F}_{p^2+ip-1}).$$

By 4.5.25 and 4.5.21, $\mathcal{F}_{p^2+ip-1} \cong \operatorname{Div}^{p-2-i}(W) \otimes \mathcal{O}_C$. By 4.4.25, $\mathcal{F}_0$ is isomorphic to the cokernel the map $\mathcal{O}_C \to \mathrm{Fr}^2_*\mathcal{O}_C$ induced by the q^2-power Frobenius morphism. Since the q-power Frobenius already acts by zero on $\mathrm{H}^1(C,\mathcal{O}_C)$, see 2.6.2, the long exact sequence in cohomology shows

$$\mathrm{H}^0(C,\mathcal{F}_0) \cong \mathrm{H}^1(C,\mathcal{O}_C) \cong \operatorname{Div}^{p-2}(W).$$

Therefore the negative terms in the inequality sum up to

$$\begin{aligned}\dim_{\mathbf{k}} \mathrm{H}^0(C,\mathscr{F}_0) + \sum\nolimits_{i=1}^{p-2} \dim_{\mathbf{k}} \mathrm{H}^0(C,\mathscr{F}_{p^2+ip-1}) &= \sum\nolimits_{i=0}^{p-2} \dim_{\mathbf{k}} \mathrm{Div}^i(W) \\ &= \sum\nolimits_{i=0}^{p-2} \binom{i+2}{2} \\ &= \binom{p+1}{3} = \binom{p}{2} + \binom{p}{3}.\end{aligned}$$

By 4.6.14, $\mathrm{H}^0(C,\mathscr{F})$ has dimension $(p^2+1)\binom{p}{2}+\binom{p}{3}$, so

$$\dim_{\mathbf{k}} \mathrm{H}^0(C,\mathbf{R}^1\varphi_*\mathscr{O}_S) \leq p^2\binom{p}{2}.$$

the short exact sequence for $\mathrm{H}^1(S,\mathscr{O}_S)$ then gives

$$\dim_{\mathbf{k}} \mathrm{H}^1(S,\mathscr{O}_S) \leq \binom{p}{2} + p^2\binom{p}{2} = \frac{1}{2}p(p-1)(p^2+1)$$

and this proves the result. ∎

4.7.28. Corollary. — *Let X be a smooth q-bic threefold and let S be its Fano surface of lines. If $p = q$, then*

$$\dim_{\mathbf{k}} \mathrm{H}^i(S,\mathscr{O}_S) = \begin{cases} 1 & \text{if } i = 0, \\ \frac{1}{2}p(p-1)(p^2+1) & \text{if } i = 1, \text{ and} \\ \frac{1}{12}p(p-1)(5p^4-2p^2-5p-2) & \text{if } i = 2. \end{cases}$$

Proof. — The first number follows from smoothness together with irreducibility of S, see 4.1.1. The second number is 4.7.27. The third number is now deduced from the Euler characteristic computation 4.1.3. ∎

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Appendix A

Generalities on Linear Projective Geometry

Some of the constructions in Chapters 2 and 4 involve fine properties of linear projection and various attendant structures. This Appendix collects these projective constructions in slightly more generality than needed. Section A.1 begins with some conventions on Grassmannian bundles. Section A.2 describes a sort of functoriality amongst such bundles. In geometric terms, this Section presents a resolution of the rational maps induced by linear projection and intersection by a linear space: see A.2.4 and A.2.9. These situations are combined in Section A.3, wherein the Subquotient Situation is introduced and the induced rational maps are resolved: see A.3.7. Finally, Section A.4 works out a relationship between extensions and projective bundles; this is crucial to the computations in 4.5.

A.1. Projective and Grassmannian bundles

This Section collects some conventions in projective and Grassmannian bundles over a base scheme B. Definitions are collected in A.1.1–A.1.3. Duality is described in A.1.4. A construction of certain rational maps to Grassmannians and a method of resolution is given in A.1.5–A.1.7. Affine subbundles to projective bundles and the behaviour of tautological structures are discussed in A.1.8–A.1.13.

A.1.1. Subbundles and quotients. — Let $\mathcal{V}$ be a finite locally free $\mathcal{O}_B$-module, alternatively referred to as a *vector bundle* over B. Given a field $\mathbf{k}$ and a morphism $x \colon \operatorname{Spec}(\mathbf{k}) \to B$, the *fibre* of $\mathcal{V}$ is the pullback $\mathcal{V}_x := x^*\mathcal{V} = \mathcal{V} \otimes_{\mathcal{O}_B} \mathbf{k}$. An injective morphism of vector bundles $\mathscr{V}' \hookrightarrow \mathscr{V}$ is called *locally split* if it admits, Zariski locally, a retraction; this is equivalent to asking for the morphism to be injective on each fibre, or for the dual morphism to be surjective. In particular, the cokernel of a locally split injection is also locally free. In the special case that $\mathscr{V}' \subset \mathscr{V}$ is the inclusion of a locally free subbmodule, $\mathscr{V}'$ is referred to as a *subbundle*.

A.1.2. Grassmannian of subbundles. — For any $\mathscr{O}_B$-module $\mathscr{F}$ and any morphism of schemes $T \to B$, write $\mathscr{F}_T$ for the $\mathscr{O}_T$-module obtained by pullback of $\mathscr{F}$. Now let $\mathscr{V}$ be a finite locally free $\mathscr{O}_B$-module, r a positive integer, and consider the functor

$$\begin{aligned} \mathbf{G}(r,\mathscr{V})\colon \mathrm{Sch}_B^{\mathrm{opp}} &\to \mathrm{Set} \\ T &\mapsto \left\{ \mathscr{V}' \subset \mathscr{V}_T \text{ subbundle of rank } r \right\}. \end{aligned}$$

Then $\mathbf{G}(r,\mathscr{V})$ is representable by a smooth projective scheme over B, see [**Stacks**, 089T]. Moreover, there is a tautological short exact sequence

$$0 \to \mathscr{S}_{\mathbf{G}(r,\mathscr{V})} \to \mathscr{V}_{\mathbf{G}(r,\mathscr{V})} \to \mathscr{Q}_{\mathbf{G}(r,\mathscr{V})} \to 0$$

where $\mathscr{S}_{\mathbf{G}(r,\mathscr{V})}$ is the universal subbundle of rank r and $\mathscr{Q}_{\mathbf{G}(r,\mathscr{V})}$ is the corresponding universal quotient bundle of corank r.

A.1.3. Grassmanian of quotients. — It will be sometimes convenient to think of the Grassmannian as parameterizing quotient bundles. To make this distinction clear, consider the following alternative functor: Let $\mathscr{V}$ be a finite locally free $\mathscr{O}_B$-module as above, let s be a positive integer, and let

$$\begin{aligned} \mathbf{G}(\mathscr{V},s)\colon \mathrm{Sch}_B^{\mathrm{opp}} &\to \mathrm{Set} \\ T &\mapsto \left\{ \mathscr{V}_T \twoheadrightarrow \mathscr{V}'' \text{ quotient bundle of rank } s \right\}. \end{aligned}$$

Since the data of a quotient bundle of rank s is equivalent to the data of a subbundle of rank $r = \mathrm{rank}_{\mathscr{O}_B}(\mathscr{V}) - s$, there is an isomorphism of functors $\mathbf{G}(\mathscr{V},s) \cong \mathbf{G}(r,\mathscr{V})$, whence the former is representable. Write

$$0 \to \mathscr{S}_{\mathbf{G}(\mathscr{V},s)} \to \mathscr{V}_{\mathbf{G}(\mathscr{V},s)} \to \mathscr{Q}_{\mathbf{G}(\mathscr{V},s)} \to 0$$

for the corresponding tautological sequence of universal bundles.

A.1.4. Duality of Grassmannians. — Since a morphism is a fibrewise injection if and only if its dual is a surjection, there are natural duality identificaitons

$$\mathbf{G}(r,\mathscr{V}) \cong \mathbf{G}(\mathscr{V}^\vee, r) \quad \text{and} \quad \mathbf{G}(\mathscr{V}, s) \cong \mathbf{G}(s, \mathscr{V}^\vee)$$

so that the universal bundles are identified as

$$\mathscr{S}^\vee_{\mathbf{G}(r,\mathscr{V})} \cong \mathscr{Q}_{\mathbf{G}(\mathscr{V}^\vee,r)} \quad \text{and} \quad \mathscr{Q}^\vee_{\mathbf{G}(\mathscr{V},s)} \cong \mathscr{S}_{\mathbf{G}(s,\mathscr{V}^\vee)}.$$

A.1.5. Rational maps to Grassmannians. — Consider the following two situations which give rational maps $[\varphi]\colon T \dashrightarrow \mathbf{G}$ over B to some Grassmannian $\mathbf{G}$ of $\mathscr{V}$:

(i) An injection $\varphi\colon \mathscr{E} \to \mathscr{V}_T$ from a bundle of rank r gives a map to $\mathbf{G}(r,\mathscr{V})$.

(ii) A generic surjection $\varphi\colon \mathscr{V}_T \to \mathscr{F}$ to a bundle of rank s gives a map to $\mathbf{G}(\mathscr{V},s)$.

Set $n := \mathrm{rank}_{\mathscr{O}_B}(\mathscr{V})$ and consider the scheme

$$D := \begin{cases} \mathrm{V}(\mathrm{Fitt}_{n-r}(\mathrm{coker}(\varphi\colon \mathscr{E} \to \mathscr{V}_T))) & \text{in situation (i), and} \\ \mathrm{V}(\mathrm{Fitt}_s(\mathrm{image}(\varphi\colon \mathscr{V}_T \to \mathscr{F}))) & \text{in situation (ii),} \end{cases}$$

defined by an appropriate Fitting ideal; this is the quasi-coherent ideal locally generated by the maximal minors of φ in the first case, and maximal minors of a locally free presentation of image(φ) in the second case; see [**Stacks**, 0C3C]. The properties of Fitting ideals imply that the cokernel of φ in the first case and the image of φ in the second is locally free away from D, hence rational map $[\varphi]$ restricts to a morphism $T \setminus D \to \mathbf{G}$.

A.1.6. — In fact, $[\varphi]$ can be resolved to a morphism on the blowup $\mathrm{bl}\colon \tilde{T} \to T$ along D. To give a precise description, recall that the *strict transform* of a quasi-coherent $\mathscr{O}_T$-module $\mathscr{G}$ along the blowup bl is the quotient of $\mathrm{bl}^*\mathscr{G}$ by the quasi-coherent submodule of sections supported on the exceptional divisor $\mathrm{bl}^{-1}(D)$, see the discussion at the beginning of [**Stacks**, 080C]. In the above situations: Let

(i) $\tilde{\mathscr{E}}$ be the kernel of the map from $\mathscr{V}_{\tilde{T}}$ to the strict transform of coker(φ), and let $\tilde{\varphi}\colon \tilde{\mathscr{E}} \hookrightarrow \mathscr{V}_{\tilde{T}}$ be the natural injection.

(ii) $\tilde{\mathscr{F}}$ be the strict transform of image(φ) and $\tilde{\varphi}\colon \mathscr{V}_{\tilde{T}} \twoheadrightarrow \tilde{\mathscr{F}}$ the natural surjection.

A.1.7. Lemma. — *There exists a morphism* $[\tilde{\varphi}] \colon \tilde{T} \to \mathbf{G}$ *resolving* $[\varphi]$ *which is characterized in the above situations as follows:*

(i) $[\tilde{\varphi}]^*(\mathscr{S}_{\mathbf{G}(r,\mathscr{V})} \to \mathscr{V}_{\mathbf{G}(r,\mathscr{V})}) = (\tilde{\varphi} : \tilde{\mathscr{E}} \to \mathscr{V}_{\tilde{T}})$, *and*

(ii) $[\tilde{\varphi}]^*(\mathscr{V}_{\mathbf{G}(\mathscr{V},s)} \to \mathscr{Q}_{\mathbf{G}(\mathscr{V},s)}) = (\tilde{\varphi} : \mathscr{V}_{\tilde{T}} \to \tilde{\mathscr{F}})$

Proof. — Set $\mathscr{G}$ to be $\operatorname{coker}(\varphi)$ in A.1.5(i) and $\operatorname{image}(\varphi)$ in A.1.5(ii). Then by the assumptions, $\mathscr{G}|_{T \setminus D}$ is locally free of rank $n-r$ and s, respectively. Thus [**Stacks**, 0CZQ] shows that the strict transform $\tilde{\mathscr{G}}$ of $\mathscr{G}$ along bl is locally free of the corresponding rank. The natural surjection $\mathscr{V}_{\tilde{T}} \to \mathrm{bl}^*\mathscr{G} \to \tilde{\mathscr{G}}$ induces a morphism from $\tilde{T}$ to $\mathbf{G}(\mathscr{V}, n-r)$ in the first case, and $\mathbf{G}(\mathscr{V}, s)$ in the second case, such that $\mathscr{V}_{\tilde{T}} \to \tilde{\mathscr{G}}$ is the pullback of the universal quotient. Now $\tilde{\mathscr{G}} = \tilde{\mathscr{F}}$ from A.1.6(ii), so this gives the conclusion in (ii). In the other case, $\tilde{\mathscr{E}} = \ker(\mathscr{V}_{\tilde{T}} \to \tilde{\mathscr{G}})$ from A.1.6(i), so the conclusion in (i) follows from the identification $\mathbf{G}(\mathscr{V}, n-r) \cong \mathbf{G}(r, \mathscr{V})$ of A.1.3. ∎

A.1.8. Affine subbundles. — Let $\pi \colon \mathbf{P}\mathscr{V} \to B$ be a projective bundle and suppose that there is a short exact sequence

$$0 \to \mathscr{U} \to \mathscr{V} \xrightarrow{\alpha} \mathscr{L} \to 0$$

where $\mathscr{L}$ is an invertible $\mathscr{O}_B$-module. Then $\mathbf{P}\mathscr{U} \subset \mathbf{P}\mathscr{V}$ is the hyperplane subbundle cut out by the morphism

$$u := \pi^*(\alpha) \circ \mathrm{eu}_\pi \colon \mathscr{O}_{\mathbf{P}\mathscr{V}}(-1) \to \pi^*\mathscr{V} \twoheadrightarrow \pi^*\mathscr{L}$$

obtained by composing the Euler section of π with the pullback of the quotient map α. Therefore its complement $\mathbf{P}\mathscr{V}^\circ := \mathbf{P}\mathscr{V} \setminus \mathbf{P}\mathscr{U}$ is an affine space bundle over B whose underlying algebra is identified as follows:

A.1.9. Lemma. — *There exists a canonical isomorphism of* $\mathscr{O}_B$*-algebras*

$$\pi_*\mathscr{O}_{\mathbf{P}\mathscr{V}^\circ} \cong \operatorname{colim}_n \operatorname{Sym}^n(\mathscr{V}^\vee \otimes \mathscr{L})$$

in which the transition maps in the colimit induced by

$$\begin{aligned}\pi_*(u^\vee)\colon \operatorname{Sym}^n(\mathscr{V}^\vee)\otimes\mathscr{L}^{\otimes n} &\xrightarrow{\alpha^\vee} \operatorname{Sym}^n(\mathscr{V}^\vee)\otimes\mathscr{V}^\vee\otimes\mathscr{L}^{\otimes n+1} \\ &\xrightarrow{\mathrm{mult}} \operatorname{Sym}^{n+1}(\mathscr{V}^\vee)\otimes\mathscr{L}^{\otimes n+1}.\end{aligned}$$

Proof. — By the description A.1.8, the ideal sheaf $\mathscr{I}$ of $\mathbf{P}\mathscr{U}$ in $\mathbf{P}\mathscr{V}$ is the image of

$$u\colon \mathscr{O}_{\mathbf{P}\mathscr{V}}(-1)\otimes\pi^*\mathscr{L}^\vee \to \mathscr{O}_{\mathbf{P}\mathscr{V}}.$$

Thus [Har77, Exercise III.3.7] gives an isomorphism between $\pi_*\mathscr{O}_{\mathbf{P}\mathscr{V}^\circ}$ and

$$\begin{aligned}\operatorname{colim}_n \pi_*\mathscr{H}\!om_{\mathscr{O}_{\mathbf{P}\mathscr{V}}}(\mathscr{I}^n,\mathscr{O}_{\mathbf{P}\mathscr{V}}) &\cong \operatorname{colim}_n \pi_*\mathscr{O}_{\mathbf{P}\mathscr{V}}(n)\otimes\mathscr{L}^{\otimes n} \\ &\cong \operatorname{colim}_n \operatorname{Sym}^n(\mathscr{V}^\vee)\otimes\mathscr{L}^{\otimes n}\end{aligned}$$

in which the transition maps are induced by u. ■

This is more explicit when the sequence in A.1.8 splits:

A.1.10. Lemma. — *A choice of splitting of the sequence in* A.1.8 *gives an isomorphism*

$$\mathbf{P}\mathscr{V}^\circ \cong \mathbf{A}(\mathscr{U}\otimes\mathscr{L}^\vee) := \operatorname{Spec}\big(\operatorname{Sym}^*(\mathscr{U}^\vee\otimes\mathscr{L})\big).$$

Proof. — Choose a section $\sigma\colon\mathscr{L}\to\mathscr{V}$ to α and let $\iota\colon\mathscr{U}\to\mathscr{V}$ be the inclusion. Let $\iota\oplus\sigma\colon\mathscr{U}\oplus\mathscr{L}\to\mathscr{V}$ be the corresponding splitting. Then $\operatorname{Sym}^n(\iota^\vee\oplus\sigma^\vee)$ gives the first isomorphism in

$$\begin{aligned}\operatorname{Sym}^n(\mathscr{V}^\vee)\otimes\mathscr{L}^{\otimes n} &\cong \operatorname{Sym}^n(\mathscr{U}^\vee\oplus\mathscr{L}^\vee)\otimes\mathscr{L}^{\otimes n} \\ &\cong \Big(\bigoplus\nolimits_{i=0}^n \operatorname{Sym}^i(\mathscr{U}^\vee)\otimes\mathscr{L}^{\vee,\otimes n-i}\Big)\otimes\mathscr{L}^{\otimes n} \\ &\cong \bigoplus\nolimits_{i=0}^n \operatorname{Sym}^i(\mathscr{U}^\vee\otimes\mathscr{L}).\end{aligned}$$

There is a commutative square

$$\begin{tikzcd} \operatorname{Sym}^n(\mathscr{V}^\vee)\otimes\mathscr{L}^{\otimes n} \arrow[r, "\operatorname{Sym}^n(\iota^\vee\oplus\sigma^\vee)"'] \arrow[d, "\alpha^\vee"'] & \operatorname{Sym}^n(\mathscr{U}^\vee\oplus\mathscr{L}^\vee)\otimes\mathscr{L}^{\otimes n} \arrow[d, "(\iota^\vee\oplus\sigma^\vee)\circ\alpha^\vee"] \\ \operatorname{Sym}^{n+1}(\mathscr{V}^\vee)\otimes\mathscr{L}^{\otimes n+1} \arrow[r, "\operatorname{Sym}^{n+1}(\iota^\vee\oplus\sigma^\vee)"] & \operatorname{Sym}^{n+1}(\mathscr{U}^\vee\oplus\mathscr{L}^\vee)\otimes\mathscr{L}^{\otimes n+1} \end{tikzcd}$$

where the vertical maps are induced by $\alpha^\vee \colon \mathscr{O}_B \to \mathscr{V}^\vee \otimes \mathscr{L}$ and the multiplication maps on symmetric powers. Since $(\iota^\vee \oplus \sigma^\vee) \circ \alpha^\vee = 0 \oplus \mathrm{id}_{\mathscr{L}^\vee}$, this shows that the transition map of A.1.9 are mapped to the natural inclusion

$$\bigoplus\nolimits_{i=0}^{n} \mathrm{Sym}^i(\mathscr{U}^\vee \otimes \mathscr{L}) \subset \bigoplus\nolimits_{i=0}^{n+1} \mathrm{Sym}^i(\mathscr{U}^\vee \otimes \mathscr{L}).$$

Thus $\mathrm{Sym}^*(\iota^\vee \oplus \sigma^\vee)$ induces an isomorphism of $\pi_*\mathscr{O}_{\mathbf{P}\mathscr{V}^\circ}$ with $\mathrm{Sym}^*(\mathscr{U}^\vee \otimes \mathscr{L})$. ■

A.1.11. Tautological bundles. — The pullback of the exact sequence in A.1.8 to $\mathbf{P}\mathscr{V}$ together with relative Euler sequence fit into a commutative diagram

$$\begin{tikzcd}
 & & \pi^*\mathscr{U} \arrow[d, hook] \arrow[dr] & & \\
0 \arrow[r] & \mathscr{O}_\pi(-1) \arrow[r, "\mathrm{eu}_\pi"] \arrow[dr, "u"'] & \pi^*\mathscr{V} \arrow[r] \arrow[d, two heads, "\pi^*\alpha"] & \mathscr{T}_\pi(-1) \arrow[r] & 0 \\
 & & \pi^*\mathscr{L} & &
\end{tikzcd}$$

in which the row and column are exact. By definition, u does not vanish on $\mathbf{P}\mathscr{V}^\circ$, and so the diagonal maps give isomorphisms of $\mathscr{O}_{\mathbf{P}\mathscr{V}^\circ}$-modules

$$\mathscr{O}_{\mathbf{P}\mathscr{V}}(-1)|_{\mathbf{P}\mathscr{V}^\circ} \cong \pi^*\mathscr{L}|_{\mathbf{P}\mathscr{V}^\circ} \quad \text{and} \quad \mathscr{T}_\pi(-1)|_{\mathbf{P}\mathscr{V}^\circ} \cong \pi^*\mathscr{U}|_{\mathbf{P}\mathscr{V}^\circ}.$$

A.1.12. Euler section. — Still in the situation of A.1.8, consider the morphism of $\mathscr{O}_B$-modules obtained by pushing forward the Euler section. Recall that the Euler section $\mathscr{O}_{\mathbf{P}\mathscr{V}} \to \pi^*\mathscr{V} \otimes \mathscr{O}_\pi(1)$ is adjoint to the dual of the trace section:

$$\pi_*\mathrm{eu}_\pi \colon \mathscr{O}_B \xrightarrow{\mathrm{tr}_{\mathscr{V}}^\vee} \mathscr{V} \otimes \mathscr{V}^\vee \cong \pi_*(\pi^*\mathscr{V} \otimes \mathscr{O}_\pi(1)).$$

Using $\mathscr{O}_{\mathbf{P}\mathscr{V}}(-1)|_{\mathbf{P}\mathscr{V}^\circ} \cong \pi^*\mathscr{L}|_{\mathbf{P}\mathscr{V}^\circ}$ from A.1.11 and taking adjoints shows that

$$\pi_*\mathrm{eu}_\pi|_{\mathbf{P}\mathscr{V}^\circ} \colon \mathscr{L} \otimes \pi_*\mathscr{O}_{\mathbf{P}\mathscr{V}^\circ} \to \mathscr{V} \otimes \pi_*\mathscr{O}_{\mathbf{P}\mathscr{V}^\circ}$$

is the map of $\pi_*\mathscr{O}_{\mathbf{P}\mathscr{V}^\circ}$-algebras induced by multiplication by $\mathrm{tr}_{\mathscr{V}}^\vee \colon \mathscr{L} \to \mathscr{V} \otimes \mathscr{V}^\vee \otimes \mathscr{L}$.

A.1.13. — This can be made more explicit in the case the short exact sequence of A.1.8 splits. Fix a splitting $\mathcal{V} \cong \mathcal{U} \oplus \mathcal{L}$ and identify $\mathbf{P}\mathcal{V}^\circ \cong \mathbf{A}(\mathcal{U} \otimes \mathcal{L}^\vee)$ as in A.1.10. Then the trace section of $\mathcal{V}$ factors as

$$\mathrm{tr}_{\mathcal{V}}^\vee \colon \mathcal{O}_B \xrightarrow{\mathrm{tr}_{\mathcal{U}}^\vee \oplus \mathrm{tr}_{\mathcal{L}}^\vee} (\mathcal{U} \otimes \mathcal{U}^\vee) \oplus (\mathcal{L} \otimes \mathcal{L}^\vee) \subset \mathcal{V} \otimes \mathcal{V}^\vee$$

and so the pushforward of the Euler section is the map

$$\pi_* \mathrm{eu}_\pi \colon \mathcal{L} \otimes \mathrm{Sym}^*(\mathcal{U}^\vee \otimes \mathcal{L}) \to \mathcal{U} \otimes \mathrm{Sym}^*(\mathcal{U}^\vee \otimes \mathcal{L})$$

of algebras induced by multiplication by

$$(\mathrm{id}_{\mathcal{L}}, \mathrm{tr}_{\mathcal{U}}^\vee) \colon \mathcal{L} \to \mathcal{L} \oplus (\mathcal{U} \otimes \mathcal{U}^\vee \otimes \mathcal{L})$$

followed by the inclusions of $\mathcal{L}$ and $\mathcal{U}$ into $\mathcal{V}$ to have target $\mathcal{V} \otimes \mathrm{Sym}^{\leq 1}(\mathcal{U}^\vee \otimes \mathcal{L})$.

A.2. Functoriality of Grassmannians

The formation of Grassmannians is, in a sense, functorial on the category of finite locally free $\mathcal{O}_B$-modules upon restricting to certain classes of morphisms. Namely, consider a morphism $\mathcal{V} \to \mathcal{W}$ of finite locally free $\mathcal{O}_B$-modules of ranks m and n, respectively. This Section is concerned with the following two situations:

(i) If $\psi \colon \mathcal{V} \to \mathcal{W}$ is surjective, then there is a rational map

$$\psi_r \colon \mathbf{G}(r, \mathcal{V}) \dashrightarrow \mathbf{G}(r, \mathcal{W}) \quad \text{for each } 1 \leq r \leq n,$$

which sends a general subbundle $\mathcal{V}' \subseteq \mathcal{V}$ of rank r to its image $\psi(\mathcal{V}') \subseteq \mathcal{W}$.

(ii) If $\varphi \colon \mathcal{V} \to \mathcal{W}$ is locally split injective, then there is a rational map

$$\varphi^s \colon \mathbf{G}(\mathcal{W}, s) \dashrightarrow \mathbf{G}(\mathcal{V}, s) \quad \text{for each } 1 \leq s \leq m,$$

which sends a general quotient $\mathcal{W} \twoheadrightarrow \mathcal{W}''$ of rank s to $\mathcal{V} \xrightarrow{\varphi} \mathcal{W} \twoheadrightarrow \mathcal{W}''$.

Both situations arise as in A.1.5 and the methods of A.1.6 and A.1.7 may be used to construct a resolution. The goal of this Section is to give a geometric description of these resolutions: see A.2.4 and A.2.9.

A.2.1. Linear projection. — Let $\psi\colon \mathscr{V} \to \mathscr{W}$ be a surjection of finite locally free sheaves of ranks m and n, respectively. Let $1 \leq r \leq n$ be an integer and consider the rational map

$$\psi_r\colon \mathbf{G}(r,\mathscr{V}) \dashrightarrow \mathbf{G}(r,\mathscr{W})$$

as in A.2(i): this is defined by the map of locally free sheaves

$$\mathscr{S}_{\mathbf{G}(r,\mathscr{V})} \subset \mathscr{V}_{\mathbf{G}(r,\mathscr{V})} \xrightarrow{\psi} \mathscr{W}_{\mathbf{G}(r,\mathscr{V})} \tag{A.2.2}$$

restricted to the open subscheme $\mathbf{G}(r,\mathscr{V})^\circ$ on which this is injective. Consider the subfunctor of $\mathbf{G}(r,\mathscr{V}) \times_B \mathbf{G}(r,\mathscr{W})$ given by

$$\begin{aligned} \mathbf{G}(r,\psi)\colon \mathrm{Sch}_B^{\mathrm{opp}} &\to \mathrm{Set} \\ T &\mapsto \big\{ (\mathscr{V}',\mathscr{W}') \bigm| \psi(\mathscr{V}') \subseteq \mathscr{W}' \big\}. \end{aligned}$$

This is representable by the closed subscheme of $\mathbf{G}(r,\mathscr{V}) \times_B \mathbf{G}(r,\mathscr{W})$ defined by the vanishing of the map

$$\mathrm{pr}_{\mathscr{V}}^* \mathscr{S}_{\mathbf{G}(r,\mathscr{V})} \subset \mathscr{V}_{\mathbf{G}(r,\mathscr{V})\times_B \mathbf{G}(r,\mathscr{W})} \xrightarrow{\psi} \mathscr{W}_{\mathbf{G}(r,\mathscr{V})\times_B \mathbf{G}(r,\mathscr{W})} \twoheadrightarrow \mathrm{pr}_{\mathscr{W}}^* \mathscr{Q}_{\mathbf{G}(r,\mathscr{W})}.$$

In particular, the pullback of A.2.2 to $\mathbf{G}(r,\psi)$ factors through the tautological map

$$\mathrm{pr}_{\mathscr{V}}^* \mathscr{S}_{\mathbf{G}(r,\mathscr{V})} \to \mathrm{pr}_{\mathscr{W}}^* \mathscr{S}_{\mathbf{G}(r,\mathscr{W})} \tag{A.2.3}$$

induced by ψ. The following summarizes the structure of the scheme $\mathbf{G}(r,\psi)$.

A.2.4. Proposition. — *In the setting of A.2.1, there is a commutative diagram*

$$\begin{tikzcd} & \mathbf{G}(r,\psi) \arrow[dl, "\mathrm{pr}_{\mathscr{V}}"'] \arrow[dr, "\mathrm{pr}_{\mathscr{W}}"] & \\ \mathbf{G}(r,\mathscr{V}) \arrow[rr, dashed, "\psi_r"] & & \mathbf{G}(r,\mathscr{W}) \end{tikzcd}$$

of schemes over B. Furthermore:

(i) $\mathrm{pr}_{\mathscr{V}}\colon \mathbf{G}(r,\psi) \to \mathbf{G}(r,\mathscr{V})$ *is the blowup along the Fitting scheme of A.2.2:*

$$D_\psi := \mathrm{V}\Big(\mathrm{Fitt}_{n-r}\big(\mathrm{coker}(\mathscr{S}_{\mathbf{G}(r,\mathscr{V})} \subset \mathscr{V}_{\mathbf{G}(r,\mathscr{V})} \xrightarrow{\psi} \mathscr{W}_{\mathbf{G}(r,\mathscr{V})})\big)\Big),$$

(ii) $\mathrm{pr}_{\mathscr{W}} \colon \mathbf{G}(r,\psi) \to \mathbf{G}(r,\mathscr{W})$ *is the Grassmannian bundle* $\mathbf{G}(r,\tilde{\mathscr{S}}) \to \mathbf{G}(r,\mathscr{W})$ *where*

$$\begin{tikzcd} \tilde{\mathscr{S}} \arrow[r, hook] \arrow[d, two heads] & \mathscr{V}_{\mathbf{G}(r,\mathscr{W})} \arrow[d, two heads, "\psi"] \\ \mathscr{S}_{\mathbf{G}(r,\mathscr{V})} \arrow[r, hook] & \mathscr{W}_{\mathbf{G}(r,\mathscr{W})} \end{tikzcd}$$

is a pullback square, and

(iii) *writing* $\mathscr{S}_{\mathrm{pr}_{\mathscr{W}}} \subset \mathrm{pr}_{\mathscr{W}}^* \tilde{\mathscr{S}}$ *for the relative tautological bundle of the Grassmannian bundle from* (ii)*, the tautological map* A.2.3 *factors as*

$$\mathrm{pr}_{\mathscr{V}}^* \mathscr{S}_{\mathbf{G}(r,\mathscr{V})} = \mathscr{S}_{\mathrm{pr}_2} \subset \mathrm{pr}_{\mathscr{W}}^* \tilde{\mathscr{S}} \twoheadrightarrow \mathrm{pr}_{\mathscr{W}}^* \mathscr{S}_{\mathbf{G}(r,\mathscr{W})}.$$

That $\mathbf{G}(r,\psi)$ fits into the commutative diagram follows from the description of its functor of points. The remaining statements will now be proved in turn.

Proof of A.2.4(i). — The functorial description of $\mathbf{G}(r,\psi)$ shows that $\mathrm{pr}_{\mathscr{V}}$ restricts to an isomorphism between the open subschemes

$$\mathbf{G}(r,\psi)^\circ := \{ (\mathscr{V}',\mathscr{W}') \mid \psi(\mathscr{V}') = \mathscr{W}' \},$$
$$\mathbf{G}(r,\mathscr{V})^\circ := \{ \mathscr{V}' \mid \mathscr{V}' \subset \mathscr{V} \xrightarrow{\psi} \mathscr{W} \text{ is injective} \},$$

where A.2.3 is an isomorphism, and where A.2.2 is injective, respectively. The complement of $\mathbf{G}(r,\mathscr{V})^\circ$, being the locus over which A.2.2 degenerates, is canonically the closed subscheme D_ψ given by the $(n-r)$-th Fitting ideal of A.2.2. Let $\mathrm{bl} \colon \tilde{\mathbf{G}}(r,\mathscr{V}) \to \mathbf{G}(r,\mathscr{V})$ be the blowup along D_ψ. Then A.1.7 shows that the strict transform of coker(A.2.2) along bl is a locally free quotient of $\mathscr{W}_{\tilde{\mathbf{G}}(r,\mathscr{V})}$ and hence defines a morphism $\tilde{\psi}_r \colon \tilde{\mathbf{G}}(r,\mathscr{V}) \to \mathbf{G}(r,\mathscr{W})$ resolving ψ_r; this induces a morphism

(A.2.5) $$\mathrm{bl}^* \mathscr{S}_{\mathbf{G}(r,\mathscr{V})} \to \tilde{\psi}_r^* \mathscr{S}_{\mathbf{G}(r,\mathscr{W})}$$

with the additional property that the vanishing of its determinant is the exceptional divisor of bl. This data provides a morphism $\Psi \colon \tilde{\mathbf{G}}(r,\mathscr{V}) \to \mathbf{G}(r,\psi)$ of schemes over $\mathbf{G}(r,\mathscr{V})$ for which the tautological map A.2.3 pulls back to A.2.5.

To construct the inverse to Ψ, consider the preimage of D_ψ in $\mathbf{G}(r,\psi)$. Base change of Fitting ideals, as in [**Stacks**, 0C3D], gives the first equality in

$$\mathrm{pr}_{\mathscr{V}}^{-1}(D_\psi) = \mathrm{V}(\mathrm{Fitt}_{n-r}(\mathrm{pr}_{\mathscr{V}}^*(\mathbf{A.2.2}))) = \mathrm{V}(\mathrm{Fitt}_0(\mathbf{A.2.3})) = \mathrm{V}(\det(\mathbf{A.2.3})).$$

The third equality is because **A.2.3** is a map between locally free sheaves of the same rank. It remains to explain the second equality. Observe that there is a factorization

$$\mathrm{pr}_{\mathscr{V}}^*(\mathbf{A.2.2}) = (\mathbf{A.2.3}) \circ \mathrm{pr}_{\mathscr{W}}^*(\mathscr{S}_{\mathbf{G}(r,\mathscr{W})} \subset \mathscr{W}_{\mathbf{G}(r,\mathscr{W})})$$

where the latter map is the tautological inclusion and hence is of maximal rank. Thus the ideal generated by the maximal minors $\mathrm{pr}_{\mathscr{V}}^*(\mathbf{A.2.2})$ coincides with that of **A.2.3**, and this is the second equality. This can also be deduced from properties of Fitting ideals using the convolution property from part (2) of [**Stacks**, 07ZA], together with the identification of Fitting ideals of locally free modules of [**Stacks**, 0C3G]. In conclusion, $\mathrm{pr}_{\mathscr{V}}^{-1}(D_\psi)$ is an effective Cartier divisor in $\mathbf{G}(r,\psi)$, so the universal property of blowing up gives a morphism $\Psi^{-1} \colon \mathbf{G}(r,\psi) \to \tilde{\mathbf{G}}(r,\mathscr{V})$ of schemes over $\mathbf{G}(r,\mathscr{V})$ such that the exceptional divisor of bl pulls back to $\mathrm{V}(\det(\mathbf{A.2.3}))$. Comparing their constructions shows that **A.2.5** pulls back to the tautological map **A.2.3**.

It remains to see that the two morphisms are mutually inverse. That $\Psi \circ \Psi^{-1}$ is the identity of $\mathbf{G}(r,\psi)$ is because the tautological maps **A.2.3** and **A.2.5** pull back to one another. For the other composite, note both maps are morphisms over $\mathbf{G}(r,\mathscr{V})$ so

$$(\Psi^{-1} \circ \Psi \circ \mathrm{bl})^{-1}(D_\psi) = (\Psi^{-1} \circ \mathrm{pr}_{\mathscr{V}})^{-1}(D_\psi) = \mathrm{bl}^{-1}(D_\psi)$$

and now its universal property shows that $\Psi^{-1} \circ \Psi$ is the identity of $\tilde{\mathbf{G}}(r,\mathscr{V})$. ■

Proof of **A.2.4**(ii) *and* (iii). — The pullback square defining $\tilde{\mathscr{S}}$ implies that the map **A.2.3** factors through $\mathrm{pr}_{\mathscr{W}}^* \tilde{\mathscr{S}}$. This gives a locally split injection $\mathrm{pr}_{\mathscr{V}}^* \mathscr{S}_{\mathbf{G}(r,\mathscr{V})} \hookrightarrow \mathrm{pr}_{\mathscr{W}}^* \tilde{\mathscr{S}}$ defining a morphism $\mathbf{G}(r,\psi) \to \mathbf{G}(r,\tilde{\mathscr{S}})$ of schemes over $\mathbf{G}(r,\mathscr{W})$.

Conversely, the relative tautological bundle $\mathscr{S}_\pi \subseteq \pi^*\tilde{\mathscr{S}}$ gives a rank r subbundle of $\mathscr{V}_{\mathbf{G}(r,\tilde{\mathscr{S}})}$ and so gives a morphism $\mathbf{G}(r,\tilde{\mathscr{S}}) \to \mathbf{G}(r,\mathscr{V})$ identifying $\mathscr{S}_\pi$ as the pullback of $\mathscr{S}_{\mathbf{G}(r,\mathscr{V})}$. But the map $\mathscr{S}_\pi \subset \pi^*\tilde{\mathscr{S}} \to \pi^*\mathscr{S}_{\mathbf{G}(r,\mathscr{W})}$ shows that the resulting morphism $\mathbf{G}(r,\tilde{\mathscr{S}}) \to \mathbf{G}(r,\mathscr{V}) \times_B \mathbf{G}(r,\mathscr{W})$ factors through a morphism $\mathbf{G}(r,\tilde{\mathscr{S}}) \to \mathbf{G}(r,\psi)$.

That these morphisms are mutually inverse and identify the tautological bundles as stated in (iii) follows from their functorial construction and the description of the points of the schemes involved. ■

A.2.6. Intersection with subbundles. — Dually, let $\varphi\colon \mathscr{V} \to \mathscr{W}$ be a locally split injection of locally free sheaves of ranks m and n, respectively. Let $1 \leq s \leq m$ be an integer and consider the rational map

$$\varphi^s\colon \mathbf{G}(\mathscr{W}, s) \dashrightarrow \mathbf{G}(\mathscr{V}, s)$$

as in A.2(ii): this is defined by the map of locally free sheaves

$$\mathscr{V}_{\mathbf{G}(\mathscr{W},s)} \xrightarrow{\varphi} \mathscr{W}_{\mathbf{G}(\mathscr{W},s)} \twoheadrightarrow \mathscr{Q}_{\mathbf{G}(\mathscr{W},s)} \tag{A.2.7}$$

restricted to the open subscheme $\mathbf{G}(\mathscr{W}, s)$ on which this is surjective. Consider the subfunctor of $\mathbf{G}(\mathscr{V}, s) \times_B \mathbf{G}(\mathscr{W}, s)$ given by

$$\begin{aligned} \mathbf{G}(\varphi, s)\colon \mathrm{Sch}_B^{\mathrm{opp}} &\to \mathrm{Set} \\ T &\mapsto \left\{ (\mathscr{V}'', \mathscr{W}'') \,\middle|\, \varphi \text{ induces a map } \mathscr{V}'' \to \mathscr{W}'' \right\}. \end{aligned}$$

The identification between Grassmanians of quotients and subbundles given in A.1.3 shows that $\mathbf{G}(\varphi, s)$ is also the subfunctor of $\mathbf{G}(m-s, \mathscr{V}) \times_B \mathbf{G}(n-s, \mathscr{W})$ given by

$$T \mapsto \left\{ (\mathscr{V}', \mathscr{W}') \,\middle|\, \varphi(\mathscr{V}') \subseteq \mathscr{W}' \right\}.$$

With either view, it follows that $\mathbf{G}(\varphi, s)$ is representable by the closed subscheme of $\mathbf{G}(\mathscr{V}, s) \times_B \mathbf{G}(\mathscr{W}, s)$ again defined by the vanishing of the map

$$\mathrm{pr}_{\mathscr{V}}^* \mathscr{S}_{\mathbf{G}(\mathscr{V},s)} \subset \mathscr{V}_{\mathbf{G}(\mathscr{V},s)\times_B \mathbf{G}(\mathscr{W},s)} \xrightarrow{\varphi} \mathscr{W}_{\mathbf{G}(\mathscr{V},s)\times_B \mathbf{G}(\mathscr{W},s)} \twoheadrightarrow \mathrm{pr}_{\mathscr{W}}^* \mathscr{Q}_{\mathbf{G}(\mathscr{W},s)}.$$

Then the pullback of A.2.7 to $\mathbf{G}(\varphi, s)$ factors through the tautological map

$$\mathrm{pr}_{\mathscr{V}}^* \mathscr{Q}_{\mathbf{G}(\mathscr{V},s)} \to \mathrm{pr}_{\mathscr{W}}^* \mathscr{Q}_{\mathbf{G}(\mathscr{W},s)} \tag{A.2.8}$$

induced by φ. The following summarizes the structure of the scheme $\mathbf{G}(\varphi, s)$.

A.2.9. Proposition. — *In the setting of A.2.6, there is a commutative diagram*

$$\begin{tikzcd}[column sep=small] & \mathbf{G}(\varphi,s) \arrow[dl, "\mathrm{pr}_{\mathscr{W}}"'] \arrow[dr, "\mathrm{pr}_{\mathscr{V}}"] & \\ \mathbf{G}(\mathscr{W},s) \arrow[rr, dashed, "\varphi^s"] & & \mathbf{G}(\mathscr{V},s) \end{tikzcd}$$

of schemes over B. Furthermore:

(i) $\mathrm{pr}_{\mathscr{W}} \colon \mathbf{G}(\varphi,s) \to \mathbf{G}(\mathscr{W},s)$ *is the blowup along the Fitting scheme of A.2.8:*

$$D_\varphi := \mathrm{V}\big(\mathrm{Fitt}_s\big(\,\mathrm{image}(\mathscr{V}_{\mathbf{G}(\mathscr{W},s)} \xrightarrow{\varphi} \mathscr{W}_{\mathbf{G}(\mathscr{W},s)} \twoheadrightarrow \mathscr{Q}_{\mathbf{G}(\mathscr{W},s)})\big)\big),$$

(ii) $\mathrm{pr}_{\mathscr{V}} \colon \mathbf{G}(\varphi,s) \to \mathbf{G}(\mathscr{V},s)$ *is the Grassmannian bundle* $\mathbf{G}(\tilde{\mathscr{Q}},s) \to \mathbf{G}(\mathscr{V},s)$ *where*

$$\begin{tikzcd} \mathscr{V}_{\mathbf{G}(\mathscr{V},s)} \arrow[r, two heads] \arrow[d, hook, "\varphi"'] & \mathscr{Q}_{\mathbf{G}(\mathscr{V},s)} \arrow[d, hook] \\ \mathscr{W}_{\mathbf{G}(\mathscr{W},s)} \arrow[r, two heads] & \tilde{\mathscr{Q}} \end{tikzcd}$$

is a pushout square, and

(iii) *writing* $\mathrm{pr}_{\mathscr{V}}^* \tilde{\mathscr{Q}} \twoheadrightarrow \mathscr{Q}_{\mathrm{pr}_{\mathscr{V}}}$ *for the relative tautological bundle of the Grassmannian bundle from (ii), the tautological map A.2.8 factors as*

$$\varphi \colon \mathrm{pr}_{\mathscr{V}}^* \mathscr{Q}_{\mathbf{G}(\mathscr{V},s)} \hookrightarrow \mathrm{pr}_{\mathscr{V}}^* \tilde{\mathscr{Q}} \twoheadrightarrow \mathscr{Q}_{\mathrm{pr}_{\mathscr{V}}} = \mathrm{pr}_{\mathscr{W}}^* \mathscr{Q}_{\mathbf{G}(\mathscr{W},s)}.$$

Proof. — This can be proved in the same way as A.2.4. More directly, this follows from A.2.4 via the duality identifications $\mathbf{G}(\mathscr{V},s) \cong \mathbf{G}(s,\mathscr{V}^\vee)$, $\mathbf{G}(\mathscr{W},s) \cong \mathbf{G}(s,\mathscr{W}^\vee)$, and $\mathbf{G}(\varphi,s) \cong \mathbf{G}(s,\varphi^\vee)$ from A.1.4. ∎

A.3. Passage to a subquotient

This Section pertains to a composite of the two situations of A.2, in which $\mathscr{V}$ is a subquotient of $\mathscr{W}$ that can be realized either as

- a quotient of a subbundle $\mathscr{W}_1$ of $\mathscr{W}$, or
- a subbundle of a quotient $\mathscr{W}_2$ of $\mathscr{W}$.

The two realizations of $\mathscr{V}$ are related by an exact commutative diagram

(A.3.1)
$$\begin{tikzcd}
 & & \mathscr{U}_1 \arrow[r, equal] & \mathscr{U}_1 & \\
0 \arrow[r] & \mathscr{U}_2 \arrow[r] & \mathscr{W} \arrow[u, two heads] \arrow[r, "\psi^2"'] & \mathscr{W}_2 \arrow[u, two heads] \arrow[r] & 0 \\
0 \arrow[r] & \mathscr{U}_2 \arrow[r] \arrow[u, equal] & \mathscr{W}_1 \arrow[u, hook, "\varphi_1"] \arrow[r, "\psi^1"] & \mathscr{V} \arrow[u, hook, "\varphi_2"'] \arrow[r] & 0\,.
\end{tikzcd}$$

Assume all modules appearing are finite locally free over $\mathscr{O}_B$ and set

$$n := \operatorname{rank}_{\mathscr{O}_B}(\mathscr{W}), \quad m := \operatorname{rank}_{\mathscr{O}_B}(\mathscr{W}_2), \quad c := \operatorname{rank}_{\mathscr{O}_B}(\mathscr{U}_1),$$

and let $r \leq m$ be a positive integer. The constructions of A.2.1 and A.2.6 give a commutative diagram of rational maps

(A.3.2)
$$\begin{tikzcd}
\mathbf{G}(\mathscr{W}, n-r) \arrow[r, phantom, "\cong"] \arrow[d, dashed, "\varphi_1^{n-r}"'] & \mathbf{G}(r, \mathscr{W}) \arrow[r, dashed, "\psi_r^2"'] & \mathbf{G}(r, \mathscr{W}_2) \arrow[r, phantom, "\cong"] & \mathbf{G}(\mathscr{W}_2, m-r) \arrow[d, dashed, "\varphi_2^{m-r}"] \\
\mathbf{G}(\mathscr{W}_1, n-r) \arrow[r, phantom, "\cong"] & \mathbf{G}(r-c, \mathscr{W}_1) \arrow[r, dashed, "\psi_{r-c}^1"] & \mathbf{G}(r-c, \mathscr{V}) \arrow[r, phantom, "\cong"] & \mathbf{G}(\mathscr{V}, m-r)
\end{tikzcd}$$

where the horizontal arrows come as pushforward along surjections, and the vertical arrows come as pullback along locally split injections. Write

$$\mathbf{H}_0 := \mathbf{G}(\mathscr{W}, n-r) \cong \mathbf{G}(r, \mathscr{W}) \quad \text{and} \quad \mathbf{G}_0 := \mathbf{G}(r-c, \mathscr{V}) \cong \mathbf{G}(\mathscr{V}, m-r)$$

for the Grassmannians on the top left and bottom right of the diagram, and let

$$\mathbf{G}_1 := \mathbf{G}(r-c, \psi^1) \quad \text{and} \quad \mathbf{G}_2 := \mathbf{G}(\varphi_2, m-r)$$

be the spaces constructed in A.2.4 and A.2.9 on which ψ^1 and φ_2, respectively, are resolved. Then the commutative diagram provides a rational map

$$\varphi_1^{n-r} \times \psi_r^2 \colon \mathbf{H}_0 \dashrightarrow \mathbf{G} := \mathbf{G}_1 \times_{\mathbf{G}_0} \mathbf{G}_2.$$

The goal of this Section is to provide a geometric resolution of this rational map, and to describe its structure over the $\mathbf{H}_0$ and $\mathbf{G}$: see A.3.7.

A.3.3. Structure of G. — Consider the fibre product $\mathbf{G} = \mathbf{G}_1 \times_{\mathbf{G}_0} \mathbf{G}_2$ of the schemes on which the rational maps ψ^1_{r-c} and φ_2^{m-r} are resolved. By A.2.1 and A.2.6, this scheme represents the functor whose value on a scheme T is the set of diagrams

$$\begin{tikzcd} & \mathscr{W}_2' \\ \mathscr{W}_1' \arrow[r, "\psi^1"] & \mathscr{V}' \arrow[u, "\varphi_2"'] \end{tikzcd} \quad \text{where} \quad \begin{array}{lcl} \mathscr{W}_2' & \in & \mathbf{G}(r, \mathscr{W}_2)(T), \\ \mathscr{V}' & \in & \mathbf{G}(r-c, \mathscr{V})(T), \\ \mathscr{W}_1' & \in & \mathbf{G}(r-c, \mathscr{W}_1)(T), \end{array}$$

consisting of subbundles of $\mathscr{W}_{2,T}$, $\mathscr{V}_T$, and $\mathscr{W}_{1,T}$ of appropriate ranks, and which are mapped to one another via the given maps ψ^1 and φ_2; here, the points of $\mathbf{G}(\varphi_2, m-r)$ are expressed in terms of subbundles.

The projection $\mathrm{pr}_{\mathscr{V}} \colon \mathbf{G} \to \mathbf{G}_0$ to the base of the fibre product exhibits $\mathbf{G}$ as a product of Grassmannian bundles over $\mathbf{G}_0$. Namely, consider the commutative diagram

(A.3.4)
$$\begin{tikzcd}[column sep=small] 0 \arrow[r] & \tilde{\mathscr{S}}_{\mathbf{G}_0} \arrow[rr] \arrow[dr] \arrow[dd] & & \mathscr{W}_{\mathbf{G}_0} \arrow[rr] \arrow[dr] & & \tilde{\mathscr{Q}}_{\mathbf{G}_0} \arrow[r] & 0 \\ & & \mathscr{W}_{1,\mathbf{G}_0} \arrow[ur] \arrow[dr] & & \mathscr{W}_{2,\mathbf{G}_0} \arrow[ur] & & \\ 0 \arrow[r] & \mathscr{S}_{\mathbf{G}_0} \arrow[rr] & & \mathscr{V}_{\mathbf{G}_0} \arrow[rr] \arrow[ur] & & \mathscr{Q}_{\mathbf{G}_0} \arrow[r] \arrow[uu, hook] & 0 \end{tikzcd}$$

in which the bottom row is the tautological exact sequence of $\mathbf{G}_0$, $\tilde{\mathscr{S}}_{\mathbf{G}_0}$ is the pullback in the left square, and $\tilde{\mathscr{Q}}_{\mathbf{G}_0}$ is the pushout of the right square; note that the top row is also exact. Then A.2.4(ii) and A.2.9(ii) together imply:

A.3.5. Corollary. — $\mathbf{G} \cong \mathbf{G}(r-c, \tilde{\mathscr{S}}_{\mathbf{G}_0}) \times_{\mathbf{G}_0} \mathbf{G}(\tilde{\mathscr{Q}}_{\mathbf{G}_0}, m-r)$ *as schemes over* $\mathbf{G}_0$. ■

A.3.6. The scheme H. — To construct the scheme on which the rational map $\varphi_1^{n-r} \times \psi_r^2 \colon \mathbf{H}_0 \dashrightarrow \mathbf{G}$ is resolved. Consider the subfunctor $\mathbf{H}$ of $\mathbf{G}(r, \mathscr{W}) \times_B \mathbf{G}$ whose points on a scheme T is the set of diagrams

$$\begin{tikzcd} \mathscr{W}' \arrow[r, "\psi^2"'] & \mathscr{W}_2' \\ \mathscr{W}_1' \arrow[u, "\varphi_1"] \arrow[r, "\psi^1"] & \mathscr{V}' \arrow[u, "\varphi_2"'] \end{tikzcd} \quad \text{where} \quad \begin{array}{lcl} \mathscr{W}' & \in & \mathbf{G}(r, \mathscr{W})(T), \\ \mathscr{W}_2' & \in & \mathbf{G}(r, \mathscr{W}_2)(T), \\ \mathscr{V}' & \in & \mathbf{G}(r-c, \mathscr{V})(T), \\ \mathscr{W}_1' & \in & \mathbf{G}(r-c, \mathscr{W}_1)(T), \end{array}$$

consisting of a subbundle of $\mathscr{W}_T$ and a T-point of $\mathbf{G}$ which are compatible with the maps φ_1 and ψ^2. This is representable by the closed subscheme of $\mathbf{G}(r,\mathscr{W}) \times_B \mathbf{G}$ given by the vanishing of the two maps

$$\begin{aligned}
\mathrm{pr}_{\mathscr{W}}^* \mathscr{S}_{\mathbf{G}(r,\mathscr{W})} \subset \mathscr{W}_{\mathbf{G}(r,\mathscr{W})\times_B \mathbf{G}} &\xrightarrow{\psi^2} \mathscr{W}_{2,\mathbf{G}(r,\mathscr{W})\times_B \mathbf{G}} \twoheadrightarrow \mathrm{pr}_{\mathscr{W}_2}^* \mathscr{Q}_{\mathbf{G}(r,\mathscr{W}_2)} \\
\mathrm{pr}_{\mathscr{W}_1}^* \mathscr{S}_{\mathbf{G}(r-c,\mathscr{W}_1)} \subset \mathscr{W}_{1,\mathbf{G}(r,\mathscr{W})\times_B \mathbf{G}} &\xrightarrow{\varphi_1} \mathscr{W}_{\mathbf{G}(r,\mathscr{W})\times_B \mathbf{G}} \twoheadrightarrow \mathrm{pr}_{\mathscr{W}}^* \mathscr{Q}_{\mathbf{G}(r,\mathscr{W})}
\end{aligned}$$

of tautological sheaves on the product.

In other words, $\mathbf{H}$ is the subscheme of the product

$$\mathbf{G}(r,\mathscr{W}) \times_B \mathbf{G}(r,\mathscr{W}_2) \times_B \mathbf{G}(r-c,\mathscr{W}_1) \times_B \mathbf{G}(r-c,\mathscr{V})$$

which is universal for the property that the morphisms φ_1, φ_2, ψ^1, and ψ^2 in the subquotient situation A.3.1 induce a commutative diagram of tautological subbundles

$$\begin{tikzcd}
\mathrm{pr}_{\mathscr{W}}^* \mathscr{S}_{\mathbf{G}(r,\mathscr{W})} \arrow[r, "\psi^2"'] & \mathrm{pr}_{\mathscr{W}_2}^* \mathscr{S}_{\mathbf{G}(r,\mathscr{W}_2)} \\
\mathrm{pr}_{\mathscr{W}_1}^* \mathscr{S}_{\mathbf{G}(r-c,\mathscr{W}_1)} \arrow[u, "\varphi_1"] \arrow[r, "\psi^1"] & \mathrm{pr}_{\mathscr{V}}^* \mathscr{S}_{\mathbf{G}(r-c,\mathscr{V})} \arrow[u, "\varphi_2"']
\end{tikzcd}$$

where $\mathrm{pr}_{\mathscr{W}}$, $\mathrm{pr}_{\mathscr{W}_2}$, $\mathrm{pr}_{\mathscr{W}_1}$, and $\mathrm{pr}_{\mathscr{V}}$ are the projections of $\mathbf{H}$ to the Grassmannian factors in the fibre product. The following describes the structure of $\mathbf{H}$.

A.3.7. Proposition. — *In the above setting, there is a commutative diagram*

$$\begin{tikzcd}
& \mathbf{H} \arrow[dl, "\mathrm{pr}_{\mathscr{W}}"'] \arrow[dr, "\mathrm{pr}_{\mathbf{G}}"] & \\
\mathbf{H}_0 \arrow[rr, dashed, "\varphi_1^{n-r} \times \psi_r^2"] & & \mathbf{G}
\end{tikzcd}$$

of schemes over B. Furthermore:

(i) $\mathrm{pr}_{\mathscr{W}} \colon \mathbf{H} \to \mathbf{H}_0$ *is projective and an isomorphism away from the union of the indeterminacy loci of* φ_1^{n-r} *and* ψ_r^2*, and*

(ii) $\mathrm{pr}_{\mathbf{G}} \colon \mathbf{H} \to \mathbf{G}$ *is the Grassmannian bundle* $\mathbf{G}(c,\bar{\mathscr{W}}) \to \mathbf{G}$*, where* $\bar{\mathscr{W}}$ *is a locally free subquotient of* $\mathscr{W}_{\mathbf{G}}$ *of rank* $n-m+c$ *which fits into a short exact sequence*

$$0 \to \mathrm{pr}_1^* \mathscr{Q}_{\mathbf{G}_1/\mathbf{G}_0} \to \bar{\mathscr{W}} \to \mathrm{pr}_2^* \mathscr{S}_{\mathbf{G}_2/\mathbf{G}_0} \to 0.$$

That the diagram in question exists and is commutative follows from the functorial descriptions of the schemes given in A.3.3 and A.3.6. The remaining statements will be established in pieces below.

A.3.8. — Let $\mathbf{H}_1 := \mathbf{G}(\varphi_1, n-r)$ and $\mathbf{H}_2 := \mathbf{G}(r, \psi^2)$ be the schemes from A.2.6 and A.2.1 which resolve φ_1^{n-r} and ψ_2, respectively. Then the fibre product $\mathbf{H}_1 \times_{\mathbf{H}_0} \mathbf{H}_2$ represents the functor that sends a scheme T over B to the set of diagrams

$$\begin{tikzcd} \mathcal{W}' \arrow[r, "\psi^2"'] & \mathcal{W}_2' \\ \mathcal{W}_1' \arrow[u, "\varphi_1"] & \end{tikzcd} \qquad \text{where} \qquad \begin{array}{rcl} \mathcal{W}' & \in & \mathbf{G}(r, \mathcal{W})(T), \\ \mathcal{W}_2' & \in & \mathbf{G}(r, \mathcal{W}_2)(T), \\ \mathcal{W}_1' & \in & \mathbf{G}(r-c, \mathcal{W}_1)(T), \end{array}$$

consisting of subbundles of $\mathcal{W}_T$, $\mathcal{W}_{1,T}$, and $\mathcal{W}_{2,T}$ of appropriate ranks, and which are compatible under the maps φ_1 and ψ^2. Comparing with A.3.6 shows that $\mathrm{pr}_{\mathcal{W}} \colon \mathbf{H} \to \mathbf{H}_0$ factors as

$$\mathbf{H} \to \mathbf{H}_1 \times_{\mathbf{H}_0} \mathbf{H}_2 \to \mathbf{H}_0.$$

The first map is birational, and the isomorphism locus is as follows:

A.3.9. Lemma. — *The morphism* $\mathbf{H} \to \mathbf{H}_1 \times_{\mathbf{H}_0} \mathbf{H}_2$ *is an isomorphism away from the closed subscheme cut out by the ideals*

$$\mathcal{I}_1 := \mathrm{Fitt}_{r-c}\big(\mathrm{image}(\psi^1 \colon \mathrm{pr}_{\mathcal{W}_1}^* \mathcal{S}_{\mathbf{G}(r-c,\mathcal{W}_1)} \to \mathcal{V}_{\mathbf{H}_1 \times_{\mathbf{H}_0} \mathbf{H}_2})\big),$$
$$\mathcal{I}_2 := \mathrm{Fitt}_{m-r}\big(\mathrm{coker}(\varphi_2 \colon \mathcal{V}_{\mathbf{H}_1 \times_{\mathbf{H}_0} \mathbf{H}_2} \to \mathrm{pr}_{\mathcal{W}_2}^* \mathcal{Q}_{\mathbf{G}(r,\mathcal{W}_2)})\big).$$

Proof. — Let T be a scheme over B and consider a T-point $(\mathcal{W}_1', \mathcal{W}', \mathcal{W}_2', \mathcal{V}')$ of $\mathbf{H}$. In the case the point lies over the complement of $\mathrm{V}(\mathcal{I}_1)$, then $\mathcal{V}'$ is determined as the rank $r-c$ locally free image of $\psi^1 \colon \mathcal{W}_1' \to \mathcal{V}_T$; similarly, if the point were over the complement of $\mathrm{V}(\mathcal{I}_2)$, then $\mathcal{V}'$ is determined as the kernel of $\mathcal{V}_T \to \mathcal{W}_{2,T}/\mathcal{W}_2'$. This shows that $\mathbf{H} \to \mathbf{H}_1 \times_{\mathbf{H}_0} \mathbf{H}_2$ is an isomorphism away from $\mathrm{V}(\mathcal{I}_1) \cap \mathrm{V}(\mathcal{I}_2)$. ■

Proof of A.3.7(i). — It suffices to show that, for every scheme T over B, the T-points of the non-isomorphism locus

$$\big\{ (\mathcal{W}_1', \mathcal{W}', \mathcal{W}_2') \,\big|\, \mathrm{rank}(\psi^1 \colon \mathcal{W}_1' \to \mathcal{V}_T) < r-c,\ \ \mathrm{rank}(\varphi_2 \colon \mathcal{V}_T \to \mathcal{W}_{2,T}/\mathcal{W}_2') < m-r \big\}$$

from A.3.9 maps into the locus

$$\{\, \mathcal{W}' \in \mathbf{H}_0(T) \;\big|\; \operatorname{rank}(\psi^2\colon \mathcal{W}' \to \mathcal{W}_{2,T}) < r \text{ or } \operatorname{rank}(\varphi_1\colon \mathcal{W}_{1,T} \to \mathcal{W}_T/\mathcal{W}') < n-r \,\}$$

of indeterminacy of ψ^2 and φ_1. But when $\mathcal{W}_1' \to \mathcal{V}_T$ does not have full rank, $\mathcal{W}_1'$ intersects the kernel of $\psi^1\colon \mathcal{W}_{1,T} \to \mathcal{V}_T$. Since $\mathcal{W}_1'$ is a subbundle of $\mathcal{W}'$, this implies that $\mathcal{W}'$ intersects the kernel of $\psi^2\colon \mathcal{W}_T \to \mathcal{W}_{2,T}$. Whence $\psi^2\colon \mathcal{W}' \to \mathcal{W}_{2,T}$ does not have full rank. ■

A.3.10. The sheaf $\bar{\mathcal{W}}$. — The sheaf appearing in A.3.7(ii) parameterizes the data of $\mathcal{W}$ that remains upon taking into account a point of $\mathbf{G}$. It is constructed as follows: Pullback the short exact sequence on the top row of A.3.4 along the projection $\mathrm{pr}_{\mathcal{V}}\colon \mathbf{G} \to \mathbf{G}_0$. Juxtaposing the resulting sequence with the tautological bundles arising from the identification A.3.5 of $\mathbf{G}$ as a product of Grassmannian bundles over $\mathbf{G}_0$ yields a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{pr}_{\mathcal{V}}^* \tilde{\mathcal{S}}_{\mathbf{G}} \arrow[r] & \mathcal{W}_{\mathbf{G}} \arrow[r] & \mathrm{pr}_{\mathcal{V}}^* \tilde{\mathcal{Q}}_{\mathbf{G}} \arrow[r] \arrow[d, two heads] & 0 \\
& \mathrm{pr}_1^* \mathcal{S}_{\mathbf{G}_1/\mathbf{G}_0} \arrow[r, hook] \arrow[u, hook] & \mathcal{W}_{\mathbf{G}} \arrow[r, two heads] \arrow[u, equal] & \mathrm{pr}_2^* \mathcal{Q}_{\mathbf{G}_2/\mathbf{G}_0} &
\end{tikzcd}$$

in which the bottom row is a complex which is not necessarily exact; let

$$\bar{\mathcal{W}} := \mathscr{H}(\mathrm{pr}_1^* \mathcal{S}_{\mathbf{G}_1/\mathbf{G}_0} \hookrightarrow \mathcal{W}_{\mathbf{G}} \twoheadrightarrow \mathrm{pr}_2^* \mathcal{Q}_{\mathbf{G}_2/\mathbf{G}_0})$$

be the cohomology sheaf of this complex. Thus $\bar{\mathcal{W}}$ is a subquotient of $\mathcal{W}_{\mathbf{G}}$ and there is an exact commutative diagram given by

$$\begin{tikzcd}
& & \mathrm{pr}_2^* \mathcal{Q}_{\mathbf{G}_2/\mathbf{G}_0} \arrow[r, equal] & \mathrm{pr}_{\mathcal{W}_2}^* \mathcal{Q}_{\mathbf{G}(\mathcal{W}_2, m-r)} & \\
0 \arrow[r] & \mathrm{pr}_1^* \mathcal{S}_{\mathbf{G}_1/\mathbf{G}_0} \arrow[r] & \mathcal{W}_{\mathbf{G}} \arrow[r] \arrow[u, two heads] & \bar{\mathcal{W}}_1 \arrow[r] \arrow[u, two heads] & 0 \\
0 \arrow[r] & \mathrm{pr}_{\mathcal{W}_1}^* \mathcal{S}_{\mathbf{G}(r-c, \mathcal{W}_1)} \arrow[r] \arrow[u, equal] & \bar{\mathcal{W}}_2 \arrow[r] \arrow[u, hook] & \bar{\mathcal{W}} \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$

where tautological bundles are identified as in A.2.4(iii) and A.2.9(iii).

Proof of A.3.7(ii). — That $\bar{\mathscr{W}}$ fits into the claimed short exact sequence follows from taking the cokernel of the left column and taking the kernel of the right column in the first diagram of A.3.10. This also shows that $\bar{\mathscr{W}}$ is locally free of rank $n-m+c$. It remains to identify $\mathbf{H}$ and $\mathbf{G}(c,\bar{\mathscr{W}})$. Construct a morphism $\mathbf{H} \to \mathbf{G}(c,\bar{\mathscr{W}})$ as follows. Consider the pullback to $\mathbf{H}$ of the tautological exact sequence on $\mathbf{G}(r,\mathscr{W})$:

$$0 \to \mathrm{pr}_{\mathscr{W}}^* \mathscr{S}_{\mathbf{G}(r,\mathscr{W})} \to \mathscr{W}_{\mathbf{H}} \to \mathrm{pr}_{\mathscr{W}}^* \mathscr{Q}_{\mathbf{G}(r,\mathscr{W})} \to 0.$$

By the characterization of $\mathbf{H}$ given in A.3.6, the inclusion of $\mathrm{pr}_{\mathscr{W}_1}^* \mathscr{S}_{\mathbf{G}(r-c,\mathscr{W}_1)}$ and the quotient to $\mathrm{pr}_{\mathscr{W}_2}^* \mathscr{Q}_{\mathbf{G}(r,\mathscr{W}_2)}$ from $\mathscr{W}_{\mathbf{H}}$ factor through maps

$$\varphi_1 \colon \mathrm{pr}_{\mathscr{W}_1}^* \mathscr{S}_{\mathbf{G}(r-c,\mathscr{W}_1)} \hookrightarrow \mathrm{pr}_{\mathscr{W}}^* \mathscr{S}_{\mathbf{G}(r,\mathscr{W})} \quad \text{and} \quad \psi^2 \colon \mathrm{pr}_{\mathscr{W}}^* \mathscr{Q}_{\mathbf{G}(r,\mathscr{W})} \twoheadrightarrow \mathrm{pr}_{\mathscr{W}_2}^* \mathscr{Q}_{\mathbf{G}(r,\mathscr{W}_2)}.$$

Comparing with the subquotient diagram of A.3.10 shows that the sheaf

$$\bar{\mathscr{S}} := \mathrm{coker}\big(\varphi_1 \colon \mathrm{pr}_{\mathscr{W}_1}^* \mathscr{S}_{\mathbf{G}(r-c,\mathscr{W}_1)} \hookrightarrow \mathrm{pr}_{\mathscr{W}}^* \mathscr{S}_{\mathbf{G}(r,\mathscr{W})}\big)$$

is a subbundle of $\bar{\mathscr{W}}$ of rank c, thereby defining a morphism $\mathbf{H} \to \mathbf{G}(c,\bar{\mathscr{W}})$.

To construct the inverse, consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \tilde{\mathscr{S}}_{\mathbf{G}(c,\bar{\mathscr{W}})} \arrow[rr] \arrow[dd, two heads] \arrow[dr] & & \mathscr{W}_{\mathbf{G}(c,\bar{\mathscr{W}})} \arrow[rr] \arrow[dr, two heads] & & \tilde{\mathscr{Q}}_{\mathbf{G}(c,\bar{\mathscr{W}})} \arrow[r] & 0 \\
& & \bar{\mathscr{W}}_{2,\mathbf{G}(c,\bar{\mathscr{W}})} \arrow[ur, hook] \arrow[dr] & & \bar{\mathscr{W}}_{1,\mathbf{G}(c,\bar{\mathscr{W}})} \arrow[ur, two heads] & & \\
0 \arrow[r] & \mathscr{S}_{\mathbf{G}(c,\bar{\mathscr{W}})} \arrow[rr] & & \bar{\mathscr{W}}_{\mathbf{G}(c,\bar{\mathscr{W}})} \arrow[rr] \arrow[ur, hook] & & \mathscr{Q}_{\mathbf{G}(c,\bar{\mathscr{W}})} \arrow[r] \arrow[uu, hook] & 0
\end{tikzcd}$$

in which the bottom row is the relative tautological sequence for $\mathbf{G}(c,\bar{\mathscr{W}}) \to \mathbf{G}$, $\tilde{\mathscr{S}}_{\mathbf{G}(c,\bar{\mathscr{W}})}$ is the pullback of the left square, and $\tilde{\mathscr{Q}}_{\mathbf{G}(c,\bar{\mathscr{W}})}$ is the pushout in the right square. Comparing with the subquotient diagram in A.3.10 shows that the composite

$$\tilde{\mathscr{S}}_{\mathbf{G}(c,\bar{\mathscr{W}})} \hookrightarrow \mathscr{W}_{\mathbf{G}(c,\bar{\mathscr{W}})} \twoheadrightarrow \mathrm{pr}_{\mathscr{W}_2}^* \mathscr{Q}_{\mathbf{G}(r,\mathscr{W}_2)}$$

vanishes and that there is a short exact sequence

$$0 \to \mathrm{pr}_{\mathscr{W}_1}^* \mathscr{S}_{\mathbf{G}(r-c,\mathscr{W}_1)} \to \tilde{\mathscr{S}}_{\mathbf{G}(c,\bar{\mathscr{W}})} \to \mathscr{S}_{\mathbf{G}(c,\bar{\mathscr{W}})} \to 0.$$

This implies that $\tilde{\mathscr{S}}_{\mathbf{G}(c,\bar{\mathscr{W}})}$ is a rank r subbundle of $\mathscr{W}_{\mathbf{G}(c,\bar{\mathscr{W}})}$ and it fits into a square

$$\begin{tikzcd}
\tilde{\mathscr{S}}_{\mathbf{G}(c,\bar{\mathscr{W}})} \arrow[r, "\psi^2"'] & \mathrm{pr}^*_{\mathscr{W}_2}\mathscr{S}_{\mathbf{G}(r,\mathscr{W}_2)} \\
\mathrm{pr}^*_{\mathscr{W}_1}\mathscr{S}_{\mathbf{G}(r-c,\mathscr{W}_1)} \arrow[u, "\varphi_1"] \arrow[r, "\psi^1"] & \mathrm{pr}^*_{\mathscr{V}}\mathscr{S}_{\mathbf{G}(r-c,\mathscr{V})} \arrow[u, "\varphi_2"']
\end{tikzcd}$$

where here, $\mathrm{pr}_{\mathscr{W}}$, $\mathrm{pr}_{\mathscr{W}_1}$, $\mathrm{pr}_{\mathscr{W}_1}$, and $\mathrm{pr}_{\mathscr{V}}$ denotes the projection from $\mathbf{G}(c,\bar{\mathscr{W}})$ to the corresponding Grassmannians. Therefore, by the description its description as a subfunctor of the quadruple product from A.3.6, yields a morphism $\mathbf{G}(c,\bar{\mathscr{W}}) \to \mathbf{H}$. That the two morphisms are mutually inverse is because the two constructions described are mutually inverse: some details omitted. ■

A.4. Extensions and projective bundles

An extension $0 \to \mathscr{V}_1 \to \tilde{\mathscr{V}} \to \mathscr{V}_2 \to 0$ of vector bundles on a scheme B induces an interesting vector bundle on any product of Grassmannian bundles of the form $\pi \colon \mathbf{G}(\mathscr{V}_1, s) \times_B \mathbf{G}(r, \mathscr{V}_2) \to B$. Namely, consider the homology sheaf

$$\mathscr{V} := \mathscr{H}(\mathscr{S}_{\pi_1} \hookrightarrow \pi^*\tilde{\mathscr{V}} \twoheadrightarrow \mathscr{Q}_{\pi_2})$$

arising from the given short exact sequence and the tautological bundles on the Grassmannian bundles. This construction arose, for instance, in the Subquotient Situation above, see A.3.7. This Section is concerned with the situation where $\mathscr{V}_i = \mathscr{W}_i \oplus L_i$ for a free $\mathscr{O}_B$-module L_i, and $r = s = 1$ so that π is a product of projective bundles and $\mathscr{V}$ is of rank 2. The hyperplane bundles $\mathscr{W}_i \subset \mathscr{V}_i$ and $\mathscr{Q}_{\pi_1} \subset \mathscr{V}$ give natural affine bundles over B and the goal of this Section is to describe the underlying $\mathscr{O}_B$-algebras. A particularly simple description is possible in the special case when, say, $\mathscr{W}_1$ is of rank 1, and this is given in A.4.8; it is also this case that is used in 4.5.

A.4.1. Setting. — Let B be a scheme and suppose given a short exact sequence

$$0 \to \mathscr{W}_1 \to \mathscr{W} \to \mathscr{W}_2 \to 0$$

of finite locally free $\mathscr{O}_B$-modules. Let L_1 and L_2 be free $\mathscr{O}_B$-modules of rank 1 and set

$$\mathscr{V}_1 := \mathscr{W}_1 \oplus L_1, \quad \mathscr{V}_2 := \mathscr{W}_2 \oplus L_2, \quad \tilde{\mathscr{V}} := \mathscr{W} \oplus L_1 \oplus L_2$$

so that $\mathscr{V}_1$, $\mathscr{V}_2$, and $\tilde{\mathscr{V}}$ still fit into a short exact sequence as above. Let

$$\mathbf{P}_1 := \mathbf{G}(\mathscr{V}_1, 1) \cong \mathbf{P}\mathscr{V}_1^\vee, \quad \mathbf{P}_2 := \mathbf{P}\mathscr{V}_2, \quad \mathbf{P} := \mathbf{P}_1 \times_B \mathbf{P}_2,$$

and let π_1, π_2, and π be their respective projections to B. For $i = 1, 2$, write $\mathscr{S}_{\pi_i}$ and $\mathscr{Q}_{\pi_i}$ for the tautological sub and quotient bundles of $\pi_i \colon \mathbf{P}_i \to B$ pulled up to $\mathbf{P}$. Thus there is a canonical complex of finite locally free $\mathscr{O}_{\mathbf{P}}$-modules

$$\mathscr{S}_{\pi_1} \to \pi^*\tilde{\mathscr{V}} \to \mathscr{Q}_{\pi_2}.$$

This complex is exact in all but the middle, and its homology sheaf $\mathscr{V}$ is a locally free $\mathscr{O}_{\mathbf{P}}$-module of rank 2 that fits into a short exact sequence

$$0 \to \mathscr{Q}_{\pi_1} \to \mathscr{V} \to \mathscr{S}_{\pi_2} \to 0.$$

Set $\mathbf{Q} := \mathbf{P}\mathscr{V}$, let $\rho \colon \mathbf{Q} \to \mathbf{P}$ be the structure morphism, and let $\varphi := \pi \circ \rho \colon \mathbf{Q} \to B$.

Consider now the affine bundles over B given by

$$\mathbf{P}_1^\circ := \mathbf{P}\mathscr{V}_1^\vee \setminus \mathbf{P}\mathscr{W}_1^\vee \cong \mathbf{A}(\mathscr{W}_1^\vee \otimes L_1) \quad \text{and} \quad \mathbf{P}_2^\circ := \mathbf{P}\mathscr{V}_2 \setminus \mathbf{P}\mathscr{W}_2 \cong \mathbf{A}(\mathscr{W}_2 \otimes L_2^\vee)$$

where the identification of the coordinate ring of the bundle comes from **A.1.10**. Set

$$\mathbf{P}^\circ := \mathbf{P}_1^\circ \times_B \mathbf{P}_2^\circ \quad \text{and} \quad \mathbf{Q}^\circ := \mathbf{P}^\circ \times_{\mathbf{P}} (\mathbf{P}\mathscr{V} \setminus \mathbf{P}\mathscr{Q}_{\pi_1}).$$

Then $\rho \colon \mathbf{Q}^\circ \to \mathbf{P}^\circ$ is also a bundle of affine spaces, and the morphism $\varphi \colon \mathbf{Q}^\circ \to B$ is affine. The goal of this Section is describe the sheaves $\mathscr{A} := \pi_*\mathscr{O}_{\mathbf{P}^\circ}$ and $\mathscr{B} := \varphi_*\mathscr{O}_{\mathbf{Q}^\circ}$ of $\mathscr{O}_B$-algebras. Their basic structure is as follows:

A.4.2. Lemma. — *There is a canonical isomorphism of bigraded $\mathscr{O}_B$-algebras*

$$\mathscr{A} \cong \mathrm{Sym}^*(\mathscr{W}_1 \otimes L_1^\vee) \otimes \mathrm{Sym}^*(\mathscr{W}_2^\vee \otimes L_2)$$

in which $L_1^\vee$ has weight $(1,0)$ and L_2 has weight $(0,1)$. The sheaf $\mathscr{B}$ is a bigraded $\mathscr{A}$-algebra with an increasing $\mathbf{Z}_{\geq 0}$ filtration with graded pieces

$$\mathrm{gr}_i\, \mathscr{B} := \mathrm{Fil}_i\, \mathscr{B} / \mathrm{Fil}_{i-1}\, \mathscr{B} \cong \mathscr{A} \otimes (L_1^\vee \otimes L_2)^{\otimes i} \quad \textit{for all } i \in \mathbf{Z}_{\geq 0}.$$

Proof. — By A.1.10 and the discussion of A.4.1, there is a canonical identification

$$\mathbf{P}^\circ \cong \mathbf{A}(\mathscr{W}_1^\vee \otimes L_1) \times_B \mathbf{A}(\mathscr{W}_2 \otimes L_2^\vee)$$

yielding the identification of $\mathscr{A}$. Note that the bigrading may be constructed by considering the $\mathbf{G}_m^2$-equivariant structure induced by a weight $(-1,0)$ action on L_1, a weight $(0,1)$ action on L_2, and trivial action on $\mathscr{W}_1$, $\mathscr{W}_2$, and B.

As for the sheaf $\mathscr{B}$, that it is an $\mathscr{A}$-algebra is due to the factorization $\varphi = \pi \circ \rho$. For the filtration, consider first the affine bundle $\mathbf{P}\mathscr{V}^\circ := \mathbf{P}\mathscr{V} \setminus \mathbf{P}\mathscr{Q}_{\pi_1}$ over $\mathbf{P}$. By A.1.9,

$$\rho_* \mathscr{O}_{\mathbf{P}\mathscr{V}^\circ} \cong \operatorname{colim}_n \operatorname{Sym}^n(\mathscr{V}^\vee \otimes \mathscr{S}_{\pi_2}).$$

There is a two step filtration on $\mathscr{V}^\vee \otimes \mathscr{S}_{\pi_2}$ induced by the short exact sequence

$$0 \to \mathscr{O}_{\mathbf{P}} \to \mathscr{V}^\vee \otimes \mathscr{S}_{\pi_2} \to \mathscr{Q}_{\pi_1}^\vee \otimes \mathscr{S}_{\pi_2} \to 0.$$

This induces an $n+1$ step filtration on each $\operatorname{Sym}^n(\mathscr{V}^\vee \otimes \mathscr{S}_{\pi_2})$. These filtrations are compatible with the maps in the colimit, so there is an induced filtration on $\rho_* \mathscr{O}_{\mathbf{P}\mathscr{V}^\circ}$ and it satisfies

$$\operatorname{gr}_i(\rho_* \mathscr{O}_{\mathbf{P}\mathscr{V}^\circ}) \cong (\mathscr{Q}_{\pi_1}^\vee \otimes \mathscr{S}_{\pi_2})^{\otimes i} \quad \text{for all } i \in \mathbf{Z}_{\geq 0}.$$

Restricting to $\mathbf{P}^\circ$ and making the identifications $\mathscr{Q}_{\pi_1}^\vee|_{\mathbf{P}^\circ} \cong \pi^* L_1^\vee$ and $\mathscr{S}_{\pi_2}|_{\mathbf{P}^\circ} \cong \pi^* L_2$ via A.1.11 then completes the statements regarding $\mathscr{B}$. ■

The main goal is to determine the components of the bigraded decompositions

$$\mathscr{A} = \bigoplus\nolimits_{a,b \in \mathbf{Z}_{\geq 0}} \mathscr{A}_{(a,b)} \otimes L_1^{\vee,\otimes a} \otimes L_2^{\otimes b} \quad \text{and} \quad \mathscr{B} = \bigoplus\nolimits_{a,b \in \mathbf{Z}_{\geq 0}} \mathscr{B}_{(a,b)} \otimes L_1^{\vee,\otimes a} \otimes L_2^{\otimes b}.$$

The description of $\mathscr{A}$ given in A.4.2 already gives:

A.4.3. Corollary. — *For each $a, b \in \mathbf{Z}_{\geq 0}$, there is a canonical isomorphism*

$$\mathscr{A}_{(a,b)} \cong \operatorname{Sym}^a(\mathscr{W}_1) \otimes \operatorname{Sym}^b(\mathscr{W}_2^\vee). \qquad ■$$

The pieces of $\mathscr{B}$ are more complicated. To begin, note that the filtration of $\mathscr{B}$ restricts to a filtration on each $\mathscr{B}_{(a,b)}$. Their graded pieces are as follows:

A.4.4. Lemma. — *For every $a, b, i \in \mathbf{Z}_{\geq 0}$, there is a canonical isomorphism*

$$\mathrm{gr}_i\, \mathscr{B}_{(a,b)} \cong \begin{cases} \mathrm{Sym}^{a-i}(\mathscr{W}_1) \otimes \mathrm{Sym}^{b-i}(\mathscr{W}_2^\vee) & \text{if } 0 \leq i \leq \min(a,b), \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. — For each $i \in \mathbf{Z}_{\geq 0}$, A.4.2 gives an isomorphism $\mathrm{gr}_i\, \mathscr{B}_{(a,b)} \cong \mathscr{A}_{(a-i,b-i)}$, from which the result follows from A.4.3. ∎

In particular, this implies that, for all $a, b \in \mathbf{Z}_{\geq 0}$,

$$\mathscr{B}_{(a,0)} \cong \mathscr{A}_{(a,0)} \cong \mathrm{Sym}^a(\mathscr{W}_1) \quad \text{and} \quad \mathscr{B}_{(0,b)} \cong \mathscr{A}_{(0,b)} \cong \mathrm{Sym}^b(\mathscr{W}_2^\vee).$$

Next, observe that $\mathscr{B}_{(a,b)}$ can always be related to a diagonal bigraded piece:

A.4.5. Lemma. — *For each $a, b \in \mathbf{Z}_{\geq 0}$ with $a \leq b$, the maps*

$$\mathscr{B}_{(a,a)} \otimes \mathscr{B}_{(0,b-a)} \to \mathscr{B}_{(a,b)}, \quad \mathscr{B}_{(a,a)} \otimes \mathscr{B}_{(b-a,0)} \to \mathscr{B}_{(b,a)}, \quad \mathrm{Sym}^a(\mathscr{B}_{(1,1)}) \to \mathscr{B}_{(a,a)}$$

induced by multiplication are strict surjections of filtered $\mathscr{O}_B$-modules.

Proof. — It follows from A.4.2 that $\mathscr{B}$ is generated in degrees $(1,0)$, $(0,1)$, and $(1,1)$. This implies that each of the maps induced by multiplication are surjective. Since multiplication is compatible with the filtration, they are strict. ∎

The following gives a sort of dual to the presentation of $\mathscr{B}_{(a,b)}$ in A.4.5:

A.4.6. Lemma. — *For each $a, b \in \mathbf{Z}_{\geq 0}$ with $a \leq b$, the maps*

$$\mathscr{B}_{(a,b)} \otimes \mathscr{B}_{(b-a,0)} \to \mathscr{B}_{(b,b)} \quad \text{and} \quad \mathscr{B}_{(b,a)} \otimes \mathscr{B}_{(0,b-a)} \to \mathscr{B}_{(b,b)}$$

are strict surjections onto $\mathrm{Fil}_a\, \mathscr{B}_{(b,b)} \subseteq \mathscr{B}_{(b,b)}$.

Proof. — Since the filtration of $\mathscr{B}_{(a,b)}$ has $a+1$ steps and since multiplication is compatible with the filtration, the multiplication maps factor through $\mathrm{Fil}_a\, \mathscr{B}_{(b,b)}$. To see that they are surjective, consider the induced maps on associated graded pieces. Using A.4.4, the map induced by $\mathscr{B}_{(a,b)} \otimes \mathscr{B}_{(b-a,0)} \to \mathscr{B}_{(b,b)}$ on the i-th associated graded piece is the multiplication map

$$\mathrm{Sym}^{a-i}(\mathscr{W}_1) \otimes \mathrm{Sym}^{b-i}(\mathscr{W}_2^\vee) \otimes \mathrm{Sym}^{b-a}(\mathscr{W}_1) \to \mathrm{Sym}^{b-i}(\mathscr{W}_1) \otimes \mathrm{Sym}^{b-i}(\mathscr{W}_2^\vee).$$

These are surjective for each $0 \leq i \leq a$, showing that $\mathscr{B}_{(a,b)} \otimes \mathscr{B}_{(b-a,0)} \to \mathscr{B}_{(b,b)}$ surjects onto $\mathrm{Fil}_a \mathscr{B}_{(b,b)}$. Similarly, the map induced by $\mathscr{B}_{(b,a)} \otimes \mathscr{B}_{(0,b-a)} \to \mathscr{B}_{(b,b)}$ on the i-th associated graded piece is the multiplication map

$$\mathrm{Sym}^{b-i}(\mathscr{W}_1) \otimes \mathrm{Sym}^{a-i}(\mathscr{W}_2^\vee) \otimes \mathrm{Sym}^{b-a}(\mathscr{W}_2^\vee) \to \mathrm{Sym}^{b-i}(\mathscr{W}_1) \otimes \mathrm{Sym}^{b-i}(\mathscr{W}_2^\vee)$$

and so this also surjects onto $\mathrm{Fil}_a \mathscr{B}_{(b,b)}$. ■

The crucial point is to determine $\mathscr{B}_{(1,1)}$. This is done as follows:

A.4.7. Proposition. — *The filtration on $\mathscr{B}_{(1,1)}$ gives an exact sequence of $\mathscr{O}_B$-modules*

$$0 \to \mathscr{W}_1 \otimes \mathscr{W}_2^\vee \to \mathscr{B}_{(1,1)} \to \mathscr{O}_B \to 0$$

whose extension class is sent to the extension class of the given sequence

$$0 \to \mathscr{W}_1 \to \mathscr{W} \to \mathscr{W}_2 \to 0$$

under the identification $\mathrm{Ext}^1_B(\mathscr{W}_2, \mathscr{W}_1) \cong \mathrm{Ext}^1_B(\mathscr{O}_B, \mathscr{W}_1 \otimes \mathscr{W}_2^\vee)$.

Proof. — That the filtration on $\mathscr{B}_{(1,1)}$ gives the claimed sequence follows from A.4.4. It remains to identify the extension class. Note that $\mathscr{B}_{(1,1)}$ is contained in

$$\mathrm{Fil}_1 \mathscr{B} \cong \pi_*(\mathrm{Fil}_1 \rho_* \mathscr{O}_{\mathbf{P}\mathscr{V}^\circ}|_{\mathbf{P}^\circ}) \cong \pi_*(\mathscr{V}^\vee \otimes \mathscr{S}_{\pi_2}|_{\mathbf{P}^\circ}) \cong \mathrm{colim}_n\, \pi_*(\mathscr{V}^\vee \otimes \mathscr{S}_{\pi_2} \otimes \mathscr{L}^{\otimes n})$$

where $\mathscr{L} := (\mathscr{Q}_{\pi_1} \otimes \pi^* L_1^\vee) \otimes (\mathscr{S}_{\pi_2}^\vee \otimes \pi^* L_2)$ is the invertible $\mathscr{O}_{\mathbf{P}}$-module corresponding to the Cartier divisor $\mathbf{P} \setminus \mathbf{P}^\circ$, and where the second and third isomorphism follow from A.1.9. In fact, the colimit is compatible with the total degree of $L_1^\vee$ and L_2, so $\mathscr{B}_{(1,1)}$ is a summand of the $n = 1$ piece of the colimit. The sheaf being pushed along π sits in the short exact sequence obtained from that of $\mathscr{V}$,

$$0 \to \mathscr{Q}_{\pi_1} \otimes \mathscr{S}_{\pi_2}^\vee \to \mathscr{V}^\vee \otimes \mathscr{Q}_{\pi_1} \to \mathscr{O}_{\mathbf{P}} \to 0$$

with an additional twist by $\pi^*(L_1^\vee \otimes L_2)$. The sheaf $\mathscr{B}_{(1,1)}$ and its filtration is now obtained by taking the weight $(1,1)$ component of the direct image of this sequence.

The local-to-global spectral sequence for Ext-groups together with the Leray spectral sequence gives a canonical isomorphism

$$\mathrm{Ext}^1_{\mathbf{P}}(\mathcal{O}_{\mathbf{P}}, \mathcal{Q}_{\pi_1} \otimes \mathcal{S}^\vee_{\pi_2}) \cong \mathrm{Ext}^1_B(\mathcal{O}_B, \mathcal{W}_1 \otimes \mathcal{W}_2^\vee)$$

such that the extension class of $\mathcal{V}^\vee \otimes \mathcal{Q}_{\pi_1}$ corresponds to that of $\mathcal{B}_{(1,1)}$.

Next, since the short exact sequence for $\mathcal{V}^\vee \otimes \mathcal{Q}_{\pi_1}$ is obtained from that of $\mathcal{V}$ by taking duals and twisting by $\mathcal{Q}_{\pi_1}$, their classes correspond under the isomorphism

$$(-)^\vee \otimes \mathcal{Q}_{\pi_1} \colon \mathrm{Ext}^1_{\mathbf{P}}(\mathcal{O}_{\mathbf{P}}, \mathcal{Q}_{\pi_1} \otimes \mathcal{S}^\vee_{\pi_2}) \cong \mathrm{Ext}^1_{\mathbf{P}}(\mathcal{S}_{\pi_2}, \mathcal{Q}_{\pi_1}).$$

Using the tautological short exact sequences and that

$$\pi_*\mathcal{S}_{\pi_1} = \pi_*\mathcal{Q}_{\pi_2} = 0, \quad \pi_*\mathcal{Q}_{\pi_1} \cong \mathcal{W}_1, \quad \pi_*\mathcal{S}^\vee_{\pi_2} \cong \mathcal{W}_2^\vee,$$

it follows that the natural map $\mathrm{Ext}^1_{\mathbf{P}}(\pi^*\mathcal{W}_2, \pi^*\mathcal{W}_1) \to \mathrm{Ext}^1_{\mathbf{P}}(\mathcal{S}_{\pi_2}, \mathcal{Q}_{\pi_1})$ given by

$$(\pi^*\mathcal{W}_2 \xrightarrow{\xi} \pi^*\mathcal{W}_1[1]) \mapsto (\mathcal{S}_{\pi_2} \subset \pi^*\mathcal{W}_2 \xrightarrow{\xi} \pi^*\mathcal{W}_1[1] \twoheadrightarrow \mathcal{Q}_{\pi_1}[1])$$

is an isomorphism.

Finally, the local-to-global spectral sequence for Ext-groups and the Leray spectral sequence give canonical and compatible isomorphisms

$$\mathrm{Ext}^1_B(\mathcal{W}_2, \mathcal{W}_1) \cong \mathrm{Ext}^1_{\mathbf{P}}(\pi^*\mathcal{W}_2, \pi^*\mathcal{W}_1) \cong \mathrm{Ext}^1_{\mathbf{P}}(\mathcal{S}_{\pi_2}, \mathcal{Q}_{\pi_1})$$

such that the classes of $\mathcal{W}$, $\pi^*\mathcal{W}$, and $\mathcal{V}$ correspond to one another. With the identifications above, this shows that the extension class of $\mathcal{B}_{(1,1)}$ corresponds with that of $\mathcal{V}$, as claimed. ■

In the case when either $\mathcal{W}_1$ or $\mathcal{W}_2$ in the setting **A.4.1** is of rank 1, all the pieces of $\mathcal{B}$ can be described quite explicitly. The following does so for when $\mathcal{W}_1$ is of rank 1:

A.4.8. Proposition. — *In the setting of* **A.4.1***, assume furthermore that $\mathcal{W}_1$ is of rank* 1. *Then there are isomorphisms of filtered $\mathcal{O}_B$-modules*

$$\mathcal{B}_{(a,b)} \cong \mathrm{Fil}_a \operatorname{Sym}^b(\mathcal{W}^\vee \otimes \mathcal{W}_1) \otimes \mathcal{W}_1^{\otimes a-b} \quad \text{for all } a, b \in \mathbf{Z}_{\geq 0}.$$

Proof. — First, when $\mathscr{W}_1$ is of rank 1, A.4.4 implies that $\mathscr{B}_{(1,1)} \cong \mathscr{W}^\vee \otimes \mathscr{W}_1$. Second, observe that the surjection $\mathrm{Sym}^a(\mathscr{B}_{(1,1)}) \to \mathscr{B}_{(a,a)}$ from A.4.5 is an isomorphism for all $a \in \mathbf{Z}_{\geq 0}$. Indeed, the induced map on the i-th associated graded piece is the canonical map

$$\mathrm{Sym}^{a-i}(\mathscr{W}_1 \otimes \mathscr{W}_2^\vee) \to \mathrm{Sym}^{a-i}(\mathscr{W}_1) \otimes \mathrm{Sym}^{a-i}(\mathscr{W}_2^\vee) \quad \text{for each } 0 \leq i \leq a,$$

where the pieces are identified via A.4.7 and A.4.4. Since $\mathscr{W}_1$ is invertible, this is an isomorphism. Third, observe that multiplication by $\mathscr{B}_{(d,0)} = \mathscr{W}_1^{\otimes d}$ is injective for any $d \in \mathbf{Z}_{\geq 0}$. This is because $\mathscr{B}$ is locally a polynomial algebra and $\mathscr{B}_{(d,0)}$ is generated by a single monomial. When $a \leq b$, this together with A.4.6 implies that the multiplication map $\mathscr{B}_{(a,b)} \otimes \mathscr{B}_{(b-a,0)} \to \mathscr{B}_{(b,b)}$ is an isomorphism onto $\mathrm{Fil}_a \mathscr{B}_{(b,b)}$. Therefore

$$\mathscr{B}_{(a,b)} \cong \mathrm{Fil}_a \mathscr{B}_{(b,b)} \otimes \mathscr{B}_{(b-a,0)}^\vee \cong \mathrm{Fil}_a \mathrm{Sym}^b(\mathscr{W}^\vee \otimes \mathscr{W}_1) \otimes \mathscr{W}_1^{\otimes a-b}.$$

Similarly, when $a \geq b$, injectivity of multiplication together with A.4.5 implies that the multiplication map $\mathscr{B}_{(b,b)} \otimes \mathscr{B}_{(a-b,0)} \to \mathscr{B}_{(a,b)}$ is an isomorphism, showing

$$\mathscr{B}_{(a,b)} \cong \mathscr{B}_{(b,b)} \otimes \mathscr{B}_{(a-b,0)} \cong \mathrm{Sym}^b(\mathscr{W}^\vee \otimes \mathscr{W}_1) \otimes \mathscr{W}_1^{\otimes a-b}.$$

Since $a \geq b$, this coincides with the a-th filtered piece, whence the result. ∎

Appendix B

Representation Theory Computations

This Appendix collects some facts and computations involving the positive characteristic representation theory of the algebraic group SL_n and the finite special unitary groups $\mathrm{SU}_n(q)$. The primary references are [Jan03, Hum06].

B.1. Setting

Throughout this Appendix, $\mathbf{k}$ denotes a field of positive characteristic $p > 0$, V is a 3-dimensional vector space over $\mathbf{k}$, and $\mathbf{SL}_3 = \mathbf{SL}(V)$ is the special linear group of automorphisms on V.

B.1.1. Root data. — Choose a maximal torus and a Borel subgroup $\mathbf{T} \subset \mathbf{B} \subset \mathbf{SL}_3$. Let

$$\begin{aligned} \mathrm{X}(\mathbf{T}) &:= \mathrm{Hom}(\mathbf{T}, \mathbf{G}_m) \cong \mathbf{Z}\{\epsilon_1, \epsilon_2, \epsilon_3\}/(\epsilon_1 + \epsilon_2 + \epsilon_3) \\ \mathrm{X}^\vee(\mathbf{T}) &:= \mathrm{Hom}(\mathbf{G}_m, \mathbf{T}) \cong \left\{ a_1\epsilon_1^\vee + a_2\epsilon_2^\vee + a_3\epsilon_3^\vee \in \mathbf{Z}\{\epsilon_1^\vee, \epsilon_2^\vee, \epsilon_3^\vee\} \;\middle|\; a_1 + a_2 + a_3 = 0 \right\} \end{aligned}$$

be the lattices of characters and cocharacters of $\mathbf{T}$; here, upon conjugating $\mathbf{T}$ to the diagonal matrices in $\mathbf{SL}_3$, the characters ϵ_i extract the i-th diagonal entry, whereas the cocharacters $\epsilon_i^\vee$ include into the i-th diagonal entry. Let

$$\langle -, - \rangle : \mathrm{X}(\mathbf{T}) \times \mathrm{X}^\vee(\mathbf{T}) \to \mathrm{Hom}(\mathbf{G}_m, \mathbf{G}_m) \cong \mathbf{Z}$$

be the natural root pairing, so that $\langle \epsilon_i, \epsilon_j^\vee \rangle = \delta_{ij}$. Let

$$\alpha_1 := \epsilon_1 - \epsilon_2, \;\; \alpha_2 := \epsilon_2 - \epsilon_3, \quad \text{and} \quad \alpha_1^\vee := \epsilon_1^\vee - \epsilon_2^\vee, \;\; \alpha_2^\vee := \epsilon_2^\vee - \epsilon_3^\vee,$$

be the simple roots and coroots corresponding to the choice of $\mathbf{B}$, so that the positive roots are $\Phi^+ := \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$. Let

$$\varpi_1 := \epsilon_1 \quad \text{and} \quad \varpi_2 := \epsilon_1 + \epsilon_2$$

be the fundamental weights, characterized as the dual basis to $\{\alpha_1^\vee, \alpha_2^\vee\}$ under the character pairing. In particular, the half sum of all positive roots is given by $\rho = \varpi_1 + \varpi_2$. The fundamental weights form a basis of the weight lattice and their nonnegative combinations form the set of dominant weights, the set of which is denoted by

$$\mathrm{X}_+(\mathbf{T}) = \left\{ a\varpi_1 + b\varpi_2 \in \mathrm{X}(\mathbf{T}) \;\middle|\; a, b \in \mathbf{Z}_{\geq 0} \right\}.$$

Highest weight theory gives a bijection between the set of simple representations of $\mathbf{SL}_3$ and the set of dominant weights; the simple representation corresponding to the dominant weight $a\varpi_1 + b\varpi_2 \in \mathrm{X}_+(\mathbf{T})$ is denoted by $L(a,b)$.

B.1.2. Flag variety. — Let $\mathrm{Flag}(V) \cong \mathbf{SL}_3/\mathbf{B}$ be the full flag variety of V. As V is 3-dimensional, the wedge product pairing yields an isomorphism $\wedge^2 V \cong V^\vee$ as $\mathbf{SL}_3$ modules. Therefore the Plücker embedding exhibits $\mathrm{Flag}(V)$ as the point-line incidence correspondence in $\mathbf{P}V \times \mathbf{P}V^\vee$; this is a divisor of bidegree $(1,1)$. Thus

$$\mathrm{Pic}(\mathrm{Flag}(V)) = \left\{ \mathscr{O}_{\mathrm{Flag}(V)}(a,b) := \mathscr{O}_{\mathbf{P}V}(a) \boxtimes \mathscr{O}_{\mathbf{P}V^\vee}(b)|_{\mathrm{Flag}(V)} \;\middle|\; a, b \in \mathbf{Z} \right\} \cong \mathbf{Z}^{\oplus 2}.$$

B.1.3. Lemma. — *For $a, b \in \mathbf{Z}$,*

$$\mathrm{pr}_{\mathbf{P}V,*}(\mathscr{O}_{\mathrm{Flag}(V)}(a,b)) = \begin{cases} \mathrm{Sym}^b(\mathscr{T}_{\mathbf{P}V}(-1)) \otimes \mathscr{O}_{\mathbf{P}V}(a) & \text{if } b \geq 0, \\ 0 & \text{if } b < 0. \end{cases}$$

Proof. — Write $\mathscr{O}(a,b) := \mathscr{O}_{\mathbf{P}V}(a) \boxtimes \mathscr{O}_{\mathbf{P}V^\vee}(b)$ for the line bundle on $\mathbf{P}V \times \mathbf{P}V^\vee$. Then $\mathrm{pr}_{\mathbf{P}V,*}\,\mathscr{O}(a,b)$ vanishes if $b < 0$ and is $\mathrm{Sym}^b(V) \otimes \mathscr{O}_{\mathbf{P}V}(a)$ otherwise. Since $\mathrm{Flag}(V)$ defined in $\mathbf{P}V \times \mathbf{P}V^\vee$ by the trace section of $\mathrm{H}^0(\mathbf{P}V \times \mathbf{P}V^\vee, \mathscr{O}(1,1)) = V^\vee \otimes V$,

$$\mathrm{pr}_{\mathbf{P}V,*}\,\mathscr{O}_{\mathrm{Flag}(V)}(a,b) \cong \mathrm{coker}\big(\mathrm{pr}_{\mathbf{P}V,*}\,\mathscr{O}(a-1,b-1) \to \mathrm{pr}_{\mathbf{P}V,*}\,\mathscr{O}(a,b)\big).$$

This vanishes when $b < 0$, and otherwise yields

$$\mathrm{pr}_{\mathbf{P}V,*}\,\mathscr{O}_{\mathrm{Flag}(V)}(a,b) \cong \mathrm{coker}\big(\mathrm{Sym}^{b-1}(V) \otimes \mathscr{O}_{\mathbf{P}V}(a-1) \to \mathrm{Sym}^b(V) \otimes \mathscr{O}_{\mathbf{P}V}(a)\big).$$

The map in the cokernel arises by applying Sym^b to the Euler sequence

$$0 \to \mathcal{O}_{\mathbf{P}V}(-1) \to V \otimes \mathcal{O}_{\mathbf{P}V} \to \mathcal{T}_{\mathbf{P}V}(-1) \to 0$$

together with a twist by $\mathcal{O}_{\mathbf{P}V}(a)$. This gives the result. ∎

B.1.4. — There is an isomorphism of abelian groups $\mathrm{X}(\mathbf{T}) \to \mathrm{Pic}(\mathrm{Flag}(V))$ given by

$$a\varpi_1 + b\varpi_2 \mapsto \mathcal{O}_{\mathrm{Flag}(V)}(a\varpi_1 + b\varpi_2) := \mathcal{O}_{\mathrm{Flag}(V)}(a,b).$$

Following the conventions of [Jan03, II.2.13(1)], the *Weyl module* corresponding to a dominant weight $a\varpi_1 + b\varpi_2 \in \mathrm{X}_+(\mathbf{T})$ is

$$\Delta(a,b) := \mathrm{H}^0(\mathrm{Flag}(V), \mathcal{O}_{\mathrm{Flag}(V)}(b,a))^\vee \cong \mathrm{H}^0(\mathbf{P}V, \mathrm{Sym}^a(\mathcal{T}_{\mathbf{P}V}(-1)) \otimes \mathcal{O}_{\mathbf{P}V}(b))^\vee$$

the isomorphism due to B.1.3. For example, $\Delta(a,0) = \mathrm{Sym}^a(V)^\vee$ and $\Delta(0,b) = \mathrm{Div}^b(V)$. Here, $\mathrm{Div}^b(V)$ is the b-th divided power of V, and is defined to be the subspace of symmetric tensors in $V^{\otimes b}$, and satisfies $\mathrm{Div}^b(V) \cong \mathrm{Sym}^b(V^\vee)^\vee$.

B.2. Borel–Weil–Bott Theorem

Let $\mathbf{G}$ be a reductive linear algebraic group and let $\mathbf{P}$ be a parabolic subgroup. The classical Borel–Weil–Bott Theorem of [Bot57, Dem68] determines the cohomology of homogeneous vector bundles on the projective variety $\mathbf{G}/\mathbf{P}$ associated with completely reducible representations of $\mathbf{P}$. The matter is rather subtle in positive characteristic: see [Jan03, II.5.5]. Remarkably, in the case $\mathbf{G} = \mathbf{SL}_3$, a complete answer can be given.

B.2.1. Definition. — Let $\lambda := a\varpi_1 + b\varpi_2 \in \mathrm{X}(\mathbf{T})$ be a weight.

(i) The weight λ is *singular* if either $a = -1$ or $b = -1$ or $a + b = -2$.

(ii) If λ is not singular, then its *index* $\mathrm{idx}(\lambda)$ is the number of negative integers in the set $\{a+1, b+1, a+b+2\}$.

(iii) The weight λ is said to *satisfy BWB* if

$$\mathbf{R}\Gamma(\mathrm{Flag}(V), \mathcal{O}_{\mathrm{Flag}(V)}(\lambda)) = \begin{cases} 0 & \text{if } \lambda \text{ is singular, and} \\ \mathrm{H}^{\mathrm{idx}(\lambda)}(\mathrm{Flag}(V), \mathcal{O}_{\mathrm{Flag}(V)}(\lambda)) & \text{if } \lambda \text{ is not singular.} \end{cases}$$

(iv) The weight λ is said to be in the *Griffith region* if $a+1$ and $b+1$ have opposite signs, and there are positive integers ν and $m<p$ such that

$$mp^{\nu} < |a+1|, |b+1| < (m+1)p^{\nu}.$$

In general, Kempf's Theorem [Kem76, Theorem 1 on p.586] shows that any dominant weight satisfies BWB; by Serre duality, any anti-dominant weight also satisfies BWB. Taking this into account, Griffith was able to completely classify those weights which satisfy BWB:

B.2.2. Theorem ([Gri80, Theorem 1.3]). — *A weight $\lambda \in \mathrm{X}(\mathbf{T})$ satisfies BWB if and only if λ is not in the Griffith region.* ∎

This allows one to compute cohomology of certain homogeneous bundles on $\mathbf{P}V = \mathbf{P}^2$. Of particular use will be:

B.2.3. Corollary. — *Let $0 \leq b \leq p-1$. Then*

(i) $\mathrm{H}^0(\mathbf{P}^2, \mathrm{Sym}^b(\mathscr{T}_{\mathbf{P}^2}(-1))(a)) = 0$ *whenever* $a<0$*, and*

(ii) $\mathrm{H}^1(\mathbf{P}^2, \mathrm{Sym}^b(\mathscr{T}_{\mathbf{P}^2}(-1))(a)) = 0$ *whenever* $a<p$.

Proof. — Consider the weight $\lambda = a\varpi_1 + b\varpi_2$. Since $0 \leq b \leq p-1$, there can be no positive integers ν and $m<p$ such that $mp^{\nu} < b+1 < (m+1)p^{\nu}$. Therefore such λ is never in the Griffith region, and so it satisfies BWB by B.2.2. If $a<0$, then either $a=-1$ and λ is singular, or $a+1<0$ and the index of λ is at least 1. If, furthermore, $a<p$, then either $a+b=-2$ when $a=-p-1$ and $b=p-1$ and λ is singular, or else $a+b+2<0$ and the index of λ is 2. Since, by B.1.3,

$$\mathbf{R}\Gamma(\mathbf{P}^2, \mathrm{Sym}^b(\mathscr{T}_{\mathbf{P}^2}(-1))(a)) = \mathbf{R}\Gamma(\mathrm{Flag}(V), \mathscr{O}_{\mathrm{Flag}(V)}(a,b))$$

this gives the result. ∎

B.3. Some simple modules

The Weyl modules $\Delta(\lambda)$, as defined in B.1.4, are generally not irreducible representations in positive characteristic. Their simple composition factors can sometimes be described using Jantzen's filtration together with a remarkable nonnegative sum formula, as described in [Jan03, II.8.19]. This is recalled in the following, and will be used to determine some simple representations of $\mathbf{SL}_3$. The Section ends with some comments as to how these results apply to representations of the finite special unitary group $\mathrm{SU}_3(p)$.

B.3.1. Jantzen filtration and sum formula. — Given a dominant weight $\lambda \in \mathrm{X}_+(\mathbf{T})$, the Jantzen filtration is a decreasing filtration

$$\Delta(\lambda) = \Delta(\lambda)^0 \supseteq \Delta(\lambda)^1 \supseteq \Delta(\lambda)^2 \supseteq \cdots$$

such that $L(\lambda) = \Delta(\lambda)/\Delta(\lambda)^1$. Furthermore, there is the *sum formula*:

$$\sum\nolimits_{i>0} \mathrm{ch}(\Delta(\lambda)^i) = \sum\nolimits_{\alpha\in\Phi^+} \sum\nolimits_{m:0<mp<\langle\lambda+\rho,\alpha^\vee\rangle} \nu_p(mp)\chi(s_{\alpha,mp}\cdot\lambda)$$

where ch extracts the $\mathbf{T}$-character of a module, $\nu_p \colon \mathbf{Z} \to \mathbf{Z}$ is the standard p-adic valuation, $s_{\alpha,mp}$ is the affine reflection on $\mathrm{X}(\mathbf{T})$ given by

$$s_{\alpha,mp}(\lambda) := \lambda + (mp - \langle\lambda,\alpha^\vee\rangle)\alpha,$$

$s_{\alpha,mp}\cdot\lambda := s_{\alpha,mp}(\lambda+\rho)-\rho$ is the dot action, and

$$\chi(\lambda) := \sum\nolimits_{i\geq 0}(-1)^i\big[\mathrm{H}^i(\mathrm{Flag}(V), \mathcal{O}_{\mathrm{Flag}(V)}(\lambda))\big]$$

is the Euler characteristic of $\mathcal{O}_{\mathrm{Flag}(V)}(\lambda)$ with values in the representation ring of $\mathbf{T}$. A simple application of this is:

B.3.2. Lemma. — *If $a, b \in \mathbf{Z}_{\geq 0}$ satisfy $a+b \leq p-2$, then $\Delta(a,b)$ is simple.*

Proof. — Consider the Jantzen filtration of $\Delta(a,b)$. Taking $\lambda = a\varpi_1 + b\varpi_2$, the root pairings appearing in the sum formula are

$$\langle\lambda+\rho,\alpha_1^\vee\rangle = a+1, \quad \langle\lambda+\rho,\alpha_2^\vee\rangle = b+1, \quad \langle\lambda+\rho,\alpha_1^\vee+\alpha_2^\vee\rangle = a+b+2.$$

Since $a+b \leq p-2$, each of these pairings is at most p, so the right hand side of the sum formula is empty. Thus $\Delta(a,b)^1 = 0$ and $\Delta(a,b) = L(a,b)$ is simple. ■

B.3.3. Lemma. — *The Weyl module of highest weight $b\varpi_2$ is $\Delta(0,b) = \mathrm{Div}^b(V)$.*

(i) *If* $0 \leq b \leq p-1$*, then* $\Delta(0,b) = L(0,b)$ *is simple.*

(ii) *If* $p \leq b \leq 2p-3$*, then* $L(0,b) = \mathrm{Fr}^*(V) \otimes \mathrm{Div}^{b-p}(V)$ *and there is a short exact sequence*

$$0 \to L(b-p+1, 2p-2-b) \to \Delta(0,b) \to L(0,b) \to 0.$$

Proof. — That $\Delta(0,b) = \mathrm{Div}^b(V)$ follows immediately from their construction in B.1.4. Case (i) follows from B.3.2. So suppose $p \leq b \leq 2p-3$. The Steinberg Tensor Product Theorem, [Jan03, II.3.17], together with (i) gives

$$L(0,b) = \mathrm{Fr}^*(L(0,1)) \otimes L(0,b-p) = \mathrm{Fr}^*(V) \otimes \mathrm{Div}^{b-p}(V).$$

As for the short exact sequence, consider the Jantzen filtration on $\Delta(0,b)$. The sum formula of B.3.1 has two terms indexed by (α_2, p) and $(\alpha_1+\alpha_2, p)$. The first is

$$\begin{aligned}\chi(s_{\alpha_2,p} \cdot b\varpi_2) &= \chi((b-p+1)\varpi_1 + (2p-2-b)\varpi_2) \\ &= \mathrm{ch}(\Delta(b-p+1, 2p-2-b))\end{aligned}$$

since $(b-p+1)\varpi_1 + (2p-2-b)\varpi_2$ is dominant. To express this Express this in terms of simple characters by considering the Jantzen filtration for $\Delta(b-p+1, 2p-b-2)$; the sum formula contains a single term indexed by $(\alpha_1+\alpha_2, p)$, and

$$s_{\alpha_1+\alpha_2,p} \cdot ((b-p+1)\varpi_1 + (2p-b-2)\varpi_2) = (b-p)\varpi_1 + (2p-3-b)\varpi_2.$$

This final weight is dominant and simple by B.3.2, so

$$\mathrm{ch}(\Delta(b-p+1, 2p-2-b)) = \mathrm{ch}(L(b-p+1, 2p-2-b)) + \mathrm{ch}(L(b-p, 2p-3-b)).$$

Consider the second term in the sum formula for $\Delta(0,b)$, indexed by $(\alpha_1+\alpha_2, p)$. Observe that

$$\begin{aligned}s_{\alpha_1+\alpha_2,p} \cdot b\varpi_2 &= (p-b-2)\varpi_1 + (p-2)\varpi_2 \\ &= s_{\alpha_1} \cdot ((b-p)\varpi_1 + (2p-3-b)\varpi_2).\end{aligned}$$

Therefore, [Jan03, II.5.9] gives the first equality in

$$\begin{aligned}\chi(s_{\alpha_1+\alpha_2,p}\cdot b\varpi_2) &= -\chi((b-p)\varpi_1+(2p-3-b)\varpi_2)\\ &= -\operatorname{ch}(L(b-p,2p-3-b)).\end{aligned}$$

Putting everything together shows that

$$\sum\nolimits_{i>0}\operatorname{ch}(\Delta(0,b)^i) = \operatorname{ch}(L(b-p+1,2p-2-b)).$$

Therefore the only composition factor in $\Delta(0,b)^1$ can be $L(b-p+1,2p-2-b)$, so it is simple and $\Delta(0,b)^2=0$. This gives the exact sequence in (ii). ■

The conclusion of (ii) means that $\ker(\mathrm{Div}^b(V)\to\mathrm{Fr}^*(V)\otimes\mathrm{Div}^{b-p}(V))$ is a simple $\mathbf{SL}_3$ representation when $p\leq b\leq 2p-3$; dually, $\mathrm{Sym}^b(V^\vee)/(\mathrm{Fr}^*(V^\vee)\otimes\mathrm{Sym}^{b-p}(V^\vee))$ is simple for $p\leq b\leq 2p-3$. This can be established in a more elementary way, as done in [Dot85].

B.3.4. Lemma. — *The Weyl module of highest weight $\varpi_1+b\varpi_2$ is given by*

$$\Delta(1,b) := \ker\big(\mathrm{ev}\colon V^\vee\otimes\mathrm{Div}^b(V)\to\mathrm{Div}^{b-1}(V)\big).$$

(i) *If $0\leq b\leq p-3$, then $\Delta(1,b)=L(1,b)$ is simple.*

(ii) *If $b=p-2$, then there is a short exact sequence*

$$0\to L(0,p-3)\to\Delta(1,p-2)\to L(1,p-2)\to 0.$$

Proof. — Its definition in B.1.4 together with the Euler sequence on $\mathbf{P}V$ gives

$$\begin{aligned}\Delta(1,b) &\cong \mathrm{H}^0(\mathbf{P}V,\mathscr{T}_{\mathbf{P}V}(b-1))^\vee\\ &\cong \ker\big(V^\vee\otimes\mathrm{H}^0(\mathbf{P}V,\mathscr{O}_{\mathbf{P}V}(b))^\vee\to\mathrm{H}^0(\mathbf{P}V,\mathscr{O}_{\mathbf{P}V}(b-1))^\vee\big)\\ &\cong \ker\big(\mathrm{ev}\colon V^\vee\otimes\mathrm{Div}^b(V)\to\mathrm{Div}^{b-1}(V)\big).\end{aligned}$$

Simplicity in case (i) follows from B.3.2. So consider case (ii) and consider the Janzten filtration on $\Delta(1,p-2)$. The sum formula of B.3.1 reads

$$\sum\nolimits_{i>0}\operatorname{ch}(\Delta(1,p-2)^i) = \chi(s_{\alpha_1+\alpha_2,p}\cdot(\varpi_1+(p-2)\varpi_2)) = \operatorname{ch}(L(0,p-3)).$$

Thus $L(0,p-3)$ is the only composition factor in $\Delta(1,p-2)^1$, giving (ii). ■

The simple submodule $L(0,p-3) \subset \Delta(1,p-2)$ in B.3.4(ii) can be explicitly identified as follows. Choose a basis $V = \langle x,y,z\rangle$ and let $\partial_x, \partial_y, \partial_z \in V^\vee$ be the associated differential operators. Then the image of the map $\mathrm{Div}^{p-3}(V) \to \mathrm{Div}^{p-2}(V) \otimes V^\vee$ given by

$$f \mapsto xf \otimes \partial_x + yf \otimes \partial_y + zf \otimes \partial_z$$

lies in the kernel of the evaluation map, showing $\mathrm{Div}^{p-3}(V) \cong L(0,p-3)$.

B.3.5. Finite unitary group. — Let (V,β) be a nondegenerate q-bic form over $\mathbf{k}$. The *special unitary group* of β is the finite étale group scheme $\mathrm{SU}(V,\beta) := \mathrm{U}(V,\beta) \cap \mathbf{SL}(V)$ defined as the subgroup of the unitary group, as in 1.3.5, contained in $\mathbf{SL}(V)$. Representations of $\mathbf{SL}(V)$ restrict to representations of $\mathrm{SU}(V,\beta)$, are often irreducible, and all irreducible representations are obtained in this way. In the case at hand:

B.3.6. Theorem. — *Let $0 \leq a,b \leq p-1$. Then the restriction of the simple $\mathbf{SL}_3$-modules $L(a,b)$ to $\mathrm{SU}_3(p)$ remain simple, are pairwise nonisomorphic, and give all isomorphism classes of simple $\mathrm{SU}_3(p)$-modules.*

Proof. — This follows from Steinberg's Restriction Theorem [Ste63]; see also [Hum06, Theorem 2.11]. ∎

By abuse of notation, $L(a,b)$ and $\Delta(a,b)$ will be used to denote the $\mathrm{SU}_3(p)$-modules obtained via restriction of the corresponding $\mathbf{SL}_3$-modules. The Steinberg Restriction Theorem together with B.3.2 and B.3.3 produces some simple modules for $\mathrm{SU}_3(p)$. The next statement identifies a certain $\mathrm{SU}_3(p)$-module as a restriction of a $\mathbf{SL}_3$-module:

B.3.7. Lemma. — *For each $0 \leq b \leq p-1$, the $\mathrm{SU}_3(p)$ representation*

$$\ker(f \colon \mathrm{Fr}^*(V) \otimes \mathrm{Div}^b(V) \xrightarrow{\beta} V^\vee \otimes \mathrm{Div}^b(V) \xrightarrow{\mathrm{ev}} \mathrm{Div}^{b-1}(V))$$

is isomorphic to the restriction of the Weyl module of $\mathbf{SL}_3$ with highest weight $\varpi_1 + b\varpi_2$.

Proof. — By construction, there is a $\mathrm{SU}_3(p)$-equivariant commutative diagram

$$\begin{tikzcd}
\mathrm{Fr}^*(V) \otimes \mathrm{Div}^b(V) \arrow[r, "f"'] \arrow[d, "\beta^\vee"'] & \mathrm{Div}^{b-1}(V) \arrow[d, equal] \\
V^\vee \otimes \mathrm{Div}^b(V) \arrow[r, "\mathrm{ev}"] & \mathrm{Div}^{b-1}(V)\,.
\end{tikzcd}$$

Thus the kernels of the two rows are isomorphic as representations of $\mathrm{SU}_3(p)$. Now the bottom row is equivariant for $\mathbf{SL}_3$, and by B.3.4, its kernel is precisely the Weyl module of weight $\varpi_1 + b\varpi_2$ for $\mathbf{SL}_3$, as claimed. ■