q-BIC THREEFOLDS AND THEIR SURFACE OF LINES

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ABSTRACT. For any power q of the positive ground field characteristic, a smooth q-bic threefold—the Fermat threefold of degree $q+1$ for example—has a smooth surface S of lines which behaves like the Fano surface of a smooth cubic threefold. I develop projective, moduli-theoretic, and degeneration techniques to study the geometry of S. Using, in addition, the modular representation theory of the finite unitary group and the geometric theory of filtrations, I compute cohomology of the structure sheaf of S when q is prime.

INTRODUCTION

The Fermat threefold of degree $q+1$ is the perhaps the most familiar example of a *q-bic threefold*, that is, a hypersurface in projective 4-space of the form

$$X := \left\{ (x_0 : x_1 : x_2 : x_3 : x_4) \in \mathbf{P}^4 : \sum\nolimits_{i,j=0}^{4} a_{ij} x_i^q x_j = 0 \right\}$$

where q is a power of the ground field characteristic $p > 0$. Such hypersurfaces have long been of recurring interest due to, for example: idiosyncrasies in their differential projective geometry [Wal56, Hef85, Bea90, KP91, Nom95]; their abundance of algebraic cycles and relation to supersingularity [Wei49, Tat65, HM78, SK79, Shi01]; and their unirationality and geometry of rational curves [Shi74, Con06, BDE$^+$13, She12]. They classically arise in relation to finite Hermitian geometries [BC66, Seg65, Hir79, Hir85]; the study of unitary Shimura varieties [Vol10, LTX$^+$22, LZ22]; and Deligne–Lusztig theory for finite unitary groups [Lus76, DL76, Han92, Li23]. Such hypersurfaces are distinguished in [KKP$^+$22] from the point of view of singularity theory as those with the lowest possible F-pure threshold. See [Che22, pp. 7–11] for a more extensive survey.

The purpose of this work is to further develop a striking analogy between the geometry of lines in q-bic and cubic hypersurfaces, and also to suggest flexible framework with which to understand some of the resulting phenomena. To explain, let $\mathbf{k}$ be, here and throughout, an algebraically closed field of characteristic $p > 0$, and q a fixed power of p. Let X be a smooth q-bic threefold over $\mathbf{k}$, and write S for its Fano scheme of lines. The main results of [Che23b], specialized to dimension 3, are:

Theorem. — *The scheme S of lines of a smooth q-bic threefold X is an irreducible, smooth, projective surface of general type. The Fano correspondence $S \leftarrow \mathbf{L} \rightarrow X$ induces purely inseparable isogenies*

$$\mathbf{Alb}_S \xrightarrow{\mathbf{L}_*} \mathbf{Ab}^2_X \xrightarrow{\mathbf{L}^*} \mathbf{Pic}^0_{S,\mathrm{red}}$$

amongst supersingular abelian varieties of dimension $\frac{1}{2}q(q-1)(q^2+1)$.

Here, $\mathbf{Ab}^2_X$ is the *intermediate Jacobian* of X, taken to be the algebraic representative for algebraically trivial 1-cycles in X, in the sense of Samuel and Murre, see [Sam60, Bea77, Mur85]; see [Che23b, **6.9**] for a resumé. The statement is analogous to the classical result [CG72, Theorem 11.19] of Clemens and Griffiths, which states that the Hodge-theoretic intermediate Jacobian of a complex cubic threefold is isomorphic to the Albanese variety of its Fano surface of lines. The geometry underlying the proofs also shares many analogies; compare especially with [Huy23, Chapter 5]. Note that when $q = 2$, X is a cubic threefold in characteristic 2, and this gives a version of the theorem of Clemens and Griffiths which is, in fact, new.

The Fano scheme S is the primary object of study in this work, and is henceforth referred to as the *Fano surface* of lines of X. To begin, refining the general techniques developed in [Che23b] gives explicit computations of some basic invariants of S. In the following statement, $\mathcal{S}$ denotes the tautological rank 2 subbundle, and $\mathcal{O}_S(1)$ is the Plücker line bundle on S.

Theorem A. — *The tangent bundle $\mathcal{T}_S$ of the Fano surface S of a smooth q-bic threefold X is isomorphic to $\mathcal{S} \otimes \mathcal{O}_S(2-q)$. Basic numerical invariants of S are*

$$c_1(S)^2 = (q+1)^2(q^2+1)(2q-3)^2, \quad c_2(S) = (q+1)^2(q^4-3q^3+4q^2-4q+3), \text{ and}$$

$$\chi(S, \mathcal{O}_S) = \frac{1}{12}(q+1)^2(5q^4-15q^3+17q^2-16q+12).$$

If $q > 2$, then S lifts neither to the second Witt vectors nor to characteristic 0.

This is an amalgamation of **1.11**, **1.12**, and **1.13**. The identification of $\mathcal{T}_S$ is an analogue of the tangent bundle theorem [CG72, Proposition 12.31] of Clemens and Griffiths, see also [AK77, Theorem 1.10]. That S does not lift to the Witt vectors is because it violates Kodaira–Akizuki–Nakano vanishing, and not to characteristic 0 because it violates the Bogomolov–Miyaoka–Yau inequality $c_1(S)^2 \leq 3c_2(S)$. As such, S is a purely positive characteristic phenomenon, and the analogy with cubics serves as a valuable guide in a setting where classical geometric intuitions are less potent.

Non-liftability also means that few vanishing theorems are available, presenting a major difficulty in coherent cohomology computations. Nonetheless, when q is the prime p itself, cohomology of the structure sheaf $\mathcal{O}_S$ behaves as well as possible:

Theorem B. — *Let X be a smooth q-bic threefold and S its Fano surface. If $q = p$, then*

$$\dim_{\mathbf{k}} \mathrm{H}^1(S, \mathcal{O}_S) = \frac{1}{2}p(p-1)(p^2+1) = \frac{1}{2}\dim_{\mathbf{Q}_\ell} \mathrm{H}^1_{\text{ét}}(S, \mathbf{Q}_\ell), \text{ and}$$
$$\dim_{\mathbf{k}} \mathrm{H}^2(S, \mathcal{O}_S) = \frac{1}{12}p(p-1)(5p^4-2p^2-5p-2).$$

In particular, the Picard scheme of S is smooth.

First cohomology is computed in 7.7, at which point second cohomology is determined by the Euler characteristic computation above. This result, together with the methods developed, provides a first step towards understanding the precise relationship amongst the abelian varieties $\mathbf{Alb}_S$, $\mathbf{Ab}^2_X$, and $\mathbf{Pic}^0_{S,\mathrm{red}} = \mathbf{Pic}^0_S$ appearing in the analogue of the Clemens–Griffiths Theorem quoted above.

The method involves a delicate degeneration argument. General considerations show that the dimension of $\mathrm{H}^1(S, \mathcal{O}_S)$ is at least half the first Betti number. A corresponding upper bound might be obtained via upper semicontinuity as follows: Specialize S to a singular surface S_0 by specializing X to a q-bic threefold X_0 with the mildest possible singularities; precisely, in terms of the classification of [Che23a, **Theorem A**], X_0 is of type $\mathbf{1}^{\oplus 3} \oplus \mathbf{N}_2$, and there is a choice of coordinates such that

$$X_0 = \left\{ (x_0 : x_1 : x_2 : x_3 : x_4) \in \mathbf{P}^4 : x_0^{q+1} + x_1^{q+1} + x_2^{q+1} + x_3^q x_4 = 0 \right\}.$$

Although S_0 is quite singular, its normalization is quite manageable. The result is as follows:

Theorem C. — *The Fano surface S_0 of a q-bic threefold X_0 of type $\mathbf{1}^{\oplus 3} \oplus \mathbf{N}_2$ is nonnormal, and its normalization S_0^ν is a projective bundle over a smooth q-bic curve C. There is a commutative diagram*

$$\begin{array}{ccc} S_0^\nu & \xrightarrow{\nu} & S_0 \\ {\scriptstyle\tilde{\varphi}_+}\downarrow & & \downarrow{\scriptstyle\varphi_-} \\ C & \xrightarrow{\phi_C} & C \end{array}$$

where $\phi_C \colon C \to C$ is conjugate to the q^2-power geometric Frobenius morphism of C. If $q = p$, then

$$\dim_{\mathbf{k}} \mathrm{H}^1(S_0, \mathcal{O}_{S_0}) = \frac{1}{2}p(p-1)(p^2+1) + \frac{1}{6}p(p-1)(p-2), \text{ and}$$
$$\dim_{\mathbf{k}} \mathrm{H}^2(S_0, \mathcal{O}_{S_0}) = \frac{1}{12}p(p-1)(5p^4-2p^2-5p-2) + \frac{1}{6}p(p-1)(p-2).$$

This collects 4.8, 4.13, and 6.12. See also 6.13 for remarks about extending the calculation to general q. The normalization S_0^ν is constructed, roughly, as the space of lines in X_0 that project to a tangent of C. Cohomology of $\mathcal{O}_{S_0}$ is determined by using the diagram to relate it to a vector bundle computation C; using group schemes $\mathbf{G}_m$ and $\boldsymbol{\alpha}_p$ in the automorphism group of S_0 to reduce to a manageable set of computations; and using modular representation theory of the finite unitary group $\mathrm{U}_3(p)$, essentially the automorphism group of C, to make explicit identifications.

Alarmingly, upon comparing Theorems B and C, one finds that the cohomology groups of the special fibre S_0 are much larger than one would hope. Upper semicontinuity therefore only shows

$$\frac{1}{2}p(p-1)(p^2+1) \leq \dim_{\mathbf{k}} \mathrm{H}^1(S, \mathcal{O}_S) \leq \frac{1}{2}p(p-1)(p^2+1) + \frac{1}{6}p(p-1)(p-2).$$

The upper bound can be refined by carefully analysing a specially chosen degeneration $S \rightsquigarrow S_0$. Specifically, there exists a flat family $\mathfrak{S} \to \mathbf{A}^1$ in which all fibres away from the central fibre S_0 are isomorphic to S, and which carries a $\mathbf{G}_m$ action compatible with the weight q^2-1 homothety on the base $\mathbf{A}^1$. The geometric theory of filtrations—the correspondence between filtrations and $\mathbf{G}_m$-equivariant objects over $\mathbf{A}^1$ as pioneered in Simpson's work [Sim91] on nonabelian Hodge theory—endows $\mathrm{H}^1(S, \mathcal{O}_S)$ with a filtration whose graded pieces are various graded pieces of $\mathrm{H}^1(S_0, \mathcal{O}_{S_0})$. A careful analysis identifies $\frac{1}{6}p(p-1)(p-2)$ cohomology classes on S_0 that do *not* lift to classes on S. With this, the upper semicontinuity bound becomes tight, completing the calculation. This use of the geometric theory of filtrations in determining cohomology appears to be new, and I expect that the method can be generalized and refined to be applied in other settings.

Crucial to the degeneration argument is the construction of a fibration $\varphi \colon S \to C$ on the smooth Fano surface that specializes to $\varphi_- \colon S_0 \to C$ as in Theorem C. This is done using a general projective geometry method, which may be adapted to other settings, exploiting the presence of *cone points*: points at which the tangent space intersects the hypersurface X at a cone over a curve C; these generalize *Eckardt points* of cubic hypersurfaces, and are sometimes also called *star points*. A combination of projection and intersection with a hyperplane induces a rational map $S \dashrightarrow C$, which can be resolved in this setting using global methods made possible by the intrinsic theory of q-bic forms developed in [Che23a]. A summary of the geometric situation is as follows:

Theorem D. — *Let X be a q-bic threefold either smooth or of type $\mathbf{1}^{\oplus 3} \oplus \mathbf{N}_2$, and S its Fano surface. A choice of cone point in X over a smooth q-bic curve C induces a canonical diagram*

$$\begin{tikzcd} & \tilde{S} \arrow[dl, "b"'] \arrow[dr, "\rho"] & \\ S \arrow[dr, "\varphi"'] & & T \arrow[dl, "\pi"] \\ & C & \end{tikzcd}$$

where $\varphi_*\mathcal{O}_S \cong \pi_*\mathcal{O}_T \cong \mathcal{O}_C$, $b \colon \tilde{S} \to S$ is a blowup along q^3+1 smooth points, and $\rho \colon \tilde{S} \to T$ is a quotient by either $\mathbf{F}_q$ or $\boldsymbol{\alpha}_q$ depending on whether X is smooth or singular. Furthermore, T is an explicit rank 1 degeneracy locus in a product of projective bundles over C.

This is a summary of 2.5, 2.10, 3.4, 3.5, 3.7, 3.9, and 3.10. The techniques developed here, and more generally in [Che22, Appendix A], may be adapted to study linear spaces in other projective varieties. For instance, projection from a cone point maps the scheme of r-planes in an n-fold to the scheme of $(r-1)$-planes in an $(n-2)$-fold, providing an inductive method to study Fano schemes. This method gives a geometric construction of some classically known structures, such as the fibration constructed in [Rou11, §3.2].

To close the Introduction, I would like to sketch a perspective with which to contextualize the results at hand. As the prime power q varies, q-bic hypersurfaces are of different degrees, and are even defined over different characteristics. Yet in many respects, they ought to be viewed as constituting a single family. One way to make sense of this might be to say that they are constructed "uniformly with respect to q." This may be formulated in terms of the shape of the defining equations, but perhaps a more flexible and geometric way is to realize a q-bic hypersurface X as the intersection

$$X = \Gamma_q \cap Z \subset \mathbf{P}^n \times_{\mathbf{k}} \mathbf{P}^n$$

between the graph Γ_q of the q-power Frobenius morphism, and a $(1,1)$-divisor Z. The Fano surface S also fits into this point of view: take the ambient variety to be the Grassmannian $\mathbf{G}$, and take Z to be the zero locus in $\mathbf{G} \times_{\mathbf{k}} \mathbf{G}$ of a general section of the vector bundle $\mathcal{S}^\vee \boxtimes \mathcal{S}^\vee$.

This point of view highlights the prime power q and the scheme Z as the parameters of construction. What should it mean that varying q yields schemes of the same family? Consider the more familiar matter of fixing q and varying Z: Traditionally, those deformations of X or S that are of the same family are those obtained by *flatly* varying Z, and this is justified in part by constancy of geometric invariants. When varying q, it is generally too much to ask for invariants to remain constant, but one could hope that invariants vary in a simple way with respect to q. One particularly pleasant, albeit optimistic, condition would be to ask for Euler characteristics or, stronger, for dimensions of cohomology groups to vary as a polynomial in q. This is true of q-bic hypersurfaces, and Theorems **B** and **C** may be understood as saying the stronger condition of constancy of cohomological invariants holds also for the Fano surfaces of q-bic threefolds, at least when q varies only over primes p. I believe it is an interesting problem to find more examples of this type of phenomenon.

Finally, observe that if $\mathbf{P}^n$ and $\mathbf{G}$ were to be replaced by more general ambient varieties P, and Z allowed to be subschemes not necessarily cut out by sections of vector bundles, even Euler characteristics cannot vary as simply as a polynomial in q: indeed, by taking Z to be the diagonal, this would count points of P over finite fields. However, the Weil conjectures show that, even then, there is structure amongst the numbers and, taken together, the give insight into the geometry of P. My hope is that, by circumscribing the class of ambient schemes P and subschemes Z, this line of study may lead to new and interesting structures in positive characteristic geometry.

Outline. — §1 begins with a summary of the theory of q-bic forms and hypersurfaces, as developed in [Che23a, Che23b]; the latter half refines these methods in order to establish Theorem **A**. §§2–3 constructs and studies a rational map $S \dashrightarrow C$ from the Fano surface to a q-bic curve, establishing Theorem **D**. These methods are applied to study q-bic threefolds of type $\mathbf{1}^{\oplus 3} \oplus \mathbf{N}_2$ in §4. §§5–6 are where most of the cohomology computation for singular q-bics takes place, and completes the proof of Theorem **C**. Finally, §7 puts everything together to prove Theorem **B**.

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1. q-BIC HYPERSURFACES

The first half of this section summarizes some of the theory and results developed in [Che23a, Che23b] regarding q-bic forms and q-bic hypersurfaces that will be freely used in this article. The second half of this section, starting in 1.10, develops a few basic results pertaining to the Fano surface of a q-bic threefold. Throughout this article, denote by $\mathbf{k}$ an algebraically closed field of characteristic $p > 0$, $q := p^e$ for some $e \in \mathbf{Z}_{>0}$, and V a $\mathbf{k}$-vector space of dimension $n+1$.

1.1. q-bic forms and hypersurfaces. — A *q-bic form* on V is a nonzero linear map

$$\beta \colon V^{[1]} \otimes_{\mathbf{k}} V \to \mathbf{k},$$

where here and elsewhere, $V^{[1]} := \mathbf{k} \otimes_{\mathrm{Fr},\mathbf{k}} V$ is the q-power Frobenius twist of V. Let $\mathbf{P}V$ be the projective space of lines associated with V. The *q-bic equation* associated with β is the map

$$f_\beta := \beta(\mathrm{eu}^{[1]}, \mathrm{eu}) \colon \mathcal{O}_{\mathbf{P}V}(-q-1) \to V^{[1]} \otimes_{\mathbf{k}} V \otimes_{\mathbf{k}} \mathcal{O}_{\mathbf{P}V} \to \mathcal{O}_{\mathbf{P}V}$$

obtained by pairing the canonical section $\mathrm{eu} \colon \mathcal{O}_{\mathbf{P}V}(-1) \to V \otimes_{\mathbf{k}} \mathcal{O}_{\mathbf{P}V}$ with its q-power via β. The *q-bic hypersurface* associated with the q-bic form β is the hypersurface defined by f_β, and may be identified as the space of lines in V isotropic for β:

$$X := X_\beta := \mathrm{V}(f_\beta) = \{[v] \in \mathbf{P}V : \beta(v^{[1]}, v) = 0\}.$$

A choice of basis $V = \langle e_0, \ldots, e_n \rangle$ makes this more explicit: Writing $\mathbf{x}^\vee := (x_0 : \cdots : x_n)$ for the corresponding projective coordinates on $\mathbf{P}V \cong \mathbf{P}^n$, and $a_{ij} := \beta(e_i^{[1]}, e_j)$ for the (i,j)-th entry of the Gram matrix of β with respect to the given basis, as in [Che23a, 1.2], the q-bic equation f_β is the homogeneous polynomial of degree $q+1$ given by

$$f_\beta(x_0, \ldots, x_n) := \mathbf{x}^{[1],\vee} \cdot \mathrm{Gram}(\beta; e_0, \ldots, e_n) \cdot \mathbf{x} = \sum\nolimits_{i,j=0}^n a_{ij} x_i^q x_j.$$

From any of the descriptions, it is easy to check that a hyperplane section of a q-bic hypersurface is a q-bic hypersurface: see [Che23b, 1.9].

To illustrate the utility of distinguishing the shape of the q-bic equation $f := f_\beta$, and to record the computation for later use, the following shows that the action of the q-power absolute Frobenius morphism $\mathrm{Fr} \colon X \to X$ is zero on higher cohomology:

1.2. Lemma. — *If* $\dim X \geq 1$, *then the map* $\mathrm{Fr} \colon \mathrm{H}^{n-1}(X, \mathcal{O}_X) \to \mathrm{H}^{n-1}(X, \mathcal{O}_X)$ *is zero.*

Proof. Consider the commutative diagram of abelian sheaves on $\mathbf{P}V$ given by

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\mathbf{P}V}(-q-1) \arrow[r, "f"'] \arrow[d, "f^{q-1}\mathrm{Fr}"] & \mathcal{O}_{\mathbf{P}V} \arrow[r] \arrow[d, "\mathrm{Fr}"] & \mathcal{O}_X \arrow[r] \arrow[d, "\mathrm{Fr}"] & 0 \\
0 \arrow[r] & \mathrm{Fr}_*(\mathcal{O}_{\mathbf{P}V}(-q-1)) \arrow[r, "f^q"] & \mathrm{Fr}_*\mathcal{O}_{\mathbf{P}V} \arrow[r] & \mathrm{Fr}_*\mathcal{O}_X \arrow[r] & 0
\end{tikzcd}$$

where Fr acts on local sections by $s \mapsto s^q$. Taking cohomology yields a commutative diagram

$$\begin{tikzcd}
\mathrm{H}^{n-1}(X, \mathcal{O}_X) \arrow[r, "\cong"'] \arrow[d, "\mathrm{Fr}"'] & \mathrm{H}^n(\mathbf{P}V, \mathcal{O}_{\mathbf{P}V}(-q-1)) \arrow[d, "f^{q-1}\mathrm{Fr}"] \\
\mathrm{H}^{n-1}(X, \mathcal{O}_X) \arrow[r, "\cong"] & \mathrm{H}^n(\mathbf{P}V, \mathcal{O}_{\mathbf{P}V}(-q-1)).
\end{tikzcd}$$

Compute the map $f^{q-1}\mathrm{Fr}$ on the right as follows: Upon choosing coordinates $(x_0 : \cdots : x_n)$ for $\mathbf{P}V = \mathbf{P}^n$, a basis for $\mathrm{H}^n(\mathbf{P}V, \mathcal{O}_{\mathbf{P}V}(-q-1))$ is given by elements

$$\xi := (x_0^{i_0} \cdots x_n^{i_n})^{-1} \text{ where } i_0 + \cdots + i_n = q+1 \text{ and each } i_j \geq 1.$$

Monomials appearing in f^{q-1} are of the form $a^q b$, where a and b are themselves monomials of degree $q-1$. Write $a = x_0^{a_0} \cdots x_n^{a_n}$ with $a_0 + \cdots + a_n = q-1$. Then $f^{q-1}\,\mathrm{Fr}(\xi)$ is a sum of terms

$$a^q b \cdot (x_0^{i_0} \cdots x_n^{i_n})^{-q} = b \cdot (x_0^{i_0 - a_0} \cdots x_n^{i_n - a_n})^{-q}.$$

Since $(i_0 - a_0) + \cdots + (i_n - a_n) = 2 \leq n$, there is some index j such that $i_j - a_j \leq 0$, and so this represents 0. Thus each potential contribution to $f^{q-1}\,\mathrm{Fr}(\xi)$ vanishes, and so $f^{q-1}\,\mathrm{Fr}(\xi) = 0$. ∎

1.3. Classification. — Isomorphic q-bic forms yield projectively equivalent q-bic hypersurfaces. Over an algebraically closed field $\mathbf{k}$, [Che23a, **Theorem A**] shows that there are finitely many isomorphism classes of q-bic forms on V, and that they may be classified via their Gram matrices: there exists a basis $V = \langle e_0, \ldots, e_n \rangle$ and nonnegative integers $a, b_m \in \mathbf{Z}_{\geq 0}$ such that

$$\mathrm{Gram}(\beta; e_0, \ldots, e_n) = \mathbf{1}^{\oplus a} \oplus \Big(\bigoplus\nolimits_{m \geq 1} \mathbf{N}_m^{\oplus b_m} \Big)$$

where $\mathbf{1}$ is the 1-by-1 matrix with unique entry 1, $\mathbf{N}_m$ is the m-by-m Jordan block with 0's on the diagonal, and $\oplus$ denotes block diagonal sums of matrices. The tuple $(a; b_m)_{m \geq 1}$ is called the *type* of β, and is the fundamental invariant of; the sum $\sum_{m \geq 1} b_m$ is its *corank*. When q-bic forms are varied in a family, the corank can jump and the type can change; the Hasse diagram of specialization relations is not linear in general, and some partial relations are given in [Che23a, **Theorem C**].

Of note is the $\mathbf{N}_1^{\oplus b_1}$ piece, which is a b_1-by-b_1 zero matrix, and will sometimes denoted by $\mathbf{0}^{\oplus b_1}$. Its underlying subspace is the *radical* of β:

$$\mathrm{rad}(\beta) = \{ v \in V : \beta(v^{[1]}, u) = \beta(u^{[1]}, v) = 0 \text{ for all } u \in V \}.$$

It is straightforward to check that X is a cone if and only if $b_1 \neq 0$, and that the vertex is the projectivization of the radical; see also [Che22, **2.4**] for an invariant treatment.

1.4. q-bic points and curves. — In low-dimensions, the classification and specialization relations are quite simple and, especially when combined with the fact that hyperplane sections of q-bics are q-bics, are geometrically very useful: see [Che22, **Chapter 3**] for more. When V has dimension 2, the possible types of q-bic forms and their specialization relations are

$$\mathbf{1}^{\oplus 2} \rightsquigarrow \mathbf{N}_2 \rightsquigarrow \mathbf{0} \oplus \mathbf{1},$$

corresponding to the subschemes in $\mathbf{P}^1$ defined by $x_0^{q+1} + x_1^{q+1}$, $x_0^q x_1$, and x_1^{q+1}. Thus q-bic points are either $q+1$ reduced points, a reduced point with a q-fold point, or one point of multiplicity $q+1$.

When V is 3-dimensional, the types and specializations excluding cones over q-bic points are

$$\mathbf{1}^{\oplus 3} \rightsquigarrow \mathbf{1} \oplus \mathbf{N}_2 \rightsquigarrow \mathbf{N}_3.$$

The corresponding subschemes of $\mathbf{P}^2$ may be described as: a smooth curve of degree $q+1$; an irreducible, geometrically rational curve with a single unibranch singularity; and a reducible curve with a linear component meeting an irreducible, geometrically rational component of degree q at its unique unibranch singularity.

1.5. Smoothness. — A q-bic hypersurface X is smooth if and only if its underlying q-bic form β is *nonsingular* in the sense that one—equivalently, both—of the adjoint maps $\beta : V \to V^{[1],\vee}$ or $\beta^\vee : V^{[1]} \to V^\vee$ is an isomorphism; with a choice of basis, this is equivalent to invertibility of a Gram matrix. The singular locus of X is supported on the projectivization of the subspace

$$V^{\perp,[-1]} := \{ v \in V : \beta(v^{[1]}, u) = 0 \text{ for every } u \in V \},$$

the Frobenius descent of the left kernel $V^\perp := \ker(\beta^\vee : V^{[1]} \to V^\vee)$: see [Che23b, **2.6**]. The embedded tangent space $\mathbf{T}_{X,x} \subset \mathbf{P}V$ to X at a point $x = \mathbf{P}L$ is the projectivization of

$$L^{[1],\perp} := \{ v \in V : \beta(u^{[1]}, v) = 0 \text{ for all } u \in L \} := \ker(\beta : V \to V^{[1],\vee} \twoheadrightarrow L^{[1],\vee}),$$

the right orthogonal of L: see [Che23b, 2.2].

1.6. Hermitian structures. — A vector $v \in V$ is said to be *Hermitian* if

$$\beta(u^{[1]}, v) = \beta(v^{[1]}, u)^q \text{ for all } u \in V.$$

A subspace $U \subset V$ is *Hermitian* if it is spanned by Hermitian vectors. The Hermitian equation implies that the left and right orthogonals of U,

$$\begin{aligned} U^{\perp,[-1]} &:= \{ v \in V : \beta(v^{[1]}, u) = 0 \text{ for all } u \in U \} \text{ and} \\ U^{[1],\perp} &:= \{ v \in V : \beta(u^{[1]}, v) = 0 \text{ for all } u \in U \}, \end{aligned}$$

coincide. When β is nonsingular, there are only finitely many Hermitian vectors, whence finitely many Hermitian subspaces, and they span V: see [Che23a, 2.1–2.6]

Continuing with β nonsingular, there is a canonical $\mathbf{k}$-vector space isomorphism

$$\sigma_\beta := \beta^{-1} \circ \beta^{[1],\vee} : V^{[2]} \to V^{[1],\vee} \to V$$

between the q^2-Frobenius twist of V and V itself; this is an $\mathbf{F}_{q^2}$ descent datum on V. Pre-composing with the universal q^2-linear map $V \to V^{[2]}$ yields a q^2-linear endomorphism $\phi : V \to V$. It preserves isotropicity so it induces an endomorphism $\phi_X : X \to X$. In a basis $V = \langle e_0, \dots, e_n \rangle$ of Hermitian vectors, the Gram matrix of β is Hermitian in the sense that

$$\operatorname{Gram}(\beta; e_0, \dots, e_n)^\vee = \operatorname{Gram}(\beta; e_0, \dots, e_n)^{[1]},$$

from which it follows that σ_β is represented by the identity matrix, and that ϕ_X is the q^2-power geometric Frobenius morphism in the corresponding coordinates: see [Che23b, 4.3].

By [Che23b, 1.3], the fixed set of ϕ_X is the subset X_{Herm} of *Hermitian points* of X, consisting of those points $x = \mathbf{P}L$ where L is spanned by a Hermitian vector. Comparing with the construction of ϕ_X, this implies that the set of Hermitian points of X is the zero locus of the map

$$\tilde{\theta} : \mathcal{O}_X(-q^2) \xrightarrow{\mathrm{eu}^{[2]}} V^{[2]} \otimes_{\mathbf{k}} \mathcal{O}_X \xrightarrow{\sigma_\beta} V \otimes_{\mathbf{k}} \mathcal{O}_X \longrightarrow \mathcal{T}_{\mathbf{P}V}(-1)|_X$$

where the final arrow arises from the dual Euler sequence. The following shows that this map factors through the tangent bundle of X. A geometric description of $\phi_X(x)$ may be extracted from the proof below, as can be found in [Che22, 2.9.9], and this is most clear when X is a curve: the tangent line $\mathbf{T}_{X,x}$ intersects X at x with multiplicity q, and the residual point of intersection is $\phi_X(x)$.

1.7. Lemma. — *Hermitian points of X are cut out by a canonical map $\theta : \mathcal{O}_X(-q^2) \to \mathcal{T}_X(-1)$.*

Proof. The tangent bundle of X is the kernel of the normal map $\delta : \mathcal{T}_{\mathbf{P}V}(-1)|_X \to \mathcal{N}_{X/\mathbf{P}V}(-1)$; this is induced via the dual Euler sequence by the map $\tilde{\delta} : V \otimes_{\mathbf{k}} \mathcal{O}_X \to \mathcal{N}_{X/\mathbf{P}V}(-1)$ which takes directional derivatives of a fixed equation of X. A simple computation shows that there is a commutative diagram

$$\begin{tikzcd} V \otimes \mathcal{O}_X \arrow[r, "\tilde{\delta}"'] \arrow[d, "\beta"'] & \mathcal{N}_{X/\mathbf{P}V}(-1) \\ V^{[1],\vee} \otimes \mathcal{O}_X \arrow[r, "\mathrm{eu}^{[1],\vee}"] & \mathcal{O}_X(q) \arrow[u, "\cong"'] \end{tikzcd}$$

and so $\delta \circ \tilde{\theta}$ factors through the map

$$\tilde{\delta} \circ \sigma_\beta \circ \mathrm{eu}^{[2]} = \mathrm{eu}^{[1],\vee} \circ \beta^{[1],\vee} \circ \mathrm{eu}^{[2]} = \beta(\mathrm{eu}^{[1]}, \mathrm{eu})^q = 0,$$

see [Che23a, 1.7] for the pentultimate relation. This implies that $\tilde{\theta}$ factors through $\mathcal{T}_X(-1)$. ∎

1.8. Scheme of lines. — The Fano scheme $\mathbf{F} := \mathbf{F}_1(X)$ of lines of a q-bic hypersurface X may be viewed as the space of 2-dimensional subspaces in V which are totally isotropic for the q-bic form β:

$$\mathbf{F} = \{\, [U] \in \mathbf{G} : \beta(u^{[1]}, u') = 0 \text{ for all } u, u' \in U \,\}.$$

This description exhibits $\mathbf{F}$ as the zero locus in the Grassmannaian $\mathbf{G} := \mathbf{G}(2, V)$ of the map

$$\beta_{\mathscr{S}} : \mathscr{S}^{[1]} \otimes_{\mathscr{O}_{\mathbf{G}}} \mathscr{S} \to \mathscr{O}_{\mathbf{G}}$$

induced by restricting β to the universal rank 2 subbundle $\mathscr{S} \subset V \otimes_{\mathbf{k}} \mathscr{O}_{\mathbf{G}}$. A direct construction shows that $\mathbf{F}$ is nonempty whenever $n \geq 3$, so this implies that $\dim \mathbf{F} \geq 2n - 6$: see [Che23b, 1.10–1.12]. Furthermore, $\mathbf{F}$ is connected whenever $n \geq 4$: see [Che23b, 2.9].

When $\mathbf{F}$ is generically smooth and of expected dimension $2n-6$, it is a local complete intersection scheme whose structure sheaf admits a Koszul resolution by $\wedge^* \mathscr{S}^{[1]} \otimes_{\mathscr{O}_{\mathbf{G}}} \mathscr{S}$, and its dualizing sheaf may be computed to be

$$\omega_{\mathbf{F}} \cong \mathscr{O}_{\mathbf{F}}(2q + 1 - n) \otimes_{\mathbf{k}} \det(V)^{\vee, \otimes 2}$$

where the twist by $\det(V)$ is useful for tracking weights of algebraic group actions: see [Che23b, 2.4]. Here, $\mathscr{O}_{\mathbf{F}}(1)$ is the Plücker line bundle; its degree is computed in [Che23b, 1.15] as

$$\deg \mathscr{O}_{\mathbf{F}}(1) = \frac{(2n-6)!}{(n-1)!(n-3)!}(q+1)^2\big((n-1)q^2 + (2n-8)q + (n-1)\big).$$

The Fano scheme $\mathbf{F}$ is singular along the subscheme of lines through the singular locus of X:

$$\operatorname{Sing} \mathbf{F} = \{\, [\ell] \in \mathbf{F} : \ell \cap \operatorname{Sing} X \neq \varnothing \,\}.$$

In particular, $\mathbf{F}$ is smooth if and only if X itself is smooth, or equivalently, when β is nonsingular: see [Che23b, 2.7]. This description moreover implies that $\mathbf{F}$ is of expected dimension if and only if the locus of lines through the singular locus of X has dimension at most $2n-6$.

1.9. q-bic surfaces. — To complete the tour in low dimensions, consider a q-bic hypersurface X of dimension 2, or briefly, a *q-bic surface*. Classification and specialization of types looks like

$$\begin{tikzcd}
\mathbf{1}^4 \arrow[r, rightsquigarrow] & \mathbf{N}_2\mathbf{1}^2 \arrow[r, rightsquigarrow] & \mathbf{N}_4 \arrow[rr, rightsquigarrow] \arrow[dr, rightsquigarrow] & & \mathbf{N}_2^2 \arrow[rr, rightsquigarrow] & & \mathbf{0N}_2\mathbf{1} \arrow[r, rightsquigarrow] & \mathbf{0N}_3 \arrow[r, rightsquigarrow] & \mathbf{0}^2\mathbf{1}^2 \arrow[r, rightsquigarrow] & \mathbf{0}^2\mathbf{N}_2 \arrow[r, rightsquigarrow] & \mathbf{0}^3\mathbf{1} \\
& & & \mathbf{N}_3\mathbf{1} \arrow[rr, rightsquigarrow] & & \mathbf{01}^3 \arrow[ur, rightsquigarrow] & & & & &
\end{tikzcd}$$

where, notably, the Hasse diagram of specialization relations is not linear. See [Che22, 3.8] for more.

The general statements of 1.8 shows that X always contains lines, and that its Fano scheme $\mathbf{F}$ has degree $(q+1)(q^3+1)$ when it is of expected dimension 0. This means, in particular, that a smooth q-bic surface has exact $(q+1)(q^3+1)$ distinct lines. The configuration of these lines is fascinating and is well-studied: see, for example, [Hir85, SSvL10, BPRS21]. Other aspects of the geometry of smooth q-bic surfaces may be found, for example, in [Shi74, Oji19].

1.10. q-bic threefolds. — In the remainder of the article, X denotes a 3-dimensional q-bic hypersurface, or a *q-bic threefold* for short. The classification and specialization relations of q-bic threefolds, at least up to the first few cones, looks as follows:

$$\begin{tikzcd}
\mathbf{1}^5 \arrow[r, rightsquigarrow] & \mathbf{1}^3\mathbf{N}_2 \arrow[r, rightsquigarrow] & \mathbf{1N}_4 \arrow[r, rightsquigarrow] \arrow[dr, rightsquigarrow] & \mathbf{N}_5 \arrow[r, rightsquigarrow] & \mathbf{1}^2\mathbf{N}_3 \arrow[r, rightsquigarrow] \arrow[dr, rightsquigarrow] & \mathbf{01}^4 \arrow[dr, rightsquigarrow] & \\
& & & \mathbf{1N}_2^2 \arrow[rr, rightsquigarrow] & & \mathbf{N}_2\mathbf{N}_3 \arrow[r, rightsquigarrow] & \mathbf{01}^2\mathbf{N}_2\,.
\end{tikzcd}$$

Notice that all non-cones have corank at most 2, which by the discussion of 1.5, implies that the singular locus of a q-bic threefold which is not a cone has dimension at most 1.

The Fano scheme $S := \mathbf{F}$ of lines of a q-bic threefold has expected dimension 2 by 1.8. It is easy to check that when X has corank at most 1 and is not a cone, the singular locus of S has dimension at most 1. Thus in all such cases, and also in the case X is smooth, S is generically smooth of expected dimension 2: see also [Che23b, 2.8]. In any of these cases, but especially when X is either smooth or of type $\mathbf{1}^{\oplus 3} \oplus \mathbf{N}_2$, S will be referred to as the *Fano surface* of X.

Begin by considering the Fano surface S of a smooth q-bic threefold X, describing the tangent bundle $\mathcal{T}_S$, and computing its Chern numbers:

1.11. Lemma. — *If X is smooth, then $\mathcal{T}_S \cong \mathcal{S} \otimes_{\mathcal{O}_S} \mathcal{O}_S(2-q)$ and the Chern numbers of S are*

$$c_1(S)^2 = (q+1)^2(q^2+1)(2q-3)^2 \text{ and } c_2(S) = (q+1)^2(q^4-3q^3+4q^2-4q+3).$$

Proof. Identify the tangent bundle of S as in [Che23b, 2.2] as $\mathcal{T}_S \cong \mathcal{H}om_{\mathcal{O}_S}(\mathcal{S}, \mathcal{S}^{[1],\perp}/\mathcal{S})$, where the sheaf in the target of the $\mathcal{H}om$ is characterized by the short exact sequence

$$0 \to \mathcal{S}^{[1],\perp}/\mathcal{S} \to \mathcal{Q} \xrightarrow{\beta} \mathcal{S}^{[1],\vee} \to 0.$$

Taking determinants shows $\mathcal{S}^{[1],\perp}/\mathcal{S} \cong \mathcal{O}_S(1-q)$, and combined with the wedge product isomorphism $\mathcal{S}^\vee \cong \mathcal{S} \otimes_{\mathcal{O}_S} \mathcal{O}_S(1)$ gives the identification of $\mathcal{T}_S$. To compute the first Chern number, note that the Plücker line bundle $\mathcal{O}_S(1)$ has degree $(q+1)^2(q^2+1)$ by taking $n=4$ in the formula given in 1.8. Combined with the identification of $\mathcal{T}_S$, this gives

$$c_1(S)^2 := \int_S c_1(\mathcal{T}_S)^2 = \int_S c_1(\mathcal{O}_S(3-2q))^2 = (2q-3)^2 \deg \mathcal{O}_S(1) = (q+1)^2(q^2+1)(2q-3)^2.$$

For the second Chern number, observe that

$$\begin{aligned} c_2(\mathcal{T}_S) &= c_2(\mathcal{S}^\vee) + c_1(\mathcal{S}^\vee)c_1(\mathcal{O}_S(1-q)) + c_1(\mathcal{O}_S(1-q))^2 \\ &= c_2(\mathcal{S}^\vee) + (q-2)(q-1)c_1(\mathcal{O}_S(1))^2. \end{aligned}$$

Since a general section of $\mathcal{S}^\vee$ cuts out the scheme of lines contained in a general hyperplane section of X, which is but a smooth q-bic surface by 1.1 and 1.3, the degree of $c_2(\mathcal{S}^\vee)$ is $(q+1)(q^3+1)$ by 1.9. Combining with the degree computation for $\mathcal{O}_S(1)$ gives $c_2(S)$. ■

This computation implies that the Fano surface S cannot lift to either the second Witt vectors of $\mathbf{k}$ or any characteristic 0 base as soon as $q > 2$. By contrast, observe that S *does* lift when $q = 2$, as is seen by taking any lift of X and by applying the Fano scheme construction in families.

1.12. Proposition. — *If X is smooth and $q > 2$, then S lifts neither to $\mathrm{W}_2(\mathbf{k})$ nor to characteristic* 0.

Proof. Dualizing the tangent bundle computation of 1.11 shows $\Omega^1_S \cong \mathcal{S}^\vee \otimes_{\mathcal{O}_S} \mathcal{O}_S(q-2)$. Since $\mathcal{S}^\vee$ has sections, this implies that Kodaira–Akizuki–Nakano vanishing fails on S, so [DI87, Corollaire 2.8] shows S cannot lift to $\mathrm{W}_2(\mathbf{k})$. The Chern number computation of 1.11 gives

$$c_1(S)^2 - 3c_2(S) = q^2(q+1)^2(q^2-3q+1).$$

The quadratic formula shows that this is positive whenever $q > 2$, and so S the Bogomolov–Miyaoka–Yau inequality of [Miy77, Theorem 4] is not satisfied on S. Since Chern numbers are constant in flat families, this implies that S cannot lift to characteristic 0. ■

The Chern number computation also gives a simple way to compute the Euler characteristic of the structure sheaf $\mathcal{O}_S$ whenever S is smooth; this holds more generally, whenever S has expected dimension, by comparing with the Koszul resolution of $\mathcal{O}_S$ on $\mathbf{G}$:

1.13. Proposition. — *If S is of its expected dimension* 2*, then*

$$\chi(S, \mathcal{O}_S) = \frac{1}{12}(q+1)^2(5q^4 - 15q^3 + 17q^2 - 16q + 12).$$

Proof. When X is a smooth, so is S, and so Noether's formula, as in [Ful98, Example 15.2.2], applies to give the first equality in

$$\chi(S,\mathcal{O}_S) = \frac{1}{12}(c_1(S)^2 + c_2(S)) = \sum\nolimits_{i=0}^{4}(-1)^i\chi(\mathbf{G},\wedge^i\mathcal{S}^{[1]}\otimes_{\mathcal{O}_{\mathbf{G}}}\mathcal{S}).$$

Substituting the Chern number computations from **1.11** gives the formula in the statement. The second equality arises from taking the Koszul resolution of $\mathcal{O}_S$ on $\mathbf{G}$. Since the Koszul resolution persists whenever S is of dimension 2, this gives the general case. ■

2. Cone situation

Study the Fano scheme S by fibering it over a q-bic curve C. This is easy to arrange rationally, as is promptly explained in **2.5**, and the main content of this section is to construct a canonical resolution of the rational map $S \dashrightarrow C$ using a mixture of projective geometry techniques and the intrinsic theory of q-bic forms from [Che23a]: see **2.11** for a summary. The setting in this section is quite general; the most important situations for the main computations of the article—where additional smoothness hypotheses are satisfied—are classified in **2.3**(ii), and are further studied in §**3**.

2.1. Cone situation. — A triple $(X,\infty,\mathbf{P}\breve{W})$ consisting of a q-bic threefold X, a point $\infty = \mathbf{P}L$ of X, and a hyperplane $\mathbf{P}\breve{W}$ such that $X \cap \mathbf{P}\breve{W}$ is a cone with vertex ∞ over a reduced q-bic curve $C \subset \mathbf{P}\overline{W}$ is called a *cone situation*; here and later, $\overline{W} := \breve{W}/L$ and $\overline{V} := V/L$. This is sometimes considered with some of the following additional geometric assumptions:

(i) there does not exist a 2-plane contained in X which passes through ∞; or
(ii) ∞ is a smooth point of X; or
(iii) C is a smooth q-bic curve.

Conditions (ii) and (iii) together imply (i). That ∞ is a vertex for $X \cap \mathbf{P}\breve{W}$ means that L lies in the radical of $\beta|_W$, so that W is contained in both orthogonals of L, see **1.3**. In particular, this means that $\mathbf{P}\breve{W}$ is contained in the embedded tangent space $\mathbf{T}_{X,\infty} = \mathbf{P}L^{[1],\perp}$ of X at ∞, see **1.5**; and if (ii) is satisfied, then equality $\mathbf{P}\breve{W} = \mathbf{T}_{X,\infty}$ necessarily holds.

2.2. Examples. — The following are some examples of cone situations:

(i) Let X be a smooth q-bic threefold and let $\infty \in X$ be a Hermitian point. Then $(X,\infty,\mathbf{T}_{X,\infty})$ is a cone situation, and it follows from **1.3** and **1.6** that all cone situations for smooth X are obtained this way. All the conditions (i), (ii), and (iii) are satisfied.

The next three examples pertain to q-bic threefolds of type $\mathbf{1}^{\oplus 3}\oplus\mathbf{N}_2$. For concreteness, let

$$X = \mathrm{V}(x_0^q x_1 + x_0 x_1^q + x_2^{q+1} + x_3^q x_4) \subset \mathbf{P}^4,$$

and set $C := \mathrm{V}(x_0^q x_1 + x_0 x_1^q + x_2^{q+1})$ in the $\mathbf{P}^2$ in which x_3 and x_4 are projected out. Write $x_- := (0:0:0:1:0)$, $x_+ := (0:0:0:0:1)$, and $\infty := (1:0:0:0:0)$.

(ii) $(X,x_-,\mathbf{T}_{X,x_-})$ is a cone situation over C satisfying each of the conditions (i), (ii), and (iii).
(iii) $(X,x_+,\mathrm{V}(x_3))$ is a cone situation over the C satisfying (i) and (iii), but not (ii).
(iv) $(X,\infty,\mathbf{T}_{X,\infty})$ is a cone situation over $\mathrm{V}(x_2^{q+1}+x_3^q x_4)$ satisfying (i) and (ii), but not (iii).

The remaining examples illustrate the range of possibilities for the cone situation.

(v) Let $X := \mathrm{V}(x_0^q x_1 + x_1^q x_2 + x_3^q x_4)$ and $\infty := (0:0:0:1:0)$. Then $(X,\infty,\mathbf{T}_{X,\infty})$ is a cone situation over $\mathrm{V}(x_0^q x_1 + x_1^q x_2)$ satisfying (ii), but not (i) nor (iii).
(vi) Let X be of type $\mathbf{0}\oplus\mathbf{1}^{\oplus 4}$, ∞ the vertex of X, and $\mathbf{P}\breve{W}$ a general hyperplane through ∞. Then $(X,\infty,\mathbf{P}\breve{W})$ is a cone situation satisfying (iii), but not (i) nor (ii).

(vii) Let X be of type $\mathbf{0}^{\oplus 2} \oplus \mathbf{1} \oplus \mathbf{N}_2$, ∞ any point of the vertex of X, and $\mathbf{P}\breve{W}$ be any hyperplane intersecting the vertex of X exactly at ∞. Then $(X, \infty, \mathbf{P}\breve{W})$ is a cone situation satisfying none of (i), (ii), nor (iii).

The following illustrates how the hypotheses of **2.1** translate to geometric consequences on X and its Fano scheme S of lines; the first statement gives sufficient conditions—although not necessary, as shown by **2.2**(vi)—for when S is of expected dimension 2, whereas the second classifies cone situations satisfying the two smoothness hypotheses on $(X, \infty, \mathbf{P}\breve{W})$.

2.3. Lemma. — *If the cone situation $(X, \infty, \mathbf{P}\breve{W})$ furthermore satisfies*

(i) *conditions* **2.1**(i) *and* **2.1**(ii)*, then* $\dim S = 2$*;*
(ii) *conditions* **2.1**(ii) *and* **2.1**(iii)*, then it is projectively equivalent to either* **2.2**(i) *or* **2.2**(ii)*.*

Proof. Suppose that $(X, \infty, \mathbf{P}\breve{W})$ satisfies **2.1**(i) and **2.1**(ii). Then X is not a cone: otherwise, its vertex x, which is different from the smooth point ∞, would span a plane $\langle x, \ell \rangle$ through ∞ with any line $\ell \subset X \cap \mathbf{P}\breve{W}$ through ∞ and not passing through x. The comments in **1.10** show that $\dim \operatorname{Sing} X \leq 1$. Suppose $x \in X$ is a singular point with underlying linear space $K \subset V$. Lines in X through x are contained in

$$X \cap \mathbf{P}K^{[1],\perp} \cap \mathbf{P}K^{\perp,[-1]} = X \cap \mathbf{T}_{X,x} \cap \mathbf{P}K^{\perp,[-1]} = X \cap \mathbf{P}K^{\perp,[-1]}$$

since the underlying linear space pairs to 0 with K on either side of β: see also [Che23b, **3.1**]. Since x is not a vertex, this is a surface which is a cone with vertex x over a q-bic curve. Thus the locus of lines through x is of dimension 1. Therefore discussion of **1.8** implies that $\dim S = 2$.

Suppose now that $(X, \infty, \mathbf{P}\breve{W})$ furthermore satisfies **2.1**(iii). If X itself is smooth, then it must be **2.2**(i), so suppose X is singular. Since ∞ is a smooth point, and since $\mathbf{P}\breve{W} = \mathbf{T}_{X,\infty}$ intersects X at a cone with a smooth base, the singular locus of X must be a single point $x = \mathbf{P}K$ disjoint from $\mathbf{P}\breve{W}$. But $W = L^{[1],\perp}$, so this means that the natural map $K \to L^{[1],\vee}$ is nonzero, whence an isomorphism. Splitting $V = \breve{W} \oplus K$, it now follows that

$$V^{[1],\perp} = \ker(\breve{W} \oplus K \to \breve{W}^{[1],\vee}) = \ker(\breve{W} \to \overline{W}^{[1],\vee}) = L.$$

Thus the restriction of β to $U := L \oplus K$ is of type $\mathbf{N}_2$. Since $U^{[1],\perp} = L^{[1],\perp}$ and $U^{\perp} = K^{\perp}$ are distinct hyperplanes, their intersection in V gives the orthogonal complement to U. Whence X is of type $\mathbf{1}^{\oplus 3} \oplus \mathbf{N}_2$ and ∞ is the smooth point of the subform $\mathbf{N}_2$. ■

Let $(X, \infty, \mathbf{P}\breve{W})$ be a cone situation over the q-bic curve C and let S be the Fano scheme of lines of X. Then there is a canonical closed immersion $C \hookrightarrow S$ identifying it with the closed subscheme

$$C_{\infty, \mathbf{P}\breve{W}} := \{ [\ell] \in S : \infty \in \ell \subset \mathbf{P}\breve{W} \}$$

of lines containing ∞ and contained in $X \cap \mathbf{P}\breve{W}$, so that the Plücker polarization $\mathcal{O}_S(1)$ pulls back to the planar polarization $\mathcal{O}_C(1)$. The hypotheses of **2.1** affect $C_{\infty, \mathbf{P}\breve{W}}$ as follows:

2.4. Lemma. — *Suppose the cone situation $(X, \infty, \mathbf{P}\breve{W})$ furthermore satisfies*

(i) *condition* **2.1**(i)*, then the closed subsets $\{ [\ell] \in S : \infty \in \ell \}$ and $\{ [\ell] \in S : \ell \subset \mathbf{P}\breve{W} \}$ coincide;*
(ii) *condition* **2.1**(ii)*, then $C_{\infty, \mathbf{P}\breve{W}}$ coincides with the subscheme C_∞ of lines in X through ∞; and*
(iii) *conditions* **2.1**(ii) *and* **2.1**(iii)*, then $\mathcal{N}_{C_{\infty, \mathbf{P}\breve{W}}/S} \cong \mathcal{O}_C(-q+1)$.*

Proof. Establish (i) through its contrapositive: To see "$\supseteq$", if there were a line $\ell \subset X \cap \mathbf{P}\breve{W}$ not containing ∞, then the cone $X \cap \mathbf{P}\breve{W}$ would contain the plane $\langle \infty, \ell \rangle$ spanned by ∞ and ℓ. To see "$\subseteq$", if there were a line $\ell \ni \infty$ not contained in $\mathbf{P}\breve{W}$, then the orthogonals of L contain not only $\breve{W}$, by the comments of **2.1**, but also the subspace underlying ℓ. Therefore L lies in the radical of β, so

X is a cone over a q-bic surface with vertex containing ∞. The result now follows since every q-bic surface contains a line by **1.9**.

For (ii), the comments following **2.1** imply that $\mathbf{P}\breve{W}$ is the embedded tangent space of X at ∞. Since any line in X through ∞ is necessarily contained in $\mathbf{T}_{X,\infty}$, the moduli problem underlying C_∞ is a closed subfunctor of that of $C_{\infty,\mathbf{P}\breve{W}}$. Since the reverse inclusion is clear, equality holds.

For (iii), observe that there is a short exact sequence of locally free sheaves

$$0 \to \mathcal{T}_C \to \mathcal{T}_S|_C \to \mathcal{N}_{C/S} \to 0$$

where $C_{\infty,\mathbf{P}\breve{W}}$ is identified with C: indeed, this is because the assumptions together with the discussion in **1.8** imply that S is smooth along C. Taking determinants then gives the normal bundle. ∎

The cone situation gives a way of transforming lines in X to points of C, inducing a rational map $S \dashrightarrow C$ in good cases. Some hypothesis on $(X, \infty, \mathbf{P}\breve{W})$ is necessary to exclude, for instance, example **2.2**(vi), wherein the locus of lines through ∞ forms an irreducible component of S. In the following, write $\mathrm{proj}_\infty \colon \mathbf{P}V \dashrightarrow \mathbf{P}\overline{V}$ for the linear projection away from ∞, and let $S^\circ := S \setminus C_{\infty,\mathbf{P}\breve{W}}$.

2.5. Lemma. — *If the cone situation $(X, \infty, \mathbf{P}\breve{W})$ satisfies* **2.1**(i)*, then there is a rational map $\varphi \colon S \dashrightarrow C$ given on points $[\ell] \in S^\circ$ by*

$$\varphi([\ell]) = \mathrm{proj}_\infty(\ell \cap \mathbf{P}\breve{W}) = \mathrm{proj}_\infty(\ell) \cap \mathbf{P}\overline{W}.$$

Proof. The point is that **2.4**(i) implies that the indeterminacy locus of the formula is a proper closed subset not containing any irreducible components of S. ∎

2.6. — Towards a resolution of φ, consider the scheme parameterizing triples

$$\mathbf{P} := \big\{ (y \mapsto y_0 \in \ell_0) : y \in \mathbf{P}\breve{W}, [\ell_0] \in \mathbf{G}(2, \overline{V}), y_0 \in C \big\}$$

consisting of a point $y \in \mathbf{P}\breve{W}$, a line $\ell_0 \subset \mathbf{P}\overline{V}$, and a point $y_0 \in \ell_0 \cap C$, such that y is contained in the line in $\mathbf{P}\breve{W}$ determined by y_0. Equivalently, $\mathbf{P}$ is the product $\mathbf{P}\mathcal{V}_1 \times_C \mathbf{P}\mathcal{V}_2$ of projective bundles over C, where $\mathcal{V}_1$ and $\mathcal{V}_2$ are defined via pullback of the left square and pushout of the right square, respectively, of the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{V}_1 \arrow[rr] \arrow[dd, two heads] \arrow[dr, hook] & & V_C \arrow[rr] \arrow[dr, two heads] & & \mathcal{V}_2 \arrow[r] & 0 \\
& & \breve{W}_C \arrow[ur, hook] \arrow[dr, two heads] & & \overline{V}_C \arrow[ur, two heads] & & \\
0 \arrow[r] & \mathcal{O}_C(-1) \arrow[rr] & & \overline{W}_C \arrow[rr] \arrow[ur, hook] & & \mathcal{T} \arrow[r] \arrow[uu, hook] & 0
\end{tikzcd}$$

where $\mathcal{T} := \mathcal{T}_{\mathbf{P}\overline{W}}(-1)|_C$, $V_C := V \otimes \mathcal{O}_C$, and similarly for the others. In particular, there are split short exact sequences

$$0 \to L_C \to \mathcal{V}_1 \to \mathcal{O}_C(-1) \to 0 \quad \text{and} \quad 0 \to \mathcal{T} \to \mathcal{V}_2 \to (V/\breve{W})_C \to 0.$$

Let $\mathbf{P}^\circ$ be the open subscheme of $\mathbf{P}$ where $y \neq \infty$ and $\ell_0 \not\subset \mathbf{P}\overline{W}$. Its closed complement is the union of two effective Cartier divisor, with points

$$\mathbf{P} \setminus \mathbf{P}^\circ = \big\{ (y \mapsto y_0 \in \ell_0) : y = \infty \text{ or } \ell_0 \subset \mathbf{P}\overline{W} \big\} = \mathbf{P}\mathcal{V}_1 \times_C \mathbf{P}\mathcal{T} \cup \mathbf{P}L_C \times_C \mathbf{P}\mathcal{V}_2.$$

An equation of $\mathbf{P} \setminus \mathbf{P}^\circ$ is as follows: Let $\pi_i \colon \mathbf{P}\mathcal{V}_i \to C$ and $\pi \colon \mathbf{P} \to C$ be the structure morphisms, let $\mathrm{pr}_i \colon \mathbf{P} \to \mathbf{P}\mathcal{V}_i$ be the projections, and for any $\mathcal{O}_{\mathbf{P}}$-module $\mathcal{F}$ and $a, b \in \mathbf{Z}$, write

$$\mathcal{F}(a,b) := \mathcal{F} \otimes \mathrm{pr}_1^* \mathcal{O}_{\pi_1}(a) \otimes \mathrm{pr}_2^* \mathcal{O}_{\pi_2}(b).$$

Then $\mathbf{P} \setminus \mathbf{P}^\circ = \mathrm{V}(u_1 u_2)$, where u_i is the map obtained by composing the Euler section eu_{π_i} with the quotient map for $\mathscr{V}_i$ in the two sequences above:

$$u_1 \colon \mathscr{O}_{\mathbf{P}}(-1,0) \hookrightarrow \pi^*\mathscr{V}_1 \twoheadrightarrow \pi^*\mathscr{O}_C(-1) \quad \text{and} \quad u_2 \colon \mathscr{O}_{\mathbf{P}}(0,-1) \hookrightarrow \pi^*\mathscr{V}_2 \twoheadrightarrow (V/\breve{W})_{\mathbf{P}}.$$

Returning to φ, observe that the morphism of 2.5 factors through a morphism $S^\circ \to \mathbf{P}^\circ$ given by

$$[\ell] \mapsto \big(\ell \cap \mathbf{P}\breve{W} \mapsto \varphi([\ell]) \in \mathrm{proj}_\infty(\ell)\big).$$

The image of this morphism can be described in terms of the geometry of $X_{\ell_0} := X \cap P_{\ell_0}$, where $P_{\ell_0} := \mathrm{proj}_\infty^{-1}(\ell_0)$ is the plane in $\mathbf{P}V$ spanned by ∞ and a line $\ell_0 \subset \mathbf{P}\overline{V}$:

2.7. Lemma. — *The image of $S^\circ \to \mathbf{P}^\circ$ is the scheme*

$$T^\circ = \big\{\, (y \mapsto y_0 \in \ell_0) \in \mathbf{P}^\circ : X_{\ell_0} \text{ is a cone with vertex } y \,\big\}.$$

If $(X, \infty, \mathbf{P}\breve{W})$ satisfies 2.1(i), then the morphism $S^\circ \to T^\circ$ is quasi-finite of degree q.

Proof. A point $(y \mapsto y_0 \in \ell_0) \in \mathbf{P}^\circ$ lies in the image T° if and only if there is a line ℓ contained in $X_{\ell_0} \setminus \infty$ through y. If $P_{\ell_0} \subset X$, then any y is a vertex of the plane X_{ℓ_0}, and any line in X_{ℓ_0} through y and not passing through ∞ witnesses membership in T°. Otherwise, X_{ℓ_0} is a q-bic curve which contains the line

$$\langle y, \infty \rangle = \mathrm{proj}_\infty^{-1}(y_0) = \mathrm{proj}_\infty^{-1}(\ell_0 \cap \mathbf{P}\overline{W}) = P_{\ell_0} \cap \mathbf{P}\breve{W}$$

spanned by y and ∞ as an irreducible component. Thus $(y \mapsto y_0 \in \ell_0)$ is contained in T° if and only if the residual curve $X_{\ell_0} - \ell_{y,\infty}$ contains a line passing through y. Classification of q-bic curves, as in 1.4, shows that this happens if and only if X_{ℓ_0} is a cone with vertex y. Finally, if the cone situation satisfies 2.1(i), then only this second case occurs, and this analysis shows that the fibres of $S^\circ \to T^\circ$ are the length q schemes parameterizing the lines in $X_{\ell_0} - \ell_{y,\infty}$. ■

2.8. Closure of T°. — To describe the closure of T° in $\mathbf{P}$, first extend the moduli description in 2.7 to all of $\mathbf{P}$ by considering the locally free $\mathscr{O}_{\mathbf{P}}$-module $\mathscr{P}$ of rank 3 defined by the diagram

$$\begin{tikzcd}
0 \arrow[r] & \pi^*\mathscr{V}_1 \arrow[r] & V_{\mathbf{P}} \arrow[r] & \pi^*\mathscr{V}_2 \arrow[r] & 0 \\
0 \arrow[r] & \pi^*\mathscr{V}_1 \arrow[r] \arrow[u, equal] & \mathscr{P} \arrow[r] \arrow[u, hook] & \mathscr{O}_{\mathbf{P}}(0,-1) \arrow[r] \arrow[u, hook, "\mathrm{eu}_{\pi_2}"] & 0
\end{tikzcd}$$

where the upper sequence is as in 2.6. The fibre at a point $(y \mapsto y_0 \in \ell_0) \in \mathbf{P}$ of

- $\mathscr{P}$ is the subspace of V underlying the plane P_{ℓ_0};
- $\pi^*\mathscr{V}_1$ is the line $\ell_{0,\infty} := \langle y_0, \infty \rangle$; and
- the tautological subbundle $\mathscr{O}_{\mathbf{P}}(-1,0) \hookrightarrow \pi^*\mathscr{V}_1$ is the point y.

Let $\beta_{\mathscr{P}} : \mathscr{P}^{[1]} \otimes \mathscr{P} \to \mathscr{O}_{\mathbf{P}}$ be the restriction of the q-bic form β. Since $\mathscr{V}_1$ is isotropic—it parameterizes lines in X!—the adjoints of $\beta_{\mathscr{P}}$ induce maps

$$\beta_{\mathscr{P}}^\vee : \pi^*\mathscr{V}_1^{[1]} \xrightarrow{\beta^\vee|_{\mathscr{V}_1}} \pi^*\mathscr{V}_2^\vee \xrightarrow{\mathrm{eu}_{\pi_2}^\vee} \mathscr{O}_{\mathbf{P}}(0,1) \quad \text{and} \quad \beta_{\mathscr{P}} : \pi^*\mathscr{V}_1 \xrightarrow{\beta|_{\mathscr{V}_1}} \pi^*\mathscr{V}_2^{[1],\vee} \xrightarrow{\mathrm{eu}_{\pi_2}^{[1],\vee}} \mathscr{O}_{\mathbf{P}}(0,q).$$

Set $v_1 := \beta_{\mathscr{P}}^\vee \circ \mathrm{eu}_{\pi_1}^{[1]}$ and $v_2 := \beta_{\mathscr{P}} \circ \mathrm{eu}_{\pi_1}$. Then $v := (v_1, v_2) \colon \mathscr{O}_{\mathbf{P}} \to \mathscr{O}_{\mathbf{P}}(q,1) \oplus \mathscr{O}_{\mathbf{P}}(1,q)$ vanishes at points where the line in $\mathscr{P}$ given by eu_{π_1} lies in the radical of $\beta_{\mathscr{P}}$, or, geometrically:

$$T' := \mathrm{V}(v) = \big\{\, (y \mapsto y_0 \in \ell_0) \in \mathbf{P} : X_{\ell_0} \text{ is a cone with vertex } y \,\big\}.$$

In particular, $T^\circ = T' \cap \mathbf{P}^\circ$.

This contains too much: for instance, T' contains the intersection of the irreducible components of $\mathbf{P}\setminus\mathbf{P}^\circ$, consisting of $(\infty\mapsto y_0\in\ell_0)$ where $\ell_0\subset\mathbf{P}\overline{W}$. In fact, the map $v=(v_1,v_2)^\vee$ often factors through $u:=(u_1,u_2)^\vee$. To explain, given splittings $\mathcal{V}_1\cong\mathcal{O}_C(-1)\oplus L_C$ and $\mathcal{V}_2\cong\mathcal{T}\oplus(V/\breve{W})_C$, write

$$u_1'\colon \mathcal{O}_{\mathbf{P}}(-1,0)\xrightarrow{\mathrm{eu}_{\pi_1}}\pi^*\mathcal{V}_1\longrightarrow L_{\mathbf{P}}\quad\text{and}\quad u_2'\colon\mathcal{O}_{\mathbf{P}}(0,-1)\xrightarrow{\mathrm{eu}_{\pi_2}}\pi^*\mathcal{V}_2\longrightarrow\pi^*\mathcal{T}$$

for the projection of the Euler sections to the subbundle.

2.9. Lemma. — *Assume there exists a 2-dimensional subspace $U\subset V$ such that*

(i) *$U\cap\breve{W}=L$, and*
(ii) *U admits an orthogonal complement W in V.*

Then the induced splittings $\mathcal{V}_1\cong\mathcal{O}_C(-1)\oplus L_C$ and $\mathcal{V}_2\cong\mathcal{T}\oplus(V/\breve{W})_C$ are such that the adjoint maps

$$\begin{aligned}\beta^\vee|_{\mathcal{V}_1}&=:\beta_1^\vee\oplus\beta_2^\vee\colon\mathcal{O}_C(-q)\oplus L_C^{[1]}\to\mathcal{T}^\vee\oplus(V/\breve{W})_C^\vee,\\ \beta|_{\mathcal{V}_1}&=:\beta_1\oplus\beta_2\colon\mathcal{O}_C(-1)\oplus L_C\to\mathcal{T}^{[1],\vee}\oplus(V/\breve{W})_C^{[1],\vee}\end{aligned}$$

are diagonal, and there exists a factorization $v=v'\circ(u_1,u_2)^\vee$ where v' is the map

$$\begin{pmatrix}u_2'\cdot\beta_1^\vee\cdot u_1^{q-1} & \beta_2^\vee\cdot u_1'^q\\ u_2'^q\cdot\beta_1 & u_2^{q-1}\cdot\beta_2\cdot u_1'\end{pmatrix}\colon\mathcal{O}_{\mathbf{P}}(1,0)\otimes\pi^*\mathcal{O}_C(-1)\oplus\mathcal{O}_{\mathbf{P}}(0,1)\otimes(V/\breve{W})_{\mathbf{P}}\to\mathcal{O}_{\mathbf{P}}(q,1)\oplus\mathcal{O}_{\mathbf{P}}(1,q).$$

Proof. The decomposition $V=W\oplus U$ induces splittings $\mathcal{V}_1\cong\mathcal{O}_C(-1)\oplus L_C$ and $\mathcal{V}_2\cong\mathcal{T}\oplus(V/\breve{W})_C$ by intersecting $\mathcal{V}_1$ with W_C as subbundles of V_C, and by projecting U_C to $\mathcal{V}_2$, respectively. The adjoint maps of β are now diagonal since W and U are orthogonal complements. The Euler sections split as $\mathrm{eu}_{\pi_1}=(u_1,u_1')^\vee$ and $\mathrm{eu}_{\pi_2}=(u_2',u_2)^\vee$, and so

$$\begin{aligned}v_1&=\mathrm{eu}_{\pi_2}^\vee\cdot\beta^\vee|_{\mathcal{V}_1}\cdot\mathrm{eu}_{\pi_1}^{[1]}=u_2'\cdot\beta_1^\vee\cdot u_1^q+u_2\cdot\beta_2^\vee\cdot u_1'^q,\text{ and}\\ v_2&=\mathrm{eu}_{\pi_2}^{[1],\vee}\cdot\beta|_{\mathcal{V}_1}\cdot\mathrm{eu}_{\pi_1}=u_2'^q\cdot\beta_1\cdot u_1+u_2^q\cdot\beta_2\cdot u_1'.\end{aligned}$$

Since u_1 and u_2 are locally multiplication by a scalar, their action commutes with the other maps present, and a direct computation shows that there is a factorization $v=v\circ(u_1,u_2)^\vee$, as claimed. ∎

The hypotheses of 2.9 are satisfied, for example, in all of the cone situations 2.2(i)–(iii), so, in particular, in all smooth cone situations as in 2.3(ii): in 2.2(i), U may be taken to be the span of L with any other isotropic Hermitian vector not lying in $\breve{W}$; in 2.2(ii) and 2.2(iii), U is necessarily the subspace supporting the subform of type $\mathbf{N}_2$. In contrast, no U exists in 2.2(v).

This setting provides a candidate for the Zariski closure of T° in $\mathbf{P}$. To state the result, write

$$\begin{aligned}\mathcal{E}_2&:=\mathcal{O}_{\mathbf{P}}(1,0)\otimes\pi^*\mathcal{O}_C(-1)\oplus\mathcal{O}_{\mathbf{P}}(0,1)\otimes(V/\breve{W})_{\mathbf{P}},\\ \mathcal{E}_1&:=\mathcal{O}_{\mathbf{P}}(q,1)\oplus\mathcal{O}_{\mathbf{P}}(1,q)\oplus\det(\mathcal{E}_2),\end{aligned}$$

and $\wedge u\colon\mathcal{E}_2\to\det(E_2)$ for the natural surjection to the torsion-free quotient of $\operatorname{coker}(u\colon\mathcal{O}_{\mathbf{P}}\to\mathcal{E}_2)$.

2.10. Proposition. — *In the setting of 2.9, the scheme $T:=T'\cap\mathrm{V}(\det(v'))$ contains the Zariski closure of T° in $\mathbf{P}$, and is the rank 1 degeneracy locus of*

$$\phi:=\left(\begin{smallmatrix}v'\\ \wedge u\end{smallmatrix}\right)\colon\mathcal{E}_2\to\mathcal{E}_1$$

If furthermore $\dim S^\circ=2$, then $\dim T=2$, T is connected, Cohen–Macaulay, and there is an exact complex of sheaves on $\mathbf{P}$ given by

$$0\longrightarrow\mathcal{E}_2(-q-1,-q-1)\xrightarrow{\phi}\mathcal{E}_1(-q-1,-q-1)\xrightarrow{\wedge^2\phi^\vee}\mathcal{O}_{\mathbf{P}}\longrightarrow\mathcal{O}_T\longrightarrow0.$$

Proof. By 2.8, T° is the vanishing locus of $v = v' \circ u$ on the open subscheme $\mathbf{P}^\circ$ where neither u_1 nor u_2 vanish. Therefore v' has rank at most 1 on T°, meaning $\det(v')|_{T^\circ} = 0$ and $T^\circ \subseteq T$. To express T as a degeneracy locus, restrict first to the vanishing locus of $\det(v')$, which is but the rank 1 locus of $v' \colon \mathcal{E}_2 \to \mathcal{O}_{\mathbf{P}}(q,1) \oplus \mathcal{O}_{\mathbf{P}}(1,q)$. Now the factorization $v = v' \circ u$ means that T is the locus where $\mathrm{im}(u) \subseteq \ker(v')$. On the one hand, $\ker(\wedge u)$ is the saturation in $\mathcal{E}_2$ of $\mathrm{im}(u)$. On the other hand, v' is a map between locally free sheaves, and so $\ker(v')$ is saturated. Therefore T is equivalently the locus in $\mathrm{V}(\det(v'))$ where $\ker(\wedge u) \subseteq \ker(v')$. Altogether, this means that T is the locus in $\mathbf{P}$ where $\phi := \left(\begin{smallmatrix} v' \\ \wedge u \end{smallmatrix}\right)$ has rank at most 1, as required.

For the remainder, note that its description as a degeneracy locus implies $\dim T \leq 2$. Suppose that $\dim S^\circ = 2$. Then its image T° has dimension at most 2. Since $T \setminus T^\circ \subseteq \mathbf{P} \setminus \mathbf{P}^\circ$, it follows that $\dim T = 2$. Since the base curve C is reduced, it is Cohen–Macaulay, and thus so is $\mathbf{P}$. With this, the remaining properties follow from the fact that T is a degeneracy locus of expected dimension: see [HE70, Theorem 1], or also [Ful98, Theorem 14.3(c)], for Cohen–Macaulayness; see [Laz04, Theorem B.2.2(ii)] for exactness of the Eagon–Northcott complex (EN_0) associated with ϕ. Finally, connectedness of T follows from that of S, see 1.8. ■

In many cases, T is the closure of T°. This is verified for smooth cone situations in 3.2.

2.11. — Write $\mathcal{T}_{\pi_i}$ for the pullback to $\mathbf{P}$ of the relative tangent bundle of $\pi_i \colon \mathbf{P}\mathcal{V}_i \to C$, and set

$$\mathcal{V} := \mathcal{H}\big(\mathcal{O}_{\mathbf{P}}(-1,0) \hookrightarrow V_{\mathbf{P}} \twoheadrightarrow \mathcal{T}_{\pi_2}(0,-1)\big) \cong \operatorname{coker}\big(\mathcal{O}_{\mathbf{P}}(-1,0) \hookrightarrow \pi^*\mathcal{V}_1 \hookrightarrow \mathcal{P}\big)$$

where $\mathcal{H}$ extracts the homology sheaf, and $\mathcal{P}$ is as in 2.8. Its associated $\mathbf{P}^1$-bundle is

$$\mathbf{P}\mathcal{V} = \big\{ ((y \in \ell) \mapsto (y_0 \in \ell_0)) : (y \mapsto y_0 \in \ell_0) \in \mathbf{P} \text{ and } \ell \text{ a line in } P_{\ell_0} \big\}.$$

Extracting the line $[\ell]$ yields a birational morphism $\mathbf{P}\mathcal{V} \to \mathbf{G}$ which is an isomorphism away from the locus where either $\infty \in \ell$ or $\ell \subset \mathbf{P}\breve{W}$. The discussion preceding 2.7 implies that S is contained in the image of $\mathbf{P}\mathcal{V}$ and is not completely contained in the non-isomorphism locus, so it has a well-defined strict transform $\tilde{S}$ along $\mathbf{P}\mathcal{V} \to \mathbf{G}$ that has a morphism $\tilde{S} \to T$; when $(X, \infty, \mathbf{P}\breve{W})$ satisfies 2.1(i), this resolves the rational map $\varphi \colon S \dashrightarrow T$ from 2.5. In summary, there is a commutative diagram:

$$\begin{tikzcd}
S \arrow[dr, dashed] & \tilde{S} \arrow[l] \arrow[r, hook] \arrow[d] & \mathbf{P}\mathcal{V} \arrow[d] \\
& T \arrow[r, hook] \arrow[dr] & \mathbf{P} \arrow[d, "\pi"] \\
& & C\,.
\end{tikzcd}$$

From now on, assume that $(X, \infty, \mathbf{P}\breve{W})$ satisfies 2.1(i). The remainder of this section is dedicated to constructing equations for $\tilde{S}$ by describing it as a bundle of q-bic points over T. Write $\rho \colon \mathbf{P}\mathcal{V}_T \to T$ for the projection of $\mathbf{P}\mathcal{V} \to \mathbf{P}$ restricted to T. Then 2.7 implies that $\tilde{S}$ is a hypersurface in $\mathbf{P}\mathcal{V}_T$. Let $(\mathcal{P}_T, \beta_{\mathcal{P}_T})$ be the restriction to T of the q-bic form $(\mathcal{P}, \beta_{\mathcal{P}})$ from 2.8. As a first step:

2.12. Lemma. — *The q-bic form $\beta_{\mathcal{P}_T}$ induces a q-bic form $\beta_{\mathcal{V}_T} \colon \mathcal{V}_T^{[1]} \otimes \mathcal{V}_T \to \mathcal{O}_T$ whose q-bic equation*

$$\beta_{\mathcal{V}_T}(\mathrm{eu}_\rho^{[1]}, \mathrm{eu}_\rho) \colon \mathcal{O}_\rho(-q-1) \hookrightarrow \rho^*\mathcal{V}_T^{[1]} \otimes \rho^*\mathcal{V}_T \to \mathcal{O}_{\mathbf{P}\mathcal{V}_T},$$

vanishes at $((y \in \ell) \mapsto (y_0 \in \ell_0)) \in \mathbf{P}\mathcal{V}_T$ if and only if ℓ is an isotropic line for β. In particular, this section vanishes on the strict transform $\tilde{S}$ of S.

Proof. By 2.8, the intersection $X_{\ell_0} = X \cap P_{\ell_0}$ is a cone with vertex y for every $(y \mapsto y_0 \in \ell_0) \in T$. The description of the fibres of $\mathcal{P}_T$ from 2.8 implies that the tautological subbundle $\mathcal{O}_T(-1,0) \hookrightarrow \mathcal{P}_T$ lies in the radical of the form $\beta_{\mathcal{P}_T}$, and so it induces a q-bic form $\beta_{\mathcal{V}_T}$ on the quotient $\mathcal{V}_T$. Comparing again with the description of the fibres of $\mathscr{P}_T$ shows that the fibre of the tautological subbundle $\mathscr{O}_\rho(-1)$ at $((y \in \ell) \mapsto (y_0 \in \ell_0))$ extracts the subspace of V underlying ℓ, whence the latter statement. ■

2.13. — Comparing the homology sheaf construction of $\mathscr{V}$ from 2.11 with the sequences for $\mathscr{V}_1$ and $\mathscr{V}_2$ from 2.6 gives a canonical short exact sequence

$$0 \to \mathscr{T}_{\pi_1}(-1,0) \to \mathscr{V} \to \mathscr{O}_{\mathbf{P}}(0,-1) \to 0.$$

The subbundle may be identified via the Euler sequence for $\mathbf{P}\mathscr{V}_1 \to C$ as:

$$\mathscr{T}_{\pi_1}(-1,0) = \operatorname{coker}(\mathrm{eu}_{\pi_1} \colon \mathscr{O}_{\mathbf{P}}(-1,0) \to \pi^*\mathscr{V}_1) \cong \det(\pi^*\mathscr{V}_1)(1,0) \cong \mathscr{O}_{\mathbf{P}}(1,0) \otimes \pi^*\mathscr{O}_C(-1) \otimes L.$$

Its inverse image under the quotient map $\mathscr{P} \to \mathscr{V}$ is the subbundle $\pi^*\mathscr{V}_1 \subset \mathscr{P}$, so its points are

$$\mathbf{P}(\mathscr{T}_{\pi_1}(-1,0)) = \{\, ((y \in \ell) \mapsto (y_0 \in \ell_0)) \in \mathbf{P}\mathscr{V} : \ell = \langle y_0, \infty \rangle \,\}.$$

Since $\ell = \langle y_0, \infty \rangle \subset \mathbf{P}\breve{W}$ whenever $(y \mapsto y_0 \in \ell_0) \in T$, this subbundle is isotropic for $\beta_{\mathscr{V}_T}$ by 2.12, yielding the following observation:

2.14. Lemma. — *The q-bic equation $\beta_{\mathscr{V}_T}(\mathrm{eu}_\rho^{[1]}, \mathrm{eu}_\rho)$ from 2.12 vanishes on $\mathbf{P}(\mathscr{T}_{\pi_1}(-1,0)|_T)$, and so it factors through the section*

$$v_3 := u_3^{-1}\beta_{\mathscr{V}_T}(\mathrm{eu}_\rho^{[1]}, \mathrm{eu}_\rho) \colon \mathscr{O}_{\mathbf{P}\mathscr{V}_T} \to \mathscr{O}_\rho(q) \otimes \rho^*\mathscr{O}_T(0,1)$$

where $u_3 \colon \mathscr{O}_\rho(-1) \to \rho^*\mathscr{V}_T \to \rho^*\mathscr{O}_T(0,-1)$ is the equation of the subbundle. ■

2.15. Lemma. — *The section v_3 vanishes on $\tilde{S}$, and $\tilde{S} \to T$ is finite flat of degree q onto its image.*

Proof. The discussion of 2.11 and 2.14 together with 2.4(i) implies that the intersection of $\tilde{S}$ with the exceptional locus of $\mathbf{P}\mathscr{V} \to \mathbf{G}$ is contained in $\mathbf{P}(\mathscr{T}_{\pi_1}(-1,0)|_T)$. Therefore v_3 vanishes on $\tilde{S}$ if and only if it vanishes on $S^\circ \cong \tilde{S} \setminus \mathbf{P}(\mathscr{T}_{\pi_1}(-1,0)|_T)$. Since u_3 is invertible on the latter open subscheme, v_3 vanishes on S° by 2.12.

Since $S^\circ \to T$ is quasi-finite of degree q by 2.7, the final statement follows upon showing that $\mathrm{V}(v_3) \to T$ is finite flat of degree q. Since v_3 is degree q on each fibre of $\mathbf{P}\mathscr{V}_T \to T$ by 2.14, it suffices to see that v_3 does not vanish on an entire fibre. But if v_3 did vanish on the fibre over $(y \mapsto y_0 \in \ell_0) \in T$, 2.12 would imply that all lines $\ell \subset P_{\ell_0}$ passing through y are isotropic for β, and hence P_{ℓ_0} would be contained in X. This is impossible with condition 2.1(i). ■

3. Smooth cone situation

The most useful cone situations that arise when studying q-bic threefolds which are either smooth or of type $\mathbf{1}^{\oplus 3} \oplus \mathbf{N}_2$ are the *smooth cone situations*: those $(X, \infty, \mathbf{P}\breve{W})$ which satisfy 2.1(ii) and 2.1(iii), and which were classified in 2.3(ii) to be as in either example 2.2(i) or 2.2(ii). In this setting, the scheme T constructed in 2.10 is the Zariski closure of T° and also a quotient of $\tilde{S}$ by a finite group scheme of order q: see 3.2 and 3.5. Furthermore, $\tilde{S}$ is a blowup of S along smooth points, see 3.7; this implies that φ from 2.5 extends to a morphism $S \to C$ and that the sheaf $\mathbf{R}^1\varphi_*\mathscr{O}_S$ carries a canonical filtration, crucial for the computations that follow: see 3.9 and 3.10.

Throughout, let $(X, \infty, \mathbf{P}\breve{W})$ be a smooth cone situation, fix an orthogonal decomposition $V \cong W \oplus U$ as in 2.9, and write $\mathscr{V}_1 \cong \mathscr{O}_C(-1) \oplus L_C$ and $\mathscr{V}_2 \cong \mathscr{T} \oplus (V/\breve{W})_C$ for the induced splittings. The Euler sections decompose as $\mathrm{eu}_{\pi_1} = (u_1, u_1')^\vee$ and $\mathrm{eu}_{\pi_2} = (u_2', u_2)^\vee$, and their vanishing loci are:

$$\mathrm{V}(u_1) = \{\, (y \mapsto y_0 \in \ell_0) \in \mathbf{P} : y = \infty \,\}, \qquad \mathrm{V}(u_2') = \{\, (y \mapsto y_0 \in \ell_0) \in \mathbf{P} : \mathbf{P}\overline{U} \in \ell_0 \,\},$$

$$\mathrm{V}(u_1') = \{\, (y \mapsto y_0 \in \ell_0) \in \mathbf{P} : y \in \mathbf{P}W \,\}, \qquad \mathrm{V}(u_2) = \{\, (y \mapsto y_0 \in \ell_0) \in \mathbf{P} : \ell_0 \subset \mathbf{P}\overline{W} \,\}.$$

Begin by describing the boundary of T° in T:

3.1. Proposition. — *The boundary $T \cap (\mathbf{P} \setminus \mathbf{P}^\circ)$ is the union of the effective Cartier divisors*

$$C' := \{(\infty \mapsto y_0 \in \ell_0) \in \mathbf{P} : y_0 = \mathbf{P}L_0 \in C \text{ and } \ell_0^{[1]} = \mathbf{P}L_0^\perp\} \text{ and}$$
$$D := \{(y \mapsto y_0 \in \mathbf{T}_{C,y_0}) \in \mathbf{P} : y_0 \in C_{\mathrm{Herm}} \text{ and } y \in \langle y_0, \infty\rangle\}.$$

Furthermore, C' is a purely inseparable multisection of degree q, and D is a disjoint union of q^3+1 smooth rational curves.

Proof. Consider the intersection between $T' = \mathrm{V}(v)$ with the irreducible components $Z_i := \mathrm{V}(u_i)$ of $Z := \mathbf{P} \setminus \mathbf{P}^\circ = \mathrm{V}(u_1u_2)$. The computation of 2.9 gives the first equalities in

$$T' \cap Z_1 = (u_1 = u_2 \cdot \beta_2^\vee \cdot u_1'^q = u_2^q \cdot \beta_2 \cdot u_1' = 0) = Z_1 \cap Z_2, \text{ and}$$
$$T' \cap Z_2 = (u_2 = u_2' \cdot \beta_1^\vee \cdot u_1^q = u_2'^q \cdot \beta_1 \cdot u_1 = 0) = (Z_1 \cap Z_2) \cup (u_2 = u_2' \cdot \beta_1^\vee = u_2'^q \cdot \beta_1 = 0).$$

The second equality is then clear for $T' \cap Z_2$. As for $T' \cap Z_1$, it is because u_1' and u_1 do not vanish simultaneously, and $\beta_2^\vee : L_C^{[1]} \to (V/\breve{W})_C^\vee$ is an isomorphism by 2.1(ii). Next, 2.9 implies that

$$-\det(v')|_{T' \cap Z} = u_2'^q \cdot \beta_1 \cdot \beta_2^\vee \cdot u_1'^q.$$

This cuts out the locus $(u_1 = u_2 = u_2'^q \cdot \beta_1 = 0)$ on $Z_1 \cap Z_2$ and vanishes on the second component of $T' \cap Z_2$. It remains to identify these with C' and D, respectively. The value of

$$u_2'^q \cdot \beta_1 : \mathcal{O}_C(-1) \to \mathcal{T}^{[1],\vee} \to \mathcal{O}_{\mathbf{P}}(0,q)$$

at a point $(y \mapsto y_0 \in \ell_0)$ is determined as follows: a local generator of $\mathcal{O}_C(-1)$ corresponds to a basis vector v of the subspace $L_0 \subset \overline{W}$ underlying y_0; β_1 maps this to the linear functional $\beta_{\overline{W}}(-,v)$ on $\overline{W}^{[1]}$; and the quotient map corresponds to restricting this to the linear space underlying ℓ_0. Thus this vanishes if and only if $\ell_0 = \mathbf{P}L_0^{\perp,[-1]}$, and so

$$(u_1 = u_2 = u_2'^q \cdot \beta_1 = 0) = \{(\infty \mapsto y_0 \in \ell_0) \in \mathbf{P} : y_0 = \mathbf{P}L_0 \in C \text{ and } \ell_0 = \mathbf{P}L_0^{\perp,[-1]}\},$$

which is precisely C'. Similarly, $u_2' \cdot \beta_1^\vee$ vanishes at points $(y \mapsto y_0 \in \ell_0)$ when $\ell_0 = \mathbf{P}L_0^{[1],\perp}$. Therefore

$$(u_2 = u_2' \cdot \beta_1^\vee = u_2'^q \cdot \beta_1 = 0) = \{(y \mapsto y_0 \in \ell_0) \in \mathbf{P} : y_0 = \mathbf{P}L_0 \in C \text{ and } \ell_0 = \mathbf{P}L_0^{[1],\perp} = \mathbf{P}L_0^{\perp,[-1]}\}.$$

This is D since $L_0^{[1],\perp} = L_0^{\perp,[-1]}$ if and only if y_0 is a Hermitian point of C, see 1.6.

Both C' and D are effective Cartier divisors on T: The analysis shows that $C' = T \cap Z_1$ and $C' + D = T \cap Z_2$; since D is the difference of effective Cartier divisors, it is Cartier itself. The moduli description easily shows that D is a union of fibres of $\mathbf{P}\mathcal{V}_1$ over the q^3+1 Hermitian points of C. To see that C' is a multisection, let $\phi_C : C \to C$ be the map that sends a point y_1 to the residual intersection point with its tangent line $\mathbf{T}_{C,y_1}$, as in 1.6. I claim that there exists a morphism

$$\phi_C' : C \to C' : y_1 \mapsto (\infty \mapsto \phi_C(y_1) \in \mathbf{T}_{C,y_1}).$$

If $y_1 = \mathbf{P}L_1$, then $L_0 := \sigma_\beta(L_1^{[2]})$ underlies $\phi_C(y_1)$. Since $\sigma_\beta = \beta^{-1} \circ \beta^{[1],\vee}$, the diagram

$$\begin{tikzcd}
V^{[1]} \arrow[r, "\beta^\vee"'] \arrow[d, equal] & V^\vee \arrow[r] \arrow[d, "\sigma_\beta^\vee"] & L_0^\vee \arrow[d, "\sigma_\beta^\vee"] \\
V^{[1]} \arrow[r, "\beta^{[1]}"] & V^{[2],\vee} \arrow[r] & L_1^{[2],\vee}
\end{tikzcd}$$

commutes, and it follows that $\mathbf{T}_{C,y_1}^{[1]}$ has underlying linear space $L_0^\perp$ and that ϕ_C' exists. Observe that the projection $C' \to C$ and ϕ_C' are both of degree q, the latter because $\mathbf{T}_{C,y_1}$ intersects C at y_1 generically with multiplicity q. Since ϕ_C is of purely inseparable degree q^2, it follows that ϕ_C' is surjective, and so C' is a purely inseparable multisection. ∎

3.2. Corollary. — *T is the Zariski closure of T° in $\mathbf{P}$.*

Proof. If not, then some irreducible component of $T \setminus T^\circ$ would be an irreducible component of T. But this is impossible: on the one hand, $T \setminus T^\circ$ is of pure dimension 1 by 3.1; on the other hand, T is connected and Cohen–Macaulay by 2.10 and is therefore equidimension 2 by [Stacks, 00OV]. ■

3.3. Corollary. — *The map $\rho \colon \tilde{S} \to T$ is surjective and finite flat of degree q, $\tilde{S}$ is the vanishing locus of v_3 in $\mathbf{P}\mathcal{V}_T$, and there is a short exact sequence of bundles on T given by*

$$0 \to \mathcal{O}_T \to \rho_*\mathcal{O}_{\tilde{S}} \to \mathrm{Div}^{q-2}(\mathcal{V}_T)(1,-2) \otimes \pi^*\mathcal{O}_C(-1) \otimes L \to 0.$$

In particular, $\rho_*\mathcal{O}_{\tilde{S}}$ has an increasing filtration whose graded pieces are

$$\mathrm{gr}_i(\rho_*\mathcal{O}_{\tilde{S}}) = \begin{cases} \mathcal{O}_T & \text{if } i = 0, \\ (\pi^*\mathcal{O}_C(-q+i) \otimes L^{\otimes q-i})(q-i, -i-1) & \text{if } 1 \leq i \leq q-1. \end{cases}$$

Proof. Since T is the closure of T° by 3.2, the map $\rho \colon \tilde{S} \to T$ is proper and dominant, whence surjective; 2.15 now implies ρ is flat of degee q, and that $\tilde{S}$ is the vanishing locus of v_3. Then 2.14 gives a short exact sequence of sheaves on $\mathbf{P}\mathcal{V}_T$:

$$0 \to \mathcal{O}_\rho(-q) \otimes \rho^*\mathcal{O}_T(0,-1) \xrightarrow{v_3} \mathcal{O}_{\mathbf{P}\mathcal{V}_T} \to \mathcal{O}_{\tilde{S}} \to 0.$$

Pushing this along ρ gives a short exact sequence of $\mathcal{O}_T$-modules

$$0 \to \mathcal{O}_T \to \rho_*\mathcal{O}_{\tilde{S}} \to \mathbf{R}^1\rho_*\mathcal{O}_\rho(-q) \otimes \mathcal{O}_T(0,-1) \to 0.$$

The Euler sequence gives $\omega_\rho \cong \rho^*\det(\mathcal{V}_T^\vee) \otimes \mathcal{O}_\rho(-2)$ and so by Grothendieck duality

$$\begin{aligned} \mathbf{R}^1\rho_*\mathcal{O}_\rho(-q) &\cong \mathbf{R}\rho_*\mathbf{R}\mathcal{H}om_{\mathcal{O}_{\mathbf{P}\mathcal{V}_T}}(\mathcal{O}_\rho(q) \otimes \omega_\rho, \omega_\rho) \cong \mathbf{R}\mathcal{H}om_{\mathcal{O}_T}(\mathbf{R}\rho_*\mathcal{O}_\rho(q-2) \otimes \det(\mathcal{V}_T^\vee), \mathcal{O}_T) \\ &\cong \mathrm{Div}^{q-2}(\mathcal{V}_T) \otimes \det(\mathcal{V}_T). \end{aligned}$$

Since $\det(\mathcal{V}_T) \cong (\pi^*\mathcal{O}_C(-1) \otimes L)(1,-1)$, this gives the exact sequence in the statement; the filtration comes from applying divided powers to the short exact sequence for $\mathcal{V}_T$ preceding 2.14. ■

The map $\rho \colon \tilde{S} \to T$ is a quotient map. To explain, let G be the subgroup scheme of $\mathbf{GL}_V$ which preserves both the flag $L \subset \breve{W} \subset V$ and the q-bic form β, and induces the identity on $\breve{W}$ and $V/\breve{W}$.

3.4. Lemma. — *If $(X, \infty, \mathbf{P}\breve{W})$ is a smooth cone situation, then*

$$G \cong \begin{cases} \mathbf{F}_q & \text{if } (X, \infty, \mathbf{P}\breve{W}) \text{ is as in 2.2(i), and} \\ \boldsymbol{\alpha}_q & \text{if } (X, \infty, \mathbf{P}\breve{W}) \text{ is as in 2.2(ii).} \end{cases}$$

Proof. A point g of G induces the identity on $\breve{W}$, so $\delta_g := g - \mathrm{id}_V$ descends to a map $V/\breve{W} \to V$; since g also induces the identity on $V/\breve{W}$, δ_g factors as a map $V/\breve{W} \to \breve{W}$. Thus the assignment $g \mapsto \delta_g$ yields a closed immersion $\delta \colon G \to \mathbf{Hom}(V/\breve{W}, \breve{W})$, the latter viewed as a vector group. In fact, δ factors through the algebraic subgroup $\mathbf{Hom}(V/\breve{W}, L)$: that G preserves β and acts as the identity on $\breve{W}$ together means that

$$\beta(\delta_g(v)^{[1]}, w) = \beta(w', \delta_g(v)) = 0$$

for every $\mathbf{k}$-algebra A, $g \in G(A)$, $v \in V \otimes_{\mathbf{k}} A$, $w \in \breve{W} \otimes_{\mathbf{k}} A$, and $w' \in (\breve{W} \otimes_{\mathbf{k}} A)^{[1]}$. Splitting $\breve{W} \cong W \oplus L$ as in 2.9 then shows that $\delta_g(v) \in L \otimes_{\mathbf{k}} A$.

Construct an equation of G in $\mathbf{Hom}(V/\breve{W}, L)$ as follows: Fix a nonzero $w \in L$, and choose $v \in V$ such that its image $\bar{v} \in V/\breve{W}$ is nonzero, so that $(\bar{v} \mapsto t \cdot w) \mapsto t$ is an isomorphism $\mathbf{Hom}(V/\breve{W}, L) \cong \mathbf{G}_a$.

If δ_g corresponds to $t \in \mathbf{G}_a(A)$ in this way, then

$$0 = \beta((g\cdot v)^{[1]}, g\cdot v) - \beta(v^{[1]}, v) \pm \beta(v^{[1]}, g\cdot v) = \beta(w^{[1]}, v)t^q + \beta(v^{[1]}, w)t.$$

Since ∞ is a smooth point, $L^{[1],\perp} = \breve{W}$ as explained in 2.1, and so $\beta(w^{[1]}, v)$ is a nonzero scalar. The scalar $\beta(v^{[1]}, w)$ is nonzero in 2.2(i), and so $G \cong \mathbf{F}_q$; whereas it is zero in 2.2(ii), and so $G \cong \boldsymbol{\alpha}_q$. ∎

The linear action of G on V induces an action on the schemes under consideration. First, it is straightforward that this action is trivial on C and T: For C, this is because G acts trivially on $\breve{W}$ and fixes L. For T, its points are triples $(y \mapsto y_0 \in \ell_0)$ where $y_0 \in C$, $y \in \mathbf{P}\breve{W}$, and $\ell_0 \subset \mathbf{P}\overline{V}$ intersects C at y_0; since G moves neither y_0 nor y, and since G maps $V/\breve{W}$ to L as in the proof of 3.4, it does not move ℓ_0. Next, comparing with the description in 2.14, this implies that G fixes the subbundle $\mathbf{P}(\mathscr{T}_{\pi_1}(-1,0))$ in $\mathbf{P}\mathscr{V}_T$. Finally, the action of G on $\tilde{S}$ is as follows:

3.5. Lemma. — *The morphism $\rho : \tilde{S} \to T$ is the quotient map for G.*

Proof. The unipotent algebraic group $\mathbf{Hom}(V/\breve{W}, L)$ acts freely on the open subscheme of $\mathbf{G}$ parameterizing lines ℓ satisfying $\infty \notin \ell$ and $\ell \not\subset \mathbf{P}\breve{W}$. The proof of 3.4 shows that G is a closed subgroup scheme of this unipotent group, and so G acts freely on the open subscheme S° of S. The result follows upon identifying S° with the open subscheme $\tilde{S}$ consisting of points $((y \in \ell) \mapsto (y_0 \in \ell_0))$ where $y \neq \infty$ and $\ell \not\subset \mathbf{P}\breve{W}$: Indeed the canonical morphism $\tilde{S}/G \to T$ is an isomorphism over T°. Since $\tilde{S} \to T$ is surjective and finite of degree q by 3.3, and lengths of fibres of finite morphisms are upper semicontinuous by Nakayama, $\tilde{S}/G \to T$ has degree 1, and so is an isomorphism. ∎

Putting 2.7, 3.1, and 3.3 together identifies the points of the complement $S \setminus S^\circ$ as follows:

3.6. Lemma. — *The complement of S° in $\tilde{S}$ the union of effective Cartier divisors*

$$\tilde{C}_\infty := \{((\infty \in \ell) \mapsto (y_0 \in \ell_0)) \in \tilde{S} : y_0 = \mathbf{P}L_0 \in C,\ \ell_0^{[1]} = \mathbf{P}L_0^\perp,\ \ell = \langle y_0, \infty\rangle\} \text{ and}$$
$$E := \{((y \in \ell) \mapsto (y_0 \in \mathbf{T}_{C,y_0})) \in \tilde{S} : y_0 \in C_{\mathrm{Herm}} \text{ and } \ell = \langle y_0, \infty\rangle\}.$$

Proof. It remains to prove that $\tilde{C}_\infty$ and E are Cartier in $\tilde{S}$. Let C_∞ be the closed subscheme of S parameterizing lines through ∞; it follows from 2.4(ii) that it is isomorphic to the smooth q-bic curve C, and that, by 1.8, S is smooth along C_∞. Therefore the description of its points shows that $\tilde{C}_\infty$ is the strict transform of the effective Cartier divisor C_∞, and so it, too, is effective Cartier. Since $\tilde{C}_\infty + E$ is the vanishing locus of the section u_3 from 2.14, it follows that E is also Cartier. ∎

Observing that the Hermitian points of C_∞ parameterize lines ℓ in X which project to Hermitian points of C and comparing with 3.6 essentially gives:

3.7. Corollary. — *The morphism $\tilde{S} \to S$ is a blowup along the Hermitian points of C_∞.*

Proof. The Hermitian points of C_∞ pullback to the effective Cartier divisor E in $\tilde{S}$, so the blowup $S' \to S$ along these points admits a canonical morphism $S' \to \tilde{S}$. Since S is smooth along C_∞, Zariski's Main Theorem as in [Har77, Corollary III.11.4] applies to show that $\tilde{S} \to S'$ is an isomorphism. ∎

In particular, this implies that $\tilde{S}$ and T are smooth along their boundary:

3.8. Corollary. — *$\tilde{S}$ is smooth along $\tilde{C}_\infty \cup E$, and T is smooth along $C' \cup D$.*

Proof. Smoothness of $\tilde{S}$ along $\tilde{C}_\infty \cup E$ follows from that of S along C_∞; that of T along $C' \cup D$ is because $\rho : \tilde{S} \to T$ is flat by 3.3, and smoothness descends along flat morphisms, see [Stacks, 05AW]. ∎

3.9. Corollary. — *The rational map $S \dashrightarrow T$ is defined away from the Hermitian points of C_∞, and the rational map $\varphi \colon S \dashrightarrow C$ extends to a morphism.*

Proof. The rational maps from S to both T and C are resolved up on $\tilde{S}$, so the statement about $S \dashrightarrow T$ follows directly from 3.7, and that about $\varphi \colon S \dashrightarrow C$ follows from 3.6 which implies that each component of the exceptional divisor E is mapped to a single point along $\tilde{S} \to C$. ■

Since the boundary divisor Z in $\mathbf{P}$ is relatively ample and since $T \cap Z \to C$ has connected fibres by 3.1, the fibres of $T \to C$ are connected. Using 3.5 and 3.6 then implies the same about the fibres of $\tilde{S} \to C$, and then 3.7 implies the same for $S \to C$. This almost implies that, for instance, $\varphi_*\mathcal{O}_S \cong \mathcal{O}_C$, but there is the matter of reduced fibres and the possibility of a factoring through a purely inseparable cover of C. At any rate, the following clarifies the structure of $\mathbf{R}\varphi_*\mathcal{O}_S$:

3.10. Lemma. — *The natural maps give isomorphisms*

$$\varphi_*\mathcal{O}_S \cong (\pi \circ \rho)_*\mathcal{O}_{\tilde{S}} \cong \pi_*\mathcal{O}_T \cong \mathcal{O}_C,$$

and $\mathbf{R}^1\varphi_*\mathcal{O}_S$ is locally free and carries a filtration with graded pieces

$$\mathrm{gr}_i(\mathbf{R}^1\varphi_*\mathcal{O}_S) \cong \begin{cases} \mathbf{R}^1\pi_*\mathcal{O}_T & \text{if } i = 0, \text{ and} \\ \mathcal{O}_C(-q+i) \otimes L^{\otimes q-i} \otimes \mathbf{R}^1\pi_*\mathcal{O}_T(q-i,-i-1) & \text{if } 1 \leq i \leq q-1. \end{cases}$$

Proof. By 2.10, the structure sheaf of T admits a resolution on $\mathbf{P}$ of the form

$$0 \to \mathcal{E}_2(-q-1,-q-1) \to \mathcal{E}_1(-q-1,-q-1) \to \mathcal{O}_{\mathbf{P}} \to \mathcal{O}_T \to 0.$$

The twists of $\mathcal{E}_1$ and $\mathcal{E}_2$ are sums of negative line bundles on a $\mathbf{P}^1 \times \mathbf{P}^2$-bundle over C, and so cohomology is in degree 3. Therefore the spectral sequence computing cohomology of $\mathcal{O}_T$ shows that the natural map $\mathcal{O}_C \to \pi_*\mathcal{O}_T$ is an isomorphism.

By 3.3, $\rho_*\mathcal{O}_{\tilde{S}}$ has a filtration with graded pieces $\mathcal{O}_T$ and $\mathcal{O}_T(q-i,-i-1) \otimes \pi^*\mathcal{O}_C(-q+i)$ where $1 \leq i \leq q-1$. The resolution above implies that the latter terms have vanishing pushforward: $\mathcal{E}_1(-i-1,-q-i-2)$ consists of line bundles which are negative on the $\mathbf{P}^2$-side of $\mathbf{P} \to C$, so cohomology is supported in degrees 2 and 3; whereas $\mathcal{E}_2(-i-1,-q-i-2)$ is negative on both factors and so cohomology is supported in degree 3. Therefore $\mathcal{O}_C \to (\pi \circ \rho)_*\mathcal{O}_{\tilde{S}}$ is an isomorphism. Finally, since $\tilde{S} \to S$ is a blowup at smooth point by 3.7, it follows that $\mathbf{R}\varphi_*\mathcal{O}_S \cong \mathbf{R}(\pi \circ \rho)_*\mathcal{O}_{\tilde{S}}$ and so $\mathcal{O}_C \to \varphi_*\mathcal{O}_S$ is an isomorphism, $\mathbf{R}^1\varphi_*\mathcal{O}_S$ is locally free by cohomology and base change, and carries the filtration by 3.3. ■

4. q-BIC THREEFOLDS OF TYPE $\mathbf{1}^{\oplus 3} \oplus \mathbf{N}_2$

This and the following two sections are concerned with the geometry of mildly singular q-bic threefolds, namely, those of type $\mathbf{1}^{\oplus 3} \oplus \mathbf{N}_2$. The purpose of this section is to construct the normalization $\nu \colon S^\nu \to S$ of the Fano surface, see 4.7, and to relate the cohomology of $\mathcal{O}_S$ with that of an $\mathcal{O}_C$-module $\mathcal{F}$ related to the quotient $\nu_*\mathcal{O}_{S^\nu}/\mathcal{O}_S$, see 4.13 and 4.14.

4.1. — Throughout §§4–6, let (V,β) be a q-bic form of type $\mathbf{1}^{\oplus 3} \oplus \mathbf{N}_2$, and let X be the associated q-bic threefold. The two kernels $L_+ := V^{\perp,[-1]}$ and $L_- := V^{[1],\perp}$ of β underlie the singular point x_+ and the special smooth point x_- of X. Setting $U := L_- \oplus L_+$, the classification of q-bic forms provides a canonical orthogonal decomposition $V = W \oplus U$ where β_W is of type $\mathbf{1}^{\oplus 3}$. The plane $\mathbf{P}W$ intersects X at the smooth q-bic curve C defined by β_W; this curve is canonically identified as the base of the cones X_- and X_+ in the cone situations

$$(X, x_-, \mathbf{P}L_-^{[1],\perp}) \quad \text{and} \quad (X, x_+, \mathbf{P}L_+^{\perp,[-1]})$$

as in 2.2(ii) and 2.2(iii).

The Fano scheme S of lines in X is of expected dimension 2: this follows, for example, from 2.3(i) applied to either cone situation above. Alternatively, by 2.4(i) and 2.4(ii), the subschemes $C_\pm \subset S$ parameterizing lines through the special points $x_\pm \in X$ are supported on a scheme isomorphic to C, so the singular locus of S has dimension 1, and the discussion of 1.8 shows S is a surface.

4.2. Automorphisms. — Consider the automorphism group scheme $\mathbf{Aut}(V,\beta)$ of the q-bic form, as introduced in [Che23a, 5.1]. Specializing the computation of [Che22, 1.3.7] with $a=1$ and $b=3$ gives the following explicit description as a closed sub-group scheme of $\mathbf{GL}_5$:

$$\mathbf{Aut}(V,\beta) \cong \left\{ \left(\begin{array}{c|cc} A & 0 & \mathbf{x} \\ \hline \mathbf{y}^\vee & \lambda^{-1} & \epsilon \\ 0 & 0 & \lambda^q \end{array} \right) \in \mathbf{GL}_5 \;\middle|\; \begin{array}{lll} \lambda \in \mathbf{G}_m, & \mathbf{x} \in \boldsymbol{\alpha}_q^3, & A \in \mathrm{U}_3(q), \\ \epsilon \in \boldsymbol{\alpha}_q, & \mathbf{y} \in \boldsymbol{\alpha}_{q^2}^3, & \lambda^q A^{\vee,[1]}\mathbf{x} = \mathbf{y}^{[1]} \end{array} \right\}.$$

A particularly useful subgroup is that which preserves the orthogonal decomposition $V = W \oplus U$:

$$\mathrm{G} := \mathbf{Aut}(W,\beta_W) \times \mathbf{Aut}(L_- \subset U, \beta_U) \cong \mathrm{U}_3(q) \times \left\{ \begin{pmatrix} \lambda^{-1} & \epsilon \\ 0 & \lambda^q \end{pmatrix} : \lambda \in \mathbf{G}_m, \epsilon \in \boldsymbol{\alpha}_q \right\}$$

Observe that the torus $\mathbf{G}_m$ in $\mathbf{Aut}(V,\beta)$ acts, in particular, on X and the Fano surface S; it is straightforward to check that its fixed schemes are $X^{\mathbf{G}_m} = \{x_-, x_+\} \cup C$ and $S^{\mathbf{G}_m} = C_- \cup C_+$, respectively.

4.3. Cone situations. — Both cone situations in 4.1 associated with the special points $x_\mp \in X$ satisfy 2.1(i) and 2.1(iii), so 2.5 gives rational maps

$$\varphi_- \colon S \longrightarrow C \quad \text{and} \quad \varphi_+ \colon S \dashrightarrow C,$$

where the orthogonal decomposition $V = W \oplus U$ identifies the target as the curve $C = X \cap \mathbf{P}W$. Since φ_- arises from a smooth cone situation, it extends to a morphism by 3.9, whereas φ_+ is defined only away from C_+. Writing $\mathrm{proj}_{\mathbf{P}U} \colon X \dashrightarrow \mathbf{P}W$ for the rational map induced by linear projection centred at $\mathbf{P}U$, the specific geometry of X offers an alternative description of the maps $\varphi_\mp$:

4.4. Lemma. — *Let $\ell \subset X$ be a line not passing through either $x_\mp$. Then $\ell_0 := \mathrm{proj}_{\mathbf{P}U}(\ell)$ is a line in $\mathbf{P}W$ tangent to C at the point $\mathrm{proj}_{\mathbf{P}U}(\ell \cap X_+)$ with residual intersection point $\mathrm{proj}_{\mathbf{P}U}(\ell \cap X_-)$, so*

$$\begin{aligned} \varphi_+([\ell]) &= \textit{point of tangency between } \ell_0 \textit{ and } C, \\ \varphi_-([\ell]) &= \textit{residual point of intersection between } \ell_0 \textit{ and } C. \end{aligned}$$

Proof. Projection from $\mathbf{P}U$ contracts only lines through $x_\mp$, so ℓ_0 is a line in $\mathbf{P}W$. Since $\mathbf{P}U$ intersects X only at the vertices $x_\mp$ of the two cones over C, and since x_+ is a singular point of multiplicity q,

$$C \cap \ell_0 = \mathrm{proj}_{\mathbf{P}U}\big(\ell \cap \mathrm{proj}_{\mathbf{P}U}^{-1}(C)\big) = \mathrm{proj}_{\mathbf{P}U}\big(\ell \cap (X_- \cup qX_+)\big).$$

Thus ℓ_0 and C are tangent at $\mathrm{proj}_{\mathbf{P}U}(\ell \cap X_+)$ and have residual point of intersection at $\mathrm{proj}_{\mathbf{P}U}(\ell \cap X_-)$. On the other hand, the definition of $\varphi_\mp$ from 2.5 gives

$$\varphi_\mp([\ell]) = \mathrm{proj}_{x_\mp}(\ell \cap X_\mp) = \mathrm{proj}_{\mathbf{P}U}(\ell \cap X_\mp)$$

with the second equality because $x_\pm \notin X_\mp$. ■

With more notation, the proof works for families of lines in $X \setminus \{x_-, x_+\}$. This leads to another proof of 3.9, that φ_- extends over C_-: 4.4 means that $\varphi_-([\ell])$ is the residual intersection point of C with its tangent line at $\mathrm{proj}_{\mathbf{P}U}(\ell \cap X_+)$, and this makes sense even when $x_- \in \ell$. More interestingly, this gives an alternative moduli interpretation of $S \setminus C_+$ as a scheme over C via φ_+. To describe it, let

$\mathscr{E}_C$ be the *embedded tangent bundle* C in $\mathbf{P}W$, which is defined via pullback of the right square in the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{O}_C(-1) & \longrightarrow & \mathscr{E}_C & \longrightarrow & \mathscr{T}_C(-1) & \longrightarrow & 0 \\
 & & \| & & \cap & & \cap & & \\
0 & \longrightarrow & \mathscr{O}_C(-1) & \longrightarrow & W_C(-1) & \longrightarrow & \mathscr{T} & \longrightarrow & 0
\end{array}$$

where, as in 2.6, $\mathscr{T} := \mathscr{T}_{\mathbf{P}W}(-1)|_C$. The fibre of $\mathscr{E}_C$ over a point $y_0 = \mathbf{P}L_0$ is the subspace $L_0^{[1],\perp}$ in W underlying the embedded tangent space $\mathbf{T}_{C,y_0}$, see 1.5. The subbundle $\mathscr{W} := \mathscr{E}_C \oplus U_C$ of V_C defines the family $\mathbf{P}\mathscr{W} \to C$ of hyperplanes in $\mathbf{P}V$ spanned by the tangent lines to C and U. Let $X_{\mathbf{P}\mathscr{W}} := X \times_{\mathbf{P}V} \mathbf{P}\mathscr{W}$ be the corresponding family of hyperplane sections of X. A reformulation of 4.4 in these terms is then:

4.5. Corollary. — *Let $\mathbf{L}^\circ$ be the restriction of the Fano correspondence $S \leftarrow \mathbf{L} \rightarrow X$ to $S \setminus C_+$. Then the morphism $\mathbf{L}^\circ \rightarrow X$ factors through $X_{\mathbf{P}\mathscr{W}}$ and fits into a commutative diagram*

$$\begin{array}{ccccc}
\mathbf{L}^\circ & \longrightarrow & X_{\mathbf{P}\mathscr{W}} & \longrightarrow & X \\
\big\downarrow & & \big\downarrow & & \\
S \setminus C_+ & \xrightarrow{\varphi_+} & C\,. & &
\end{array}$$

Proof. Away from $C_- \subset S$, this follows from 4.4. It remains to observe that the lines through x_- correspond to the subbundle $\mathscr{O}_C(-1) \oplus L_{-,C}$ of $\mathscr{W}$. ■

A line $\ell \subset X \setminus x_+$ such that $\varphi_+([\ell]) = y_0$ intersects X_+ along the line $\langle x_+, y_0\rangle$. These lines are globally parameterized by the subbundle $\mathscr{W}' := \mathscr{O}_C(-1) \oplus L_{+,C}$; together with 4.4 and 4.5, this implies that $\mathbf{L}^\circ \times_{X_{\mathbf{P}\mathscr{W}}} \mathbf{P}\mathscr{W}'$ projects isomorphically to $S \setminus C_+$. This suggests that φ_+ may be resolved by considering all lines in $X_{\mathbf{P}\mathscr{W}}$ that pass through $\mathbf{P}\mathscr{W}'$. To proceed, consider the morphism $X_{\mathbf{P}\mathscr{W}} \to C$:

4.6. Lemma. — *$X_{\mathbf{P}\mathscr{W}} \to C$ is a family of corank 2 q-bic surfaces with singular locus $\mathbf{P}\mathscr{W}'$.*

Proof. This family of q-bic surfaces over C is determined by the q-bic form $\beta_{\mathscr{W}} \colon \mathscr{W}^{[1]} \otimes \mathscr{W} \to \mathscr{O}_C$ obtained by restricting β to $\mathscr{W}$. The orthogonal splitting $V = W \oplus U$ restricts to a decomposition

$$(\mathscr{W}, \beta_{\mathscr{W}}) = (\mathscr{E}_C, \beta_{W,\tan}) \oplus (U_C, \beta_U)$$

where $\beta_{W,\tan}$ is q-bic form induced by β_W on the embedded tangent spaces to C. Observe that β_U is of constant type $\mathbf{N}_2$, and $\beta_{W,\tan}$ is of type $\mathbf{0} \oplus \mathbf{1}$ or $\mathbf{N}_2$ depending on whether or not the point of tangency is a Hermitian point of C; in particular, $\beta_{\mathscr{W}}$ is everywhere of corank 2. By [Che23b, 2.6], the nonsmooth locus of $X_{\mathbf{P}\mathscr{W}} \to C$ corresponds to the subbundle

$$\mathscr{W}^\perp = \mathscr{E}_C^{\perp_{\beta_{W,\tan}}} \oplus U_C^{\perp_{\beta_U}} = \mathscr{O}_C(-q) \oplus L_{+,C}^{[1]} \subset \mathscr{W}^{[1]}$$

where the two orthogonals can be computed geometrically by noting that $X \cap \mathbf{P}U$ is singular at x_+, and the intersection between C and its tangent line at y_0 is singular at y_0. This bundle descends through Frobenius to $\mathscr{W}'$ as in the statement, and so $X_{\mathbf{P}\mathscr{W}}$ itself is singular along thereon. ■

Construct the family of lines in $X_{\mathbf{P}\mathscr{W}}$ over C through $\mathbf{P}\mathscr{W}'$ as follows: Set $\mathscr{W}'' := \mathscr{W}/\mathscr{W}'$, let $S^\nu := \mathbf{P}\mathscr{W}''$ be the associated $\mathbf{P}^1$-bundle, and write $\tilde{\varphi}_+ \colon S^\nu \to C$ for the structure map. Linear projection over C with centre $\mathbf{P}\mathscr{W}'$ produces a rational map $X_{\mathbf{P}\mathscr{W}} \dashrightarrow S^\nu$ which is resolved on the blowup $\tilde{X}_{\mathbf{P}\mathscr{W}}$ along $\mathbf{P}\mathscr{W}'$. Since $X_{\mathbf{P}\mathscr{W}}$ is a family of q-bic surfaces over C, its singular locus has multiplicity q, the fibres of $\tilde{X}_{\mathbf{P}\mathscr{W}} \to S^\nu$ are curves of degree 1 in $\mathbf{P}V$. In other words, this is a family of lines in X, and so defines a morphism $\nu \colon S^\nu \to S$.

4.7. Proposition. — *The morphism $\nu\colon S^\nu \to S$ is the normalization, fits into a commutative diagram*

$$\begin{tikzcd} S^\nu \arrow[rr, "\nu"] \arrow[dr, "\tilde\varphi_+"'] & & S \arrow[dl, dashed, "\varphi_+"] \\ & C & \end{tikzcd}$$

and satisfies $\nu^*\mathcal{O}_S(1) = \tilde\varphi_+^*\mathcal{O}_C(1) \otimes \mathcal{O}_{\tilde\varphi_+}(q+1) \otimes L_+^\vee$.

Proof. Note $\mathcal{W}'' \cong \mathcal{T}_C(-1) \oplus L_{-,C}$ and the fibres of $\tilde{X}_{\mathbf{P}\mathcal{W}} \to S^\nu$ over $C_+^\nu := \mathbf{P}(\mathcal{T}_C(-1))$ are those lines through $x_+ \in X$. The statement of and the comments following 4.5 imply that $S \setminus C_+$ and $S^\nu \setminus C_+^\nu$ represent the same moduli problem over C, and this implies the first two statements.

To compute the pullback of the Plücker line bundle, describe the $\mathbf{P}^1$-bundle $\tilde{X}_{\mathbf{P}\mathcal{W}} \to S^\nu$ more precisely: The blowup of $\mathbf{P}\mathcal{W}$ along $\mathbf{P}\mathcal{W}'$ is canonically the $\mathbf{P}^2$-bundle over S^ν associated with $\tilde{\mathcal{W}}$ formed in the pullback diagram

$$\begin{tikzcd} 0 \arrow[r] & \tilde\varphi_+^*\mathcal{W}' \arrow[r] \arrow[d, equal] & \tilde{\mathcal{W}} \arrow[r] \arrow[d, hook] & \mathcal{O}_{\tilde\varphi_+}(-1) \arrow[r] \arrow[d, hook, "\mathrm{eu}_{\tilde\varphi_+}"] & 0 \\ 0 \arrow[r] & \tilde\varphi_+^*\mathcal{W}' \arrow[r] & \tilde\varphi_+^*\mathcal{W} \arrow[r] & \tilde\varphi_+^*\mathcal{W}'' \arrow[r] & 0 \end{tikzcd}$$

and so the exceptional divisor of $\mathbf{P}\tilde{\mathcal{W}} \to \mathbf{P}\mathcal{W}$ is the subbundle $\mathbf{P}(\tilde\varphi_+^*\mathcal{W}') \subset \mathbf{P}\tilde{\mathcal{W}}$. The inverse image $X_{\mathbf{P}\tilde{\mathcal{W}}} := X_{\mathbf{P}\mathcal{W}} \times_{\mathbf{P}\mathcal{W}} \mathbf{P}\tilde{\mathcal{W}}$ of $X_{\mathbf{P}\mathcal{W}}$ along this blowup is the bundle of q-bic curves over S^ν defined by the q-bic form $\beta_{\tilde{\mathcal{W}}}\colon \tilde{\mathcal{W}}^{[1]} \otimes \tilde{\mathcal{W}} \to \mathcal{O}_{S^\nu}$ obtained by restricting $\tilde\varphi_+^*\beta_{\mathcal{W}}$. Since $\mathcal{W}^\perp = \mathcal{W}'^{[1]}$ by 4.6, the diagram above implies that $\tilde\varphi_+^*\mathcal{W}'^{[1]} = \tilde{\mathcal{W}}^\perp$. Thus by [Che23a, 1.5], there is an exact sequence

$$0 \to \mathcal{K} \to \tilde{\mathcal{W}} \xrightarrow{\beta_{\tilde{\mathcal{W}}}} \tilde{\mathcal{W}}^{[1],\vee} \to \tilde\varphi_+^*\mathcal{W}'^{[1],\vee} \to 0$$

where $\mathcal{K}$ is the rank 2 subbundle defining the $\mathbf{P}^1$-bundle $\tilde{X}_{\mathbf{P}\mathcal{W}} \to S^\nu$. So since $\nu^*\mathcal{S} = \mathcal{K}$,

$$\begin{aligned} \nu^*\mathcal{O}_S(1) \cong \nu^*\det(\mathcal{S})^\vee \cong \det(\mathcal{K})^\vee &\cong \det(\tilde{\mathcal{W}})^{\vee,\otimes q+1} \otimes \tilde\varphi_+^*\det(\mathcal{W}')^{\otimes q} \\ &\cong \tilde\varphi_+^*\det(\mathcal{W}')^\vee \otimes \mathcal{O}_{\tilde\varphi_+}(q+1) \\ &\cong \tilde\varphi_+^*\mathcal{O}_C(1) \otimes \mathcal{O}_{\tilde\varphi_+}(q+1) \otimes L_+^\vee \end{aligned}$$

since $\mathcal{W}' = \mathcal{O}_C(-1) \oplus L_{+,C}$. ■

The following statement summarizes 4.4 and 4.7 and gives a direct geometric relationship between the maps $\tilde\varphi_+$ and φ_-. For this, let $\phi_C\colon C \to C$ be the endomorphism that sends a point x to the residual intersection point between C and its tangent line at x, as described in 1.6.

4.8. Corollary. — *There is a commutative diagram of morphisms*

$$\begin{tikzcd} S^\nu \arrow[r, "\nu"'] \arrow[d, "\tilde\varphi_+"'] & S \arrow[d, "\varphi_-"] \\ C \arrow[r, "\phi_C"] & C\,. \end{tikzcd}$$

4.9. Conductors. — Consider the conductor ideals associated with the normalization $\nu\colon S^\nu \to S$:

$$\mathrm{cond}_{\nu,S} := \mathcal{A}nn_{\mathcal{O}_S}(\nu_*\mathcal{O}_{S^\nu}/\mathcal{O}_S) \subset \mathcal{O}_S \quad\text{and}\quad \mathrm{cond}_{\nu,S^\nu} := \nu^{-1}\mathrm{cond}_{\nu,S}\cdot\mathcal{O}_{S^\nu} \subset \mathcal{O}_{S^\nu}.$$

The corresponding conductor subschemes $D \subset S$ and $D^\nu \subset S^\nu$ are thickenings of the curves C_+ and C_+^ν, respectively, and fit into a commutative diagram

$$\begin{tikzcd} D^\nu \arrow[r, "\nu"'] \arrow[d, "\tilde\varphi_+"'] & D \arrow[d, "\varphi_-"] \\ C \arrow[r, "\phi_C"] & C \end{tikzcd}$$

where the vertical maps are finite by properness of S and S^ν over C. Algebraically, the conductor ideal of S is characterized as the largest ideal of $\mathcal{O}_S$ which is also an ideal of $\nu_*\mathcal{O}_{S^\nu}$, so there is a commutative diagram of exact sequences of sheaves on S:

$$\begin{tikzcd} & \mathrm{cond}_{\nu,S} \arrow[r, "\cong"'] \arrow[d, hook] & \nu_*\mathrm{cond}_{\nu,S^\nu} \arrow[d, hook] & & \\ 0 \arrow[r] & \mathcal{O}_S \arrow[r, "\nu^\#"] \arrow[d] & \nu_*\mathcal{O}_{S^\nu} \arrow[r] \arrow[d] & \nu_*\mathcal{O}_{S^\nu}/\mathcal{O}_S \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & \mathcal{O}_D \arrow[r, "\nu^\#"] & \nu_*\mathcal{O}_{D^\nu} \arrow[r] & \nu_*\mathcal{O}_{D^\nu}/\mathcal{O}_D \arrow[r] & 0\,. \end{tikzcd}$$

Duality theory for $\nu\colon S^\nu \to S$ identifies the conductor ideal $\mathrm{cond}_{\nu,S^\nu}$ with the relative dualizing sheaf $\omega_{S^\nu/S} \cong \omega_{S^\nu} \otimes \nu^*\omega_S^\vee$, compare [Stacks, 0FKW] and [Rei94, Proposition 2.3].

4.10. Proposition. — *The conductor ideal of S^ν is isomorphic to*

$$\mathrm{cond}_{\nu,S^\nu} \cong \mathcal{O}_{\tilde\varphi_+}(-\delta-1)\otimes(L_+^{\otimes 2q-1}\otimes L_-) \quad \textit{where } \delta := 2q^2-q-2.$$

In particular, the conductor subscheme D^ν is the δ-order neighbourhood of C_+^ν.

Proof. Since S^ν is the $\mathbf{P}^1$-bundle on $\mathscr{W}'' \cong \mathscr{T}_C(-1)\oplus L_{-,C}$, as in 4.7, the relative Euler sequence gives

$$\omega_{S^\nu} \cong \omega_{S^\nu/C}\otimes\tilde\varphi_+^*\omega_C \cong \mathcal{O}_{\tilde\varphi_+}(-2)\otimes\tilde\varphi_+^*(\omega_C^{\otimes 2}\otimes\mathcal{O}_C(1))\otimes L_-^\vee.$$

By 1.8, $\omega_S \cong \mathcal{O}_S(2q-3)\otimes(L_+\otimes L_-)^{\vee,\otimes 2}$, so 4.7 gives

$$\begin{aligned} \nu^*\omega_S &\cong \nu^*\mathcal{O}_S(2q-3)\otimes(L_+\otimes L_-)^{\vee,\otimes 2} \\ &\cong \tilde\varphi_+^*\mathcal{O}_C(2q-3)\otimes\mathcal{O}_{\tilde\varphi_+}\big((2q-3)(q+1)\big)\otimes L_+^{\vee,\otimes 2q-1}\otimes L_-^{\vee,\otimes 2}. \end{aligned}$$

Since C is a plane curve of degree $q+1$, $\omega_C^{\otimes 2}\otimes\mathcal{O}_C(1)\cong\mathcal{O}_C(2q-3)$, and so

$$\omega_{S^\nu/S}\cong\omega_{S^\nu}\otimes\nu^*\omega_S^\vee\cong\mathcal{O}_{\tilde\varphi_+}(-\delta-1)\otimes(L_+^{\otimes 2q-1}\otimes L_-). \qquad \blacksquare$$

4.11. The sheaf $\mathscr{F}$. — Let $\mathscr{D}$ and $\mathscr{D}^\nu$ be the coherent $\mathcal{O}_C$-algebras by pushing structure sheaves along the finite morphisms $\varphi_-\colon D\to C$ and $\phi_C\circ\tilde\varphi_+\colon D^\nu\to C$, respectively. Restricting ν to the conductors induces an injective map $\nu^\#\colon\mathscr{D}\to\mathscr{D}^\nu$ of $\mathcal{O}_C$-algebras. Its cokernel

$$\mathscr{F} := \mathscr{D}/\mathscr{D}^\nu \cong \varphi_{-,*}(\nu_*\mathcal{O}_{D^\nu}/\mathcal{O}_D)\cong\varphi_{-,*}(\nu_*\mathcal{O}_{S^\nu}/\mathcal{O}_S),$$

identifications arising from the diagram of 4.9, is an $\mathcal{O}_C$-module related to S as follows:

4.12. Lemma. — *There is an exact sequence of finite locally free $\mathcal{O}_C$-modules*

$$0\to\mathcal{O}_C\to\phi_{C,*}\mathcal{O}_C\to\mathscr{F}\to\mathbf{R}^1\varphi_{-,*}\mathcal{O}_S\to 0$$

in which the map $\mathscr{F}\to\mathbf{R}^1\varphi_{-,*}\mathcal{O}_S$ splits.

Proof. Push the middle row in the diagram of 4.9 along φ_- to obtain the sequence, noting that $\varphi_{-,*}\mathcal{O}_S\cong\tilde\varphi_{+,*}\mathcal{O}_{S^\nu}\cong\mathcal{O}_C$ by 3.10, and that $\mathbf{R}^1\tilde\varphi_{+,*}\mathcal{O}_{S^\nu}=0$ since $\tilde\varphi_+$ is a projective bundle over C. Observe that $\mathscr{F}$ is locally free since it is an extension of the locally free $\mathcal{O}_C$-modules $\mathbf{R}^1\varphi_{-,*}\mathcal{O}_S$ is and $\phi_{C,*}\mathcal{O}_C/\mathcal{O}_C$: the former is locally free by 3.10; for the latter, this follows by Kunz's Theorem [Kun69, Theorem 2.1] or [Stacks, 0EC0], since ϕ_C is, up to an automorphism, the q^2-Frobenius of the regular curve C by 1.6. The $\mathbf{G}_m$-action from 4.2 provides $\mathscr{D}$ and $\mathscr{D}^\nu$ with gradings such that $\mathscr{O}_C$ and $\phi_{C,*}\mathscr{O}_C$, being the constant functions, make up the degree 0 components. Therefore the positively graded components of $\mathscr{F}$ map isomorphically to $\mathbf{R}^1\varphi_{-,*}\mathscr{O}_S$ and provide the desired splitting. ■

The importance of $\mathscr{F}$ is in the following relation with the cohomology of the structure sheaf of S:

4.13. Proposition. — *The cohomology of $\mathscr{O}_S$ is given by*

$$\mathrm{H}^i(S,\mathscr{O}_S) \cong \begin{cases} \mathrm{H}^0(C,\mathscr{O}_C) & \text{if } i=0, \\ \mathrm{H}^0(C,\mathscr{F}) & \text{if } i=1, \text{ and} \\ \mathrm{H}^1(C,\mathscr{F})/\mathrm{H}^1(C,\mathscr{O}_C) & \text{if } i=2. \end{cases}$$

Proof. Consider the cohomology sequence associated with the exact sequence

$$0 \to \mathscr{O}_S \to \nu_*\mathscr{O}_{S^\nu} \to \nu_*\mathscr{O}_{S^\nu}/\mathscr{O}_S \to 0.$$

Note that $\varphi_{-,*}\mathscr{O}_S \cong \mathscr{O}_C$ by 3.10 and, for each $i = 0,1,2$,

$$\mathrm{H}^i(S,\nu_*\mathscr{O}_{S^\nu}/\mathscr{O}_S) \cong \mathrm{H}^i(C,\mathscr{F}) \quad \text{and} \quad \mathrm{H}^i(S^\nu,\mathscr{O}_{S^\nu}) \cong \mathrm{H}^i(C,\mathscr{O}_C)$$

since $\nu_*\mathscr{O}_{S^\nu}/\mathscr{O}_S$ is supported on D and $D \to C$ is affine, and and the fact that $\tilde{\varphi}_+ \colon S^\nu \to C$ is a projective bundle. Thus $\mathrm{H}^0(S,\mathscr{O}_S) \cong \mathrm{H}^0(S^\nu,\mathscr{O}_{S^\nu}) \cong \mathrm{H}^0(C,\mathscr{O}_C)$ and there is an exact sequence

$$0 \to \mathrm{H}^0(C,\mathscr{F}) \xrightarrow{a} \mathrm{H}^1(S,\mathscr{O}_S) \xrightarrow{b} \mathrm{H}^1(S,\nu_*\mathscr{O}_{S^\nu}) \to \mathrm{H}^1(C,\mathscr{F}) \to \mathrm{H}^2(S,\mathscr{O}_S) \to 0.$$

The result will follow upon verifying that $b \colon \mathrm{H}^1(S,\mathscr{O}_S) \to \mathrm{H}^1(S,\nu_*\mathscr{O}_{S^\nu})$ vanishes. Since $\varphi_{-,*}\mathscr{O}_S \cong \mathscr{O}_C$, the Leray spectral sequence gives a short exact sequence

$$0 \to \mathrm{H}^1(C,\mathscr{O}_C) \xrightarrow{c} \mathrm{H}^1(S,\mathscr{O}_S) \xrightarrow{d} \mathrm{H}^0(C,\mathbf{R}^1\varphi_{-,*}\mathscr{O}_S) \to 0.$$

The splitting in 4.12 implies that the composite $d \circ a \colon \mathrm{H}^0(C,\mathscr{F}) \to \mathrm{H}^0(C,\mathbf{R}^1\varphi_{-,*}\mathscr{O}_S)$ is a surjection. So exactness of the long sequence means it remains to show that $b \circ c \colon \mathrm{H}^1(C,\mathscr{O}_C) \to \mathrm{H}^1(S,\nu_*\mathscr{O}_{S^\nu})$ vanishes. Pushing down to C along φ_- and applying 4.8 shows that this is

$$\phi_C \colon \mathrm{H}^1(C,\mathscr{O}_C) \to \mathrm{H}^1(C,\phi_{C,*}\mathscr{O}_C)$$

which, as in 4.12, is the map induced by the q^2-power Frobenius up to an automorphism, and this is the zero map by 1.2. ■

Structure sheaf cohomology of S is therefore reduced to that of $\mathscr{F}$, which will be computed when $q = p$ in 6.12. Combined with the Euler characteristic computation in 1.13, this gives Theorem B. In turn, cohomology of $\mathscr{F}$ is determined by describing its structure, and this is achieved in the next section via the following duality relationship with the algebra $\mathscr{D}$:

4.14. Proposition. — *There is a canonical isomorphism of graded $\mathscr{O}_C$-modules*

$$\mathscr{F} \cong \mathscr{D}^\vee \otimes \mathscr{O}_C(-q+1) \otimes L_+^{\otimes 2q-1} \otimes L_-^{\otimes 2}.$$

Proof. Applying $\mathbf{R}\mathscr{H}om_{\mathscr{O}_S}(-,\mathscr{O}_S)$ to the ideal sheaf sequence of the conductor $D \hookrightarrow S$ yields a triangle in the derived category of S

$$\mathbf{R}\mathscr{H}om_{\mathscr{O}_S}(\mathscr{O}_D,\mathscr{O}_S) \to \mathscr{O}_S \to \nu_*\mathscr{O}_{S^\nu} \xrightarrow{+1}$$

since $\mathbf{R}\mathscr{H}om_{\mathscr{O}_S}(\mathscr{O}_S,\mathscr{O}_S) = \mathscr{O}_S$ and $\mathbf{R}\mathscr{H}om_{\mathscr{O}_S}(\mathrm{cond}_{\nu,S},\mathscr{O}_S) = \nu_*\mathscr{O}_{S^\nu}$, upon identifying $\mathrm{cond}_{\nu,S}$ with $\nu_*\omega_{S^\nu/S}$ as in 4.9 and using duality for ν. The map $\mathscr{O}_S \to \nu_*\mathscr{O}_{S^\nu}$ is dual to evaluation at 1, and hence is the $\mathscr{O}_S$-module map determined by $1 \mapsto 1$; in other words, this is the map $\nu^\#$, and so

$$\nu_*\mathscr{O}_{S^\nu}/\mathscr{O}_S \cong \mathbf{R}\mathscr{H}om_{\mathscr{O}_S}(\mathscr{O}_D,\mathscr{O}_S)[1]$$

in the derived category of S. Applying $\mathbf{R}\varphi_{-,*}$ and applying relative duality for $\varphi_- \colon S \to C$ yields

$$\mathscr{F} \cong \mathbf{R}\varphi_{-,*}\mathbf{R}\mathscr{H}om_{\mathscr{O}_S}(\mathscr{O}_D,\mathscr{O}_S)[1] \cong \mathbf{R}\mathscr{H}om_{\mathscr{O}_C}(\mathbf{R}\varphi_{-,*}(\mathscr{O}_D \otimes \omega_{\varphi_-}),\mathscr{O}_C)[1]$$

where $\omega_{\varphi_-} = (\omega_S \otimes \varphi_-^*\omega_C^\vee)[1]$. By 1.8,

$$\mathscr{O}_D \otimes \omega_S \cong \mathscr{O}_S(2q-3)|_D \otimes L_+^{\vee,\otimes 2} \otimes L_-^{\vee,\otimes 2} \cong \varphi_-^*\mathscr{O}_C(2q-3)|_D \otimes L_+^{\vee,\otimes 2q-1} \otimes L_-^{\vee,\otimes 2}$$

where $\mathscr{O}_S(1)|_D = \varphi_-^*\mathscr{O}_C(1)|_D \otimes L_+^\vee$ since $\varphi_- \colon D \to C$ is induced by the line subbundle of $\mathscr{S}|_D$ obtained by intersecting with $(L_- \oplus W)_D$ by 2.5, so there is a short exact sequence

$$0 \to \varphi_-^*\mathscr{O}_C(-1)|_D \to \mathscr{S}|_D \to L_{+,D} \to 0,$$

and taking determinants yields the desired identification. Combining with $\omega_C \cong \mathscr{O}_C(q-2)$ gives

$$\mathbf{R}\varphi_{-,*}(\mathscr{O}_D \otimes \omega_{\varphi_-}) = (\mathbf{R}\varphi_{-,*}\mathscr{O}_D) \otimes \mathscr{O}_C(q-1) \otimes L_+^{\vee,\otimes 2q-1} \otimes L_-^{\vee,\otimes 2}[1].$$

Since $D \to C$ is of relative dimension 0, $\mathbf{R}\varphi_{-,*}\mathscr{O}_D = \varphi_{-,*}\mathscr{O}_D = \mathscr{D}$, yielding the result. ■

4.15. The algebra $\mathscr{D}^\nu$. — Before turning to $\mathscr{D}$, consider the simpler algebra $\mathscr{D}^\nu$ associated with $\phi_C \circ \tilde{\varphi}_+ \colon D^\nu \to C$. Since D^ν is disjoint from the curve $C_-^\nu = \mathbf{P}L_{-,C}$ parameterizing lines through $x_- \in X$, it is contained in the $\mathbf{A}^1$-bundle over C given by

$$S^{\nu,\circ} := S^\nu \setminus C_-^\nu = \mathbf{P}(\mathscr{T}_C(-1) \oplus L_{-,C}) \setminus \mathbf{P}L_{-,C} \cong \mathbf{A}(\Omega_C^1(1) \otimes L_-),$$

the identification obtained by taking graphs of a linear function $\mathscr{T}_C(-1) \to L_{-,C}$. Then 4.10 means that D^ν is the δ-order neighbourhood of the zero section C_+^ν, so this identifies the graded $\mathscr{O}_C$-module underlying the algebra of D^ν as

$$\mathscr{D}^\nu = (\phi_C \circ \tilde{\varphi}_+) * \mathscr{O}_{D^\nu} \cong \phi_{C,*}\Big(\bigoplus\nolimits_{i=0}^{\delta}(\mathscr{T}_C(-1) \otimes L_-^\vee)^{\otimes i}\Big).$$

5. Structure of $\mathscr{D}$ and $\mathscr{F}$

The purpose of this section is to describe the structure of the sheaves $\mathscr{D}$ and—especially!—$\mathscr{F}$ in terms of more familiar sheaves on the curve C. To give the main statement regarding $\mathscr{F}$, some notation: View the restricted Euler sequence

$$0 \to \Omega^1_{\mathbf{P}W}|_C \to W^\vee \otimes \mathscr{O}_C(-1) \to \mathscr{O}_C \to 0$$

as a 2-step filtration on $W^\vee \otimes \mathscr{O}_C(-1)$, with the sheaf of differentials the 0-th filtered piece. This induces a $(d+1)$-step filtration on the symmetric powers $\mathrm{Sym}^d(W^\vee) \otimes \mathscr{O}_C(-d)$ with graded pieces

$$\mathrm{gr}_i\big(\mathrm{Sym}^d(W^\vee) \otimes \mathscr{O}_C(-d)\big) \cong \mathrm{Sym}^{d-i}(\Omega^1_{\mathbf{P}W}|_C) \text{ for } 0 \leq i \leq d.$$

Generally, given any locally free sheaf $\mathscr{E}$ and $d \geq q$, $\mathrm{Sym}^d(\mathscr{E})$ contains a subsheaf $\mathscr{E}^{[1]} \otimes \mathrm{Sym}^{d-q}(\mathscr{E})$ consisting of products of q-powers and monomials of degree $d-q$. Write

$$\mathrm{Sym}^d_{\mathrm{red}}(\mathscr{E}) := \mathrm{Sym}^d(\mathscr{E})/(\mathscr{E}^{[1]} \otimes \mathrm{Sym}^{d-q}(\mathscr{E}))$$

for the quotient by this subsheaf, with the convention that $\mathrm{Sym}^d_{\mathrm{red}} = \mathrm{Sym}^d$ when $d < q$. Finally, write $\mathrm{Div}^d_{\mathrm{red}}(\mathscr{E}) := \mathrm{Sym}^d_{\mathrm{red}}(\mathscr{E}^\vee)^\vee$ for the reduced d-th divided power of $\mathscr{E}$. The result is:

5.1. Proposition. — *The graded $\mathcal{O}_C$-module $\mathcal{F}$ carries a q-step filtration such that*

$$\mathrm{Fil}_0 \mathcal{F} \cong \bigoplus\nolimits_{b=0}^{q-2} \bigoplus\nolimits_{a=0}^{q-1} \mathrm{Div}^{q-2-b}(\mathcal{T}) \otimes \mathcal{O}_C(-a) \otimes L_+^{\otimes b} \otimes L_-^{\vee,\otimes a}.$$

For each $0 \leq b \leq 2q-3$, there is a canonical short exact sequence of filtered bundles

$$0 \to \mathrm{Div}^{2q-3-b}_{\mathrm{red}}(W) \otimes \mathcal{O}_C \to \mathcal{F}_{bq+q-1} \to \mathrm{Div}^{q-3-b}(W) \otimes \mathcal{O}_C(q-1) \to 0.$$

There is a degree $-q-1$ map $\partial \colon \mathcal{F} \to \mathcal{F}$ such that, for each $0 \leq i \leq q-1$ and $0 \leq d \leq \delta - q - 1$, $\partial(\mathrm{Fil}_i \mathcal{F}) \subseteq \mathrm{Fil}_{i-1} \mathcal{F}$ and $\mathrm{gr}_i \partial \colon \mathrm{Fil}_i \mathcal{F}_{d+q+1} \to \mathrm{Fil}_{i-1} \mathcal{F}_d$ is an isomorphism if $p \nmid i$ and zero otherwise.

This is proved in 5.13 at the end of the section.

5.2. Affine bundles. — The duality relation 4.14 relates $\mathcal{F}$ with $\mathcal{D}$, and the latter is a quotient of coordinate rings of schemes affine over C: namely, $S^\circ := S \setminus C_-$ and T° as in 2.7 with respect to the smooth cone situation $(X, x_-, \mathbf{P}L_-^{[1],\perp})$. These lie in affine space bundles

$$\mathbf{A}_1 := \mathbf{P}\mathcal{V}_1 \setminus \mathbf{P}L_{-,C}, \qquad \mathbf{A}_2 := \mathbf{P}\mathcal{V}_2 \setminus \mathbf{P}\mathcal{T}, \qquad \mathbf{B} := \mathbf{P}\mathcal{V} \setminus \mathbf{P}(\mathcal{T}_{\pi_1}(-1,0))|_{\mathbf{A}}$$

with $\mathcal{V}_1$ and $\mathcal{V}_2$ as in 2.6, and $\mathcal{V}$ is as in 2.11; set $\mathbf{A} := \mathbf{A}_1 \times_C \mathbf{A}_2$. Comparing the diagram in 2.6 with the description of the boundaries $T \setminus T^\circ$ from 3.1 implies that there is a commutative diagram of affine schemes over C given by

$$\begin{tikzcd}
D \arrow[r, hook] \arrow[dr] & S^\circ \arrow[r, hook] \arrow[d] & \mathbf{B} \arrow[d, "\rho"] \arrow[dd, bend left, "\varphi"] \\
& T^\circ \arrow[r, hook] \arrow[dr] & \mathbf{A} \arrow[d, "\pi"] \\
& & C\,.
\end{tikzcd}$$

Observe that the relative Euler sequences for these affine bundles give canonical isomorphisms

$$\mathcal{O}_{\mathbf{A}}(-1,0) \cong \pi^*\mathcal{O}_C(-1), \qquad \mathcal{T}_{\pi_2}(0,-1)|_{\mathbf{A}} \cong \pi^*\mathcal{T}, \qquad \mathcal{O}_\rho(-1) \cong \rho^*\mathcal{O}_{\mathbf{A}}(0,-1) \cong L_{+,\mathbf{B}},$$
$$\mathcal{T}_{\pi_1}(-1,0)|_{\mathbf{A}} \cong L_{-,\mathbf{A}}, \qquad \mathcal{O}_{\mathbf{A}}(0,-1) \cong L_{+,\mathbf{A}}, \qquad \mathcal{T}_\rho(-1)|_{\mathbf{B}} \cong \rho^*\mathcal{T}_{\pi_1}(-1,0)|_{\mathbf{B}} \cong L_{-,\mathbf{B}}.$$

This identifies the $\mathcal{O}_C$-algebras $\mathcal{A} := \pi_*\mathcal{O}_{\mathbf{A}}$ and $\mathcal{B} := \varphi_*\mathcal{O}_{\mathbf{B}}$ as follows:

5.3. Lemma. — *The q-bic form β induces an isomorphism*

$$\mathcal{A} \cong \mathrm{Sym}^*(\mathcal{O}_C(-1) \otimes L_-^\vee) \otimes \mathrm{Sym}^*(\Omega^1_{\mathbf{P}W}(1)|_C \otimes L_+),$$

and endows the $\mathcal{A}$-algebra $\mathcal{B}$ with an increasing filtration whose associated graded pieces are

$$\mathrm{gr}_i \mathcal{B} := \mathrm{Fil}_i \mathcal{B} / \mathrm{Fil}_{i-1} \mathcal{B} \cong \mathcal{A} \otimes (L_-^\vee \otimes L_+)^{\otimes i} \quad \textit{for all } i \in \mathbf{Z}_{\geq 0}.$$

Proof. The splittings $\mathcal{V}_1 \cong \mathcal{O}_C(-1) \oplus L_{-,C}$ and $\mathcal{V}_2 \cong \mathcal{T} \oplus L_{+,C}$ as in §3 give canonical relative projective coordinates on $\mathbf{P}\mathcal{V}_i$ over C, and identifies the associated affine bundles as

$$\mathbf{A} \cong \mathbf{A}(\mathcal{O}_C(1) \otimes L_-) \times_C \mathbf{A}(\mathcal{T} \otimes L_+^\vee).$$

This identifies $\mathcal{A} = \pi_*\mathcal{O}_{\mathbf{A}}$ as claimed.

For $\mathcal{B}$, begin with the affine bundle $\rho \colon \mathbf{B} \to \mathbf{A}$ obtained as the complement of $\mathbf{P}(\mathcal{T}_{\pi_1}(-1,0))$ in $\mathbf{P}\mathcal{V}$ over $\mathbf{A}$. Dualizing the short exact sequence in 2.13 and restricting to $\mathbf{A}$ gives a sequence

$$0 \to \mathcal{O}_{\mathbf{A}}(0,1) \to \mathcal{V}^\vee|_{\mathbf{A}} \to \Omega^1_{\pi_1}(1,0)|_{\mathbf{A}} \to 0.$$

View $\mathcal{V}^\vee|_{\mathbf{A}}$ as the linear functions on $\mathbf{P}\mathcal{V}|_{\mathbf{A}}$ over $\mathcal{A}$. A local generator for the subbundle $\mathcal{O}_{\mathbf{A}}(0,1)$ is then a linear equation defining $\mathbf{P}(\mathcal{T}_{\pi_1}(-1,0)|_{\mathbf{A}})$, and so becomes invertible on $\mathbf{B}$. Therefore

$$\rho_*\mathcal{O}_{\mathbf{B}} \cong \mathrm{colim}_n\, \mathrm{Sym}^n(\mathcal{V}^\vee(0,-1))|_{\mathbf{A}}$$

where the transition maps are induced by multiplication by a local generator for the subbundle $\mathcal{O}_{\mathbf{A}} \hookrightarrow \mathcal{V}^\vee(0,-1)|_{\mathbf{A}}$. The 2-step filtration on $\mathcal{V}^\vee(0,-1)|_{\mathbf{A}}$ starting with $\mathcal{O}_{\mathbf{A}}$ and followed by the entire bundle induces filtrations on the symmetric powers, compatible with the transition maps, whence a filtration on $\rho_*\mathcal{O}_{\mathbf{B}}$ with graded pieces

$$\mathrm{gr}_i\,\rho_*\mathcal{O}_{\mathbf{B}} \cong \Omega^1_{\pi_1}(1,-1)|_{\mathbf{A}}^{\otimes i} \cong \mathcal{O}_{\mathbf{A}} \otimes (L_-^\vee \otimes L_+)^{\otimes i} \quad \text{for all } i \in \mathbf{Z}_{\geq 0}$$

upon applying the identifications of 5.2. Pushing along $\pi \colon \mathbf{A} \to C$ then gives the result. ■

5.4. Symmetries. — Consider the linear algebraic subgroup of $\mathbf{GL}(V)$ which preserves the decomposition $V = W \oplus U$, fixes the point $x_- = \mathbf{P}L_-$, and which acts via automorphisms on C:

$$\mathbf{Aut}(L_- \subset U) \times \mathbf{Aut}(W, \beta_W) \cong \left\{ \begin{pmatrix} \lambda_-^{-1} & \epsilon \\ 0 & \lambda_+ \end{pmatrix} \in \mathbf{GL}(L_- \oplus L_+) \right\} \times \mathrm{U}_3(q).$$

Via its natural linear action on V, it acts on each of the sheaves $\mathcal{V}_1$, $\mathcal{V}_2$, and $\mathcal{V}$, and preserves the subbundles excised in defining the tower of affine bundles $\mathbf{B} \to \mathbf{A} \to C$. As such, the $\mathcal{O}_C$-algebras $\mathcal{A}$ and $\mathcal{B}$ are equivariant for this group. Two pieces of structure now leap to the forefront:

First, the action of the maximal torus $(\lambda_-^{-1}, \lambda_+)$ endows $\mathcal{A}$ and $\mathcal{B}$ with a bigrading, normalized so that $L_-^{\vee,\otimes a} \otimes L_+^{\otimes b}$ has weight $(a,b) \in \mathbf{Z}^2_{\geq 0}$. The bigraded pieces are as follows:

5.5. Lemma. — *The form β induces isomorphisms of filtered bundles*

$$\mathcal{B}_{(a,0)} \cong \mathcal{A}_{(a,0)} \cong \mathcal{O}_C(-a), \;\; \mathcal{B}_{(0,b)} \cong \mathcal{A}_{(0,b)} \cong \mathrm{Sym}^b(\Omega_{\mathbf{P}W}(1))|_C, \;\; \mathcal{B}_{(1,1)} \cong W^\vee \otimes \mathcal{O}_C(-1),$$

and $\mathcal{B}_{(a,b)} \cong \mathrm{Fil}_a\big(\mathrm{Sym}^b(W^\vee) \otimes \mathcal{O}_C(-a)\big)$ *for all* $a, b \in \mathbf{Z}_{\geq 0}$.

Proof. Matching weights in 5.3 identifies the bigraded pieces of weights $(a,0)$ and $(0,b)$ as claimed, and shows that the $(1,1)$ piece of $\mathcal{B}$ sits as an extension

$$0 \to \Omega^1_{\mathbf{P}W}|_C \to \mathcal{B}_{(1,1)} \to \mathcal{O}_C \to 0.$$

Begin by considering the extension class of $\mathcal{V}$ in the sequence 2.13: By its construction in 2.11, the class $[\mathcal{V}]$ is the image under the map, induced by pullback and pushforward along tautological maps,

$$\mathrm{Ext}^1_{\mathbf{P}}(\pi^*\mathcal{V}_2, \pi^*\mathcal{V}_1) \to \mathrm{Ext}^1_{\mathbf{P}}(\mathcal{O}_{\mathbf{P}}(0,-1), \pi^*\mathcal{V}_1) \to \mathrm{Ext}^1_{\mathbf{P}}(\mathcal{O}_{\mathbf{P}}(0,-1), \mathcal{T}_{\pi_1}(-1,0))$$

of the class of $[V_{\mathbf{P}}]$ from the top sequence in the diagram 2.6. The kernel of the two maps in question are $\mathrm{Ext}^1_{\mathbf{P}}(\mathcal{T}_{\pi_2}(0,-1), \pi^*\mathcal{V}_1)$ and $\mathrm{Ext}^1_{\mathbf{P}}(\mathcal{O}_{\mathbf{P}}(0,-1), \mathcal{O}_{\mathbf{P}}(-1,0))$, both of which vanish. Therefore the class $[\mathcal{V}]$ is nonzero, and is the image of the class of the dual Euler sequence on $\mathbf{P}W$. Since $\mathcal{B}_{(1,1)}$ is obtained by pushing a twist of the dual of the sequence for $\mathcal{V}$ in 2.13, its extension class is that of the Euler sequence, so $\mathcal{B}_{(1,1)} \cong W^\vee \otimes \mathcal{O}_C(-1)$.

The remaining bigraded pieces are obtained via multiplication: Since $\mathcal{B}$ is locally a polynomial algebra with two generators of degree $(0,1)$, one of degree $(1,0)$, and one of degree $(1,1)$, it follows that the multiplication maps, for $d, e \in \mathbf{Z}_{\geq 0}$,

$$\mathrm{Sym}^d(\mathcal{B}_{(1,1)}) \to \mathcal{B}_{(d,d)}, \quad \mathcal{B}_{(d,d)} \otimes \mathcal{B}_{(e,0)} \to \mathcal{B}_{(d+e,d)}, \quad \mathcal{B}_{(d,d+e)} \otimes \mathcal{B}_{(e,0)} \to \mathcal{B}_{(d+e,d+e)}$$

are isomorphisms in the first two cases, and an isomorphism onto the d-th filtered piece of $\mathcal{B}_{(d+e,d+e)}$. Combined with the identification of the low degree pieces completes the proof. ■

Second, the additive group $\mathbf{G}_a$ acts through ϵ, the unipotent radical of $\mathbf{Aut}(L_- \subset U)$. This action sends L_+ to L_-, so it is trivial on $\mathcal{V}_1$ and $\mathcal{V}_2$, whence on $\mathcal{A}$. Therefore $\mathcal{B}$ is $\mathcal{A}$-linearly $\mathbf{G}_a$-equivariant. This structure is described algebraically via an $\mathcal{A}$-comodule structure

$$\mathcal{B} \to \mathcal{B} \otimes \mathbf{k}[\epsilon] \colon z \mapsto \sum\nolimits_{j=0}^{\infty} \partial_j(z) \otimes \epsilon^j$$

where $\mathbf{G}_a := \operatorname{Spec} \mathbf{k}[\epsilon]$, and the $\partial_j \colon \mathscr{B} \to \mathscr{B}$ are $\mathscr{A}$-linear maps such that a given local section z lies in the kernel of all but finitely many ∂_j, $\partial_0 = \mathrm{id}$, and

$$\partial_j \circ \partial_k = \binom{j+k}{j} \partial_{j+k} \text{ for all } j,k \in \mathbf{Z}_{\geq 0}.$$

See [Jan03, I.7.3, I.7.8, and I.7.12] for details. Of particular importance is the operator $\partial := \partial_1$, and its salient features are as follows:

5.6. Lemma. — *The map $\partial \colon \mathscr{B} \to \mathscr{B}$ is of bidegree $(-1,-1)$ and satisfies*

$$\partial(\mathrm{Fil}_i \mathscr{B}) \subseteq \mathrm{Fil}_{i-1} \mathscr{B} \text{ for each } i \in \mathbf{Z}_{\geq 0}.$$

The associated graded map $\mathrm{gr}_i \partial \colon \mathrm{gr}_i \mathscr{B} \to \mathrm{gr}_{i-1} \mathscr{B}$ is an isomorphism if $p \nmid i$ and is zero otherwise.

Proof. The action of $\mathbf{G}_a$ on V corresponds in degree 1 to the linear map $\partial \colon V \to V$ which is an isomorphism between the components $L_+ \to L_-$, and zero elsewhere. Tracing through the construction of $\mathscr{V}$ from 2.6 and 2.11, it is straightforward that upon restricting to $\mathbf{A}$ and in using the identifications in 5.2 to write the short exact sequence in 2.13 as

$$0 \to \mathscr{O}_{\mathbf{A}} \to \mathscr{V}^\vee(0,-1)|_{\mathbf{A}} \to \mathscr{O}_{\mathbf{A}} \otimes (L_-^\vee \otimes L_+) \to 0,$$

∂ acts on $\mathscr{V}^\vee(0,-1)|_{\mathbf{A}}$ as 0 on the subbundle $\mathscr{O}_{\mathbf{A}}$, and sends the quotient $\mathscr{O}_{\mathbf{A}} \otimes (L_-^\vee \otimes L_+)$ isomorphically to the subbundle via the isomorphism $\partial \colon L_-^\vee \otimes L_+ \to \mathbf{k}$. Taking symmetric powers shows that

$$\partial(\mathrm{Fil}_i \operatorname{Sym}^n(\mathscr{V}^\vee(0,-1))|_{\mathbf{A}}) \subseteq \mathrm{Fil}_{i-1} \operatorname{Sym}^n(\mathscr{V}^\vee(0,-1))|_{\mathbf{A}} \text{ for each } 0 \leq i \leq n,$$

and that the associated graded map $\mathrm{gr}_i \partial \colon \mathscr{O}_{\mathbf{A}} \otimes (L_-^\vee \otimes L_+)^{\otimes i} \to \mathscr{O}_{\mathbf{A}} \otimes (L_-^\vee \otimes L_+)^{\otimes i-1}$ is multiplication by i. Passing to the colimit as in 5.3 and pushing along $\pi \colon \mathbf{A} \to C$ gives the lemma. ∎

5.7. Coordinate rings. — The coordinate rings of T° and S° over C as quotients of $\mathscr{A}$ and $\mathscr{B}$: By 2.8, $T^\circ \subset \mathbf{A}$ is the codimension 2 complete intersection cut out by the sections

$$v_1 := \mathrm{eu}_{\pi_2}^\vee \circ \beta^\vee \circ \mathrm{eu}_{\pi_1}^{[1]} \colon \mathscr{O}_{\mathbf{P}}(-q,-1) \to \mathscr{O}_{\mathbf{P}}$$
$$v_2 := \mathrm{eu}_{\pi_2}^{[1],\vee} \circ \beta \circ \mathrm{eu}_{\pi_1} \colon \mathscr{O}_{\mathbf{P}}(-1,-q) \to \mathscr{O}_{\mathbf{P}}$$

restricted to $\mathbf{A}$. By 2.14, S° is the hypersurface in $\mathbf{B} \times_{\mathbf{A}} T^\circ$ cut out by the restriction of the section

$$v_3 := u_3^{-1} \beta_{\mathscr{V}_T}(\mathrm{eu}_\rho^{[1]}, \mathrm{eu}_\rho) \colon \mathscr{O}_\rho(-q) \otimes \rho^* \mathscr{O}_T(0,-1)|_{\mathbf{P}\mathscr{V}_T} \to \mathscr{O}_{\mathbf{P}\mathscr{V}_T}.$$

Pushing forward to C and using the identifications from 5.2 then gives presentations

$$\pi_* \mathscr{O}_{T^\circ} \cong \operatorname{coker}\left(v_1 \oplus v_2 \colon (\mathscr{A}(-q) \otimes L_+) \oplus (\mathscr{A}(-1) \otimes L_+^{\otimes q}) \to \mathscr{A}\right), \text{ and}$$
$$\varphi_* \mathscr{O}_{S^\circ} \cong \operatorname{coker}\left(v_3 \colon \mathscr{B}_T \otimes L_+^{\otimes q+1} \to \mathscr{B}_T\right) \text{ where } \mathscr{B}_T := \mathscr{B} \otimes_{\mathscr{A}} \pi_* \mathscr{O}_{T^\circ}.$$

The coordinate rings carry a grading induced by the torus in the group G from 4.2. This torus is the subgroup $(\lambda^{-1}, \lambda^q)$ of the $\mathbf{G}_m^2$ from 5.4, meaning that the gradings on $\pi_* \mathscr{O}_{T^\circ}$ and $\varphi_* \mathscr{O}_{S^\circ}$ are related to the bigradings of $\mathscr{A}$ and $\mathscr{B}$ via

$$\mathscr{A}_d = \bigoplus\nolimits_{a+bq=d} \mathscr{A}_{(a,b)} \quad \text{and} \quad \mathscr{B}_d = \bigoplus\nolimits_{a+bq=d} \mathscr{B}_{(a,b)}.$$

Similarly, the action of $\mathbf{G}_a$ on $\mathbf{B}$ is related to that of $\boldsymbol{\alpha}_q$ on S°; in particular, the operator $\partial \colon \mathscr{B} \to \mathscr{B}$ from 5.6 induces a $\pi_* \mathscr{O}_{T^\circ}$-module endomorphism $\partial \colon \varphi_* \mathscr{O}_{S^\circ} \to \varphi_* \mathscr{O}_{S^\circ}$ which is of degree $-q-1$ and shifts the induced filtration down by 1.

To proceed, examine the equations v_1, v_2, and v_3 in detail. Begin with v_1: This is of degree q and of the form $x^q + \delta$, where x is the degree 1 generator of $\mathscr{B}$ and δ involves the degree q generators. Using this to eliminate the q-th power of the degree 1 coordinate results in the following:

5.8. Lemma. — *Let* $\mathscr{B}' := \operatorname{coker}(v_1 \colon \mathscr{B}(-q) \otimes L_+ \to \mathscr{B})$*. Then for* $b \in \mathbf{Z}_{\geq 0}$ *and* $0 \leq a < q$*,*

$$\mathscr{B}'_{bq+a} \cong \operatorname{Fil}_a\big(\operatorname{Sym}^b(W^\vee) \otimes \mathscr{O}_C(-a)\big).$$

Proof. More globally, v_1 is obtained by pairing q-powers of the coordinates of $\mathbf{P}\mathscr{V}_1$ with the coordinates of $\mathbf{P}\mathscr{V}_2$ via β, so it acts as the isomorphism $\beta_U \colon L_+ \to L_-^{\vee,\otimes q}$ on the components in U, and as

$$\delta = \beta_W^\vee \circ \mathrm{eu}_{\mathbf{P}W}^{[1]} \colon \mathscr{O}_C(-q) \to W^{[1]} \otimes \mathscr{O}_C \xrightarrow{\sim} W^\vee \otimes \mathscr{O}_C$$

on the components in W; furthermore, since C is the locus over which $\mathrm{eu}_{\mathbf{P}W}$ is isotropic for β_W, δ factors through $\Omega^1_{\mathbf{P}W}(1)|_C \subset W^\vee \otimes \mathscr{O}_C$. Therefore v_1 is induced by the map

$$(\beta_U, \delta) \colon \mathscr{O}_C(-q) \otimes L_+ \to \mathscr{O}_C(-q) \otimes L_-^{\vee,\otimes q} \oplus \Omega^1_{\mathbf{P}W}(1)|_C \otimes L_+,$$

and generally maps the degree $(d, e-1)$ bigraded piece of $\mathscr{B}$ to the degrees $(d+q, e-1)$ and (d,e) pieces. But $\mathscr{B}_{(d,e-1)}$ and $\mathscr{B}_{(d+q,e-1)}$ differ only by a twist of $\mathscr{O}_C(-q) \otimes L_-^{\vee,\otimes q}$ by 5.5, so v_1 followed by projection yields an isomorphism between the two sheaves. Considering now all bigraded pieces that lie in total degree $bq+a$, this implies that the map

$$\mathscr{B}_{(b-1)q+a}(-q) \otimes L_+ \xrightarrow{v_1} \mathscr{B}_{bq+a} \twoheadrightarrow \bigoplus\nolimits_{e=0}^{b-1} \mathscr{B}_{((b-e)q+a,e)}$$

obtained by composing v_1 with projection to its first d bigraded components is an isomorphism, and so $\mathscr{B}'_{bq+a} \cong \mathscr{B}_{(a,b)}$. The result now follows from 5.5. ■

The same argument shows that $\operatorname{coker}(v_1 \colon \mathscr{A}(-q) \otimes L_+ \to \mathscr{A})_{bq+a} \cong \mathscr{A}_{(a,b)}$. Since v_2 and v_3 have degrees at least q^2, this identifies the low degree components of $\pi_*\mathscr{O}_{T^\circ}$ and $\varphi_*\mathscr{O}_{S^\circ}$:

5.9. Corollary. — *For each* $0 \leq a, b \leq q-1$*, there are canonical identifications*

$$(\pi_*\mathscr{O}_{T^\circ})_{bq+a} \cong \operatorname{Sym}^b(\Omega_{\mathbf{P}W}(1))|_C \otimes \mathscr{O}_C(-a) \;\text{ and }\; (\varphi_*\mathscr{O}_{S^\circ})_{bq+a} \cong \operatorname{Fil}_a\big(\operatorname{Sym}^b(W^\vee) \otimes \mathscr{O}_C(-a)\big). \quad ■$$

Consider the equation v_3 next: By its construction in 2.14, it arises from a map induced by the q-bic form $\beta_{\mathscr{V}_T}$. However, the identifications from 5.2 together with the sequence from 2.13 shows that $\mathscr{V}|_{\mathbf{A}}$ is canonically identified as an extension

$$0 \to L_{-,\mathbf{A}} \to \mathscr{V}_{\mathbf{A}} \to L_{+,\mathbf{A}} \to 0.$$

Therefore β already induces a q-bic form on $\mathscr{V}_{\mathbf{A}}$, meaning that v_3 extends to all of $\mathbf{A}$. From now on, view v_3 as a map $\mathscr{B}' \otimes L_+^{\otimes q+1} \to \mathscr{B}'$. Since v_3 is locally a degree q polynomial in the fibre coordinate of $\rho \colon \mathbf{B} \to \mathbf{A}$, and since the filtration of $\mathscr{B}$ from 5.3 is by degree of the fibre coordinate of ρ, it follows that v_3 shifts the filtration up by q steps. In fact:

5.10. Lemma. — *The section* v_3 *maps* $\mathscr{B}' \otimes L_+^{\otimes q+1}$ *isomorphically onto* $\mathscr{B}'/\operatorname{Fil}_{q-1}\mathscr{B}'$*.*

Proof. If x and y are local coordinates of the subbundles $\mathbf{P}L_{-,\mathbf{A}}$ and $\mathbf{P}L_{+,\mathbf{A}}$ in $\mathbf{P}\mathscr{V}_{\mathbf{A}}$, then x/y is the local fibre coordinate of the affine bundle $\mathbf{B}$, and v_3 is $(x/y)^q$. Globally, this means that v_3 acts through the isomorphism $\beta_U \colon L_+ \to L_-^{\vee,\otimes q}$, and that it maps $\rho_*\mathscr{O}_{\mathbf{B}} \otimes L_+^{\otimes q+1}$ onto the principal ideal of $\rho_*\mathscr{O}_{\mathbf{B}}$ generated in degree q by $(L_-^\vee \otimes L_+)^{\otimes q}$. Comparing with 5.3 gives the result. ■

Finally, consider the equation v_2. Arguing as in 5.8 shows that it is induced by the map

$$\beta_W \circ \mathrm{eu}_{\mathbf{P}W} \colon \mathscr{O}_C(-1) \otimes L_+^{\otimes q} \to \Omega_{\mathbf{P}W}(1)|_C^{[1]} \otimes L_+^{\otimes q}.$$

This makes higher degree components of $\varphi_*\mathscr{O}_{S^\circ}$ less straightforward to describe. Components of the form $(\varphi_*\mathscr{O}_{S^\circ})_{dq+q-1}$ for $d < 2q$ are a notable and useful exception, as they may be exhibited as an extension of two rather simple bundles:

5.11. Proposition. — *For each $q \leq b \leq 2q-1$, there are short exact sequences of filtered bundles*

$$0 \to \mathrm{Sym}^{b-q}(W^\vee) \otimes \mathcal{O}_C(-2b+q) \to (\varphi_*\mathcal{O}_{S^\circ})_{bq+q-1} \to \mathrm{Sym}^b_{\mathrm{red}}(W^\vee) \otimes \mathcal{O}_C(-2b+2q-1) \to 0.$$

Proof. Consider the degree $b(q+1)$ component of $\varphi_*\mathcal{O}_{S^\circ}$: First, the degree $b(q+1)$ component of $\mathcal{B}$ modulo v_1 and v_3 is identified with $\mathrm{Fil}_{q-1}\big(\mathrm{Sym}^b(W^\vee) \otimes \mathcal{O}_C(-b)\big)$ using 5.3, 5.8, and 5.10. Next, there is a short exact sequence

$$0 \to \mathrm{Fil}_{q-1}\big(\mathrm{Sym}^{b-q}(W^\vee) \otimes \mathcal{O}_C(-b+q-1)\big) \to \mathrm{Fil}_{q-1}\big(\mathrm{Sym}^b(W^\vee) \otimes \mathcal{O}_C(-b)\big) \to (\varphi_*\mathcal{O}_{S^\circ})_{b(q+1)} \to 0$$

where the first map is induced by v_2 and multiplication. Since Frobenius stretches filtrations by a factor of q, the inclusion $\Omega^1_{\mathbf{P}W}(1)|_C^{[1]} \hookrightarrow W^{\vee,[1]} \otimes \mathcal{O}_C$ induces an equality

$$\mathrm{Fil}_{q-1}\big(\Omega^1_{\mathbf{P}W}(1)|_C^{[1]} \otimes \mathrm{Sym}^{b-q}(W^\vee) \otimes \mathcal{O}_C(-b+q)\big) = \mathrm{Fil}_{q-1}\big(W^{\vee,[1]} \otimes \mathrm{Sym}^{b-q}(W^\vee) \otimes \mathcal{O}_C(-b+q)\big).$$

Factoring the first map in the short exact sequence through v_2, using this identification of filtered pieces, and applying the snake lemma with $\mathrm{coker}(v_2) \cong \mathcal{O}_C(-q+1)$ yields a short exact sequence

$$0 \to \mathrm{Fil}_{q-1}\big(\mathrm{Sym}^{b-q}(W^\vee) \otimes \mathcal{O}_C(-b+1)\big) \to (\varphi_*\mathcal{O}_{S^\circ})_{q(b+1)} \to \mathrm{Fil}_{q-1}\big(\mathrm{Sym}^b_{\mathrm{red}}(W^\vee) \otimes \mathcal{O}_C(-b+q)\big) \to 0.$$

Since $b-q < q$, Fil_{q-1} gives the entire subbundle. Since $\Omega^1_{\mathbf{P}W}(1)|_C$ lies in the 0-th step of the filtration of $W^\vee \otimes \mathcal{O}_C(-1)$, the linear algebra fact 5.12 shows the same for the quotient. Finally, multiplication with $(\varphi_*\mathcal{O}_{S^\circ})_1^{\otimes b-q+1} \cong \mathcal{O}_C(b-q+1)$ maps $(\varphi_*\mathcal{O}_{S^\circ})_{bq+q-1}$ injectively into $(\varphi_*\mathcal{O}_{S^\circ})_{b(q+1)}$, and this is an isomorphism because the relations of $\varphi_*\mathcal{O}_{S^\circ}$ lie in degrees q, q^2, and $q(q+1)$, meaning that the ranks of the two bundles match. Twisting then gives the result. ■

The following is a simple observation about how Frobenius twists interact with filtrations:

5.12. Lemma. — *Let V be a finite dimensional vector space with a two step filtration* $\mathrm{Fil}_0 V \subseteq \mathrm{Fil}_1 V = V$. *If $\mathrm{gr}_1 V$ is one-dimensional, then, for all integers $b \geq q$, the map*

$$V^{[1]} \otimes \mathrm{Sym}^{b-q}(V) \to \mathrm{Sym}^b(V)/\mathrm{Fil}_{q-1}\mathrm{Sym}^b(V)$$

induced by multiplication is surjective, and it induces a canonical isomorphism

$$\mathrm{Fil}_{q-1}\mathrm{Sym}^b(V)/\mathrm{Fil}_{q-1}(V^{[1]} \otimes \mathrm{Sym}^{b-q}(V)) \cong \mathrm{Sym}^b_{\mathrm{red}}(V).$$

Proof. Choose a basis $\mathrm{Fil}_0 V = \langle v_1, \ldots, v_n \rangle$ and extend it to a basis of V with a lift $w \in V$ of a basis vector of $\mathrm{gr}_1 V$. Then $\mathrm{Sym}^b(V)$ has a basis given by the monomials of degree b in the $v_1, \ldots, v_n, w$, and the $(q-1)$-st piece of the induced filtration is

$$\mathrm{Fil}_{q-1}\mathrm{Sym}^b(V) = \langle v_1^{i_1} \cdots v_n^{i_n} w^j \mid i_1 + \cdots i_n + j = b \text{ and } j \leq q-1 \rangle.$$

Therefore each member of the monomial basis of $\mathrm{Sym}^b(V)/\mathrm{Fil}_{q-1}\mathrm{Sym}^b(V)$ is a product of w^q with an element of $\mathrm{Sym}^{b-q}(V)$, so the multiplication map in question is surjective. Since

$$\mathrm{Fil}_{q-1}(V^{[1]} \otimes \mathrm{Sym}^{b-q}(V)) = \ker\big(V^{[1]} \otimes \mathrm{Sym}^{b-q}(V) \to \mathrm{Sym}^b(V)/\mathrm{Fil}_{q-1}\mathrm{Sym}^b(V)\big)$$

the second statement now follows from the five lemma. ■

5.13. Proof of 5.1. — It remains to put everything together. First, by 4.10, the coordinate ring $\mathcal{D}$ of the conductor subscheme $D \hookrightarrow S$ is the truncation of $\varphi_*\mathcal{O}_{S^\circ}$ at degrees larger than $\delta = 2q^2 - q - 2$. Second, 4.14 gives a canonical isomorphism

$$\mathcal{F} \cong \mathcal{D}^\vee \otimes \mathcal{O}_C(-q+1) \otimes L_+^{\otimes 2q-1} \otimes L_-^{\otimes 2}.$$

Gradings are related by $\mathscr{F}_d \cong \mathscr{D}_{\delta-d}^\vee \otimes \mathcal{O}_C(-q+1)$, so since $\delta-(bq+q-1)=(2q-b-1)q-q-1$, combined with 5.11, this gives the exact sequence of 5.1. Third, since the filtration on $\mathscr{D}$ has only q steps by 5.10, it induces a filtration on $\mathscr{F}$ via

$$\mathrm{Fil}_i\,\mathscr{F} := (\mathscr{D}/\mathrm{Fil}_{q-2-i}\,\mathscr{D})^\vee \otimes \mathcal{O}_C(-q+1)\otimes L_+^{\otimes 2q-1}\otimes L_-^{\otimes 2} \text{ for } 0\leq i\leq q-1.$$

In particular, $\mathrm{Fil}_0\,\mathscr{F}$ is related to the $(q-1)$-st graded piece of $\mathscr{D}$, which by 5.9 is

$$\begin{aligned}\mathrm{gr}_{q-1}\,\mathscr{D} &\cong (\pi_*\mathcal{O}_{T^\circ}\otimes(L_-^\vee\otimes L_+)^{\otimes q-1})_{\leq 2q^2-q-2}\cong(\pi_*\mathcal{O}_{T^\circ})_{\leq q^2-q-1}\\ &\cong \bigoplus\nolimits_{b=0}^{q-2}\bigoplus\nolimits_{a=0}^{q-1}\mathrm{Sym}^b(\Omega^1_{\mathbf{P}W}(1))|_C\otimes\mathcal{O}_C(-a)\otimes L_+^{\otimes b}\otimes L_-^{\vee,\otimes a}.\end{aligned}$$

Dualizing, twisting, inverting summation indices, and effacing a global factor of $(L_+\otimes L_-)^{\otimes q+1}$ identifies $\mathrm{Fil}_0\,\mathscr{F}$ as in 5.1. Finally, $\partial\colon\mathscr{D}\to\mathscr{D}$ from the $\boldsymbol{\alpha}_q$-action fits into a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Fil}_{q-1-i}\,\mathscr{D} \arrow[r]\arrow[d,"\partial"] & \mathscr{D} \arrow[r]\arrow[d,"\partial"] & \mathscr{D}/\mathrm{Fil}_{q-1-i}\,\mathscr{D} \arrow[r]\arrow[d,"\partial"] & 0\\
0 \arrow[r] & \mathrm{Fil}_{q-2-i}\,\mathscr{D} \arrow[r] & \mathscr{D} \arrow[r] & \mathscr{D}/\mathrm{Fil}_{q-2-i}\,\mathscr{D} \arrow[r] & 0
\end{tikzcd}$$

and so there is an induced map $\partial\colon\mathscr{F}\to\mathscr{F}$ which, by 5.6, is of degree $-q-1$, satisfies $\partial(\mathrm{Fil}_i\,\mathscr{F})\subseteq\mathrm{Fil}_{i-1}\,\mathscr{F}$ for each $0\leq i\leq q-1$, and such that $\mathrm{gr}_i\,\partial\colon\mathrm{gr}_i\,\mathscr{F}_{d+q+1}\to\mathrm{gr}_{i-1}\,\mathscr{F}_d$ is an isomorphism if $p\nmid i$ and zero otherwise. This completes the proof of 5.1. ■

6. Cohomology of $\mathscr{F}$

The purpose of this section is to compute cohomology of $\mathscr{F}$ when $q=p$ using the results of §5: see 6.11. The strategy is to identify each graded component $\mathrm{H}^0(C,\mathscr{F}_i)$ as a representation of the special unitary group $\mathrm{SU}_3(p):=\mathrm{U}_3(p)\cap\mathbf{SL}(W)$, and this is achieved in three steps: First, many global sections are constructed for pieces of the form $\mathscr{F}_{bp+p-1}$ using the exact sequence in 5.1. Second, the action of $\boldsymbol{\alpha}_p$ on $\mathscr{F}$ gives maps

$$\mathscr{F}_{bp+p-1}\xrightarrow{\partial}\mathscr{F}_{bp+p-1-(p+1)}\xrightarrow{\partial}\cdots\xrightarrow{\partial}\begin{cases}\mathscr{F}_{p-1-b} & \text{if } 0\leq b\leq p-2, \text{ and}\\ \mathscr{F}_{(b-p+1)p} & \text{if } p-1\leq b\leq 3p-3.\end{cases}$$

Each of the maps are injective on global sections by 6.2, and so this gives a lower bound on the space of sections of each component, see 6.3. Third, a corresponding upper bound is determined for those rightmost sheaves, see 6.8 and 6.10.

Begin with a simple, but crucial, computation of sections lying in the 0-th filtered piece of $\mathscr{F}$:

6.1. Lemma. — $\mathrm{H}^0(C,\mathrm{Fil}_0\,\mathscr{F})\cong\bigoplus_{b=0}^{p-2}\mathrm{Div}^{p-2-b}(W)\otimes L_+^{\otimes b}$.

Proof. The filtration statement of 5.1 shows that

$$\mathrm{H}^0(C,\mathrm{Fil}_0\,\mathscr{F})\cong\bigoplus\nolimits_{b=0}^{p-2}\bigoplus\nolimits_{a=0}^{p-1}\mathrm{H}^0(C,\mathrm{Div}^{p-2-b}(\mathscr{T})\otimes\mathcal{O}_C(-a))\otimes L_+^{\otimes b}\otimes L_-^{\vee,\otimes a}.$$

All divided powers appearing have exponent less than p, so may be replaced by symmetric powers. Then Griffith's Borel–Weil–Bott vanishing A.4 applies to give the result. ■

Consider now the operator $\partial\colon\mathscr{F}\to\mathscr{F}$ induced by the action of $\boldsymbol{\alpha}_p$ on S. The final statement of 5.1 implies that

$$\ker(\partial\colon\mathscr{F}\to\mathscr{F})=\bigoplus\nolimits_{i=0}^{p}\mathscr{F}_i\oplus\bigoplus\nolimits_{i=p+1}^{\delta}\mathrm{Fil}_0\,\mathscr{F}_i.$$

Taking global sections and comparing with 6.1 shows that ∂ acts injectively on most of the graded components of $\mathrm{H}^0(C,\mathscr{F})$:

6.2. Lemma. — $\ker\big(\partial\colon \mathrm{H}^0(C,\mathscr{F}) \to \mathrm{H}^0(C,\mathscr{F})\big) = \bigoplus_{a=0}^{p-1} \mathrm{H}^0(C,\mathscr{F}_a) \oplus \bigoplus_{b=1}^{p-2} \mathrm{H}^0(C,\mathrm{Fil}_0\,\mathscr{F}_{bp})$. ∎

Iterating ∂ gives a lower bound on the sections of the $\mathscr{F}_i$:

6.3. Lemma. — *For each $0 \leq b \leq 2p-3$ and $0 \leq a \leq \min(b,p-1)$,*

$$\mathrm{Div}^{2p-3-b}_{\mathrm{red}}(W) \subseteq \mathrm{H}^0(C,\mathscr{F}_{bp+p-1-a(p+1)}).$$

Proof. When $a=0$, this concerns $\mathscr{F}_{bp+p-1}$, and the statement follows from the exact sequence in 5.1. Iteratively applying ∂ then gives the result for $a>0$ in view of injectivity from 6.2. ∎

It remains to give a matching upper bound. Generic injectivity of $\partial\colon \mathrm{H}^0(C,\mathscr{F}) \to \mathrm{H}^0(C,\mathscr{F})$ from 6.2 means it suffices to determine $\mathrm{H}^0(C,\mathscr{F}_i)$ when $0 \leq i \leq p-1$, and when $i = jp$ for $0 \leq j \leq 2p-2$. The cases $0 \leq i \leq p$ are dealt with an explicit cohomology computation, see 6.7 and 6.8; the remaining cases then follow from this explicit calculation by further analysing the action of ∂ on global sections, see 6.10.

6.4. — Consider the degree $0 \leq i \leq p$ components of the defining presentation of $\mathscr{F}$ from 4.11:

$$0 \to \mathscr{D}_i \xrightarrow{\nu_i^\#} \mathscr{D}_i^\nu \to \mathscr{F}_i \to 0.$$

Identifying low degree pieces of $\mathscr{D}$ and $\mathscr{D}^\nu$ via 5.11 and 4.15, respectively, and taking the long exact sequence in cohomology shows that

$$\mathrm{H}^0(C,\mathscr{F}_i) \cong \begin{cases} \ker\big(\nu_i^\#\colon \mathrm{H}^1(C,\mathscr{O}_C(-i)) \to \mathrm{H}^1(C,\phi_{C,*}(\mathscr{T}_C(-1)^{\otimes i}))\big) & \text{if } 0 \leq i \leq p-1, \text{ and} \\ \ker\big(\nu_p^\#\colon \mathrm{H}^1(C,\Omega^1_{\mathbf{P}W}(1)|_C) \to \mathrm{H}^1(C,\phi_{C,*}(\mathscr{T}_C(-1)^{\otimes p}))\big) & \text{if } i = p. \end{cases}$$

An explicit description of $\nu_1^\#$ can be given in terms of the canonical section $\theta\colon \mathscr{O}_C(-p^2) \to \mathscr{T}_C(-1)$ from 1.7 determining the Hermitian points of C:

6.5. Lemma. — *The sheaf map $\nu_1^\#\colon \mathscr{D}_1 \to \mathscr{D}_1^\nu$ is identified with*

$$\phi_{C,*}(\theta) \circ \phi_C^\#\colon \mathscr{O}_C(-1) \to \phi_{C,*}\phi_C^*(\mathscr{O}_C(-1)) \to \phi_{C,*}(\mathscr{T}_C(-1))$$

Proof. The degree 1 generators of $\mathscr{D}$ and $\mathscr{D}^\nu$ correspond to the fibre coordinates over C of the affine bundles $\mathbf{A}_1$ from 5.2 and $S^{\nu,\circ}$ from 4.15, respectively. So consider the commutative diagram

$$\begin{tikzcd} S^\nu \arrow[r, "\nu"'] \arrow[d, "\tilde\varphi_+"] & S \arrow[r, dashed, "\psi"'] \arrow[d, "\varphi_-"] & \mathbf{P}\mathscr{V}_1 \arrow[d, "\pi_1"] \\ C \arrow[r, "\phi_C"] & C \arrow[r, "\mathrm{id}_C"] & C \end{tikzcd}$$

obtained by putting 2.6 and 4.8 together. The rational map ψ restricts to a morphism $S^\circ \to \mathbf{A}_1$, and sends a line $\ell \subset X \setminus x_-$ to its point of intersection $\ell \cap \mathbf{P}L_-^{[1],\perp}$ with the tangent hyperplane to X at x_-. Thus the pullback of $\mathscr{O}_{\pi_1}(-1)$ via ψ, at least on S°, is identified with the subsheaf

$$\mathscr{L} := \mathscr{S} \cap L_{-,S}^{[1],\perp} \subset V_S.$$

Moreover, $\mathscr{L}$ lies inside $\varphi_-^*\mathscr{V}_1 = \varphi_-^*\mathscr{O}_C(-1) \oplus L_{-,S}$, the subbundle of V_S parameterizing the lines spanned by points of C and x_-, and projection away from the $L_{-,S}$ factor yields the identification between the fibre coordinate of $\mathscr{A}_1$ with $\varphi_-^*\mathscr{O}_C(-1)$ on S°, as explained in 5.2.

Pulling back to S^ν produces a subsheaf $\nu^*\mathscr{L} \hookrightarrow \mathscr{K}$, where $\mathscr{K}$ is as in the proof of of 4.7. The map $\nu_1^\#$ in question is obtained from the composition

$$\nu^*\mathscr{L} \hookrightarrow \mathscr{K} \twoheadrightarrow \mathscr{O}_{\tilde\varphi_+}(-1)$$

by projecting out L_- and restricting to $S^{\nu,\circ}$. Projection away from L_- maps $\mathscr{K}$ onto the subbundle of $(W \oplus L_+)_{S^\nu}$ underlying planes spanned by the tangent lines to C and x_+; since $\nu^*\mathscr{L}$ and $\mathcal{O}_{\tilde{\varphi}_+}(-1)$ project to line bundles on C, the map in question is pulled back via $\tilde{\varphi}_+^*$ of the map

$$\phi_C^*(\mathcal{O}_C(-1)) \hookrightarrow \mathcal{E}_C \twoheadrightarrow \mathcal{T}_C(-1)$$

which, as explained in 1.7, is given by θ. Pushing along $\phi_C \colon C \to C$ now gives the result. ∎

6.6. — Since $\nu^\#$ is a map of algebras, the composition $\nu_i^\# \circ \mu_i \colon \mathscr{D}_1^{\otimes i} \hookrightarrow \mathscr{D}_i \to \mathscr{D}_i^\nu$ of the i-fold multiplication map μ_i followed by $\nu_i^\#$ is given by the i-th power $\phi_{C,*}(\theta^i) \circ \phi_C^\#$ of the map appearing in 6.5. The action of this map on $\mathrm{H}^1(C, \mathcal{O}_C(-i))$ can be determined as follows: Let

$$f \in \mathrm{H}^0(\mathbf{P}W, \mathcal{O}_{\mathbf{P}W}(p+1)) \quad \text{and} \quad \tilde{\theta} \in \mathrm{H}^0(\mathbf{P}W, \mathcal{O}_{\mathbf{P}^2}(p^2-p+1))$$

be an equation for C and any lift of $\theta \in \mathrm{H}^0(C, \mathcal{O}_C(p^2-p+1))$, respectively. Then there is a commutative diagram of sheaves on $\mathbf{P}W$ with exact rows given by

$$\begin{tikzcd}
\mathcal{O}_{\mathbf{P}W}(-i-p-1) \arrow[r, hook, "f"'] \arrow[d, "\phi_C^\#"] & \mathcal{O}_{\mathbf{P}W}(-i) \arrow[r, two heads] \arrow[dd, "\phi_C^\#"] & \mathcal{O}_C(-i) \arrow[dd, "\phi_C^\#"] \\
\phi_{C,*}\mathcal{O}_{\mathbf{P}W}(-(i+p+1)p^2) \arrow[d, "f^{p^2-1}"] & & \\
\phi_{C,*}\mathcal{O}_{\mathbf{P}W}(-ip^2-p-1) \arrow[r, hook, "\phi_{C,*}(f)"] \arrow[d, "\phi_{C,*}(\tilde{\theta}^i)"] & \phi_{C,*}\mathcal{O}_{\mathbf{P}W}(-ip^2) \arrow[r, two heads] \arrow[d, "\phi_{C,*}(\tilde{\theta}^i)"] & \phi_{C,*}\mathcal{O}_C(-ip^2) \arrow[d, "\phi_{C,*}(\theta^i)"] \\
\phi_{C,*}\mathcal{O}_{\mathbf{P}W}(-i(p-1)-p-1) \arrow[r, hook] & \phi_{C,*}\mathcal{O}_{\mathbf{P}W}(-i(p-1)) \arrow[r, two heads] & \phi_{C,*}(\mathcal{T}_C(-1)^{\otimes i}).
\end{tikzcd}$$

Taking cohomology and explicitly computing with the cohomology of $\mathbf{P}W$ gives:

6.7. Lemma. — $\phi_{C,*}(\theta^i) \circ \phi_C^\# \colon \mathrm{H}^1(C, \mathcal{O}_C(-i)) \to \mathrm{H}^1(C, \phi_{C,*}(\mathcal{T}_C(-1)^{\otimes i}))$ *is nonzero for* $2 \leq i \leq p$.

Proof. Choose coordinates $(x:y:z)$ on $\mathbf{P}W = \mathbf{P}^2$ so that $f = x^p y + x y^p - z^{p+1}$. The following is a lift of θ, as can be verified by showing it vanishes on the $\mathbf{F}_{p^2}$ points of C, as done in [Che22, 3.5.4]:

$$\tilde{\theta} := \frac{x^{p^2}y - xy^{p^2}}{x^p y + x y^p} z = \big(x^{p(p-1)} - x^{(p-1)(p-1)}y^{p-1} + \cdots - y^{p(p-1)}\big)z.$$

View the cohomology groups of $\mathbf{P}^2$ as a module over its homogeneous coordinate ring as explained in [Stacks, 01XT], and consider a class

$$\xi := \frac{1}{xyz}\frac{1}{x^{i+p-2}} \in \mathrm{H}^2(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(-i-p-1)).$$

This acts on homogeneous polynomials by contraction, see [Stacks, 01XV]. Observe that $\xi \cdot f = 0$ since f does not contain a pure power of x, and so ξ represents a class in $\mathrm{H}^1(C, \mathcal{O}_C(-i))$. I claim that $\phi_{C,*}(\theta^i)(\xi) \neq 0$. Indeed, since f is a Hermitian q-bic equation, ϕ_C is the p^2-power Frobenius by 1.6, and the diagram of 6.6 shows that $\phi_{C,*}(\theta^i)(\xi)$ is represented by the product

$$\xi^{p^2} \cdot (f^{p^2-1}\tilde{\theta}^i) = \Big(\frac{1}{xyz}\frac{1}{x^{d+p-2}}\Big)^{p^2} \cdot \Big((x^p y + xy^p - z^{p+1})^{p^2-1}\Big(\frac{x^{p^2}y - xy^{p^2}}{x^p y + xy^p}\Big)^i z^i\Big).$$

Consider the coefficient of $z^{(p+1)(p-2)+i}$ in $f^{p^2-1}\tilde{\theta}^i$: Since $0 < i < p+1$, this is the coefficient of z^i in $\tilde{\theta}^i$ multiplied by the coefficient of $z^{(p+1)(p-2)}$ in f^{p^2-1}. Writing

$$f^{p^2-1} = \big((x^p y + xy^p)^p - z^{p(p+1)}\big)^{p-1}\big((x^p y + xy^p) - z^{p+1}\big)^{p-1}$$

shows that the latter is $-(x^p y + xy^p)^{p^2-p+1}$. Therefore $\xi^{p^2} \cdot (f^{p^2-1}\tilde{\theta}^i)$ has as a summand

$$\frac{1}{x^{(i+p-1)p^2} y^{p^2} z^{p+2-i}} \cdot \Big(-(x^p y + xy^p)^{p^2-p+1} \Big(\frac{x^{p^2} y - xy^{p^2}}{x^p y + xy^p} \Big)^i \Big)$$
$$= \frac{-1}{x^{(i+p-2)p^2+p-1} y^{p-1} z^{p+2-i}} \cdot \Big((x^{p-1} + y^{p-1})^{p^2-p+1-i} (x^{p^2-1} - y^{p^2-1})^i \Big).$$

Since all monomials in y involve at least y^{p-1}, the only potentially nonzero contribution is the pure power of x, so this is equal to

$$\frac{-1}{x^{(i+p-1)p^2+p-1} y^{p-1} z^{p+2-i}} \cdot x^{(p-1)(p^2-p+1-i)+(p^2-1)i} = \frac{-1}{x^{(i-1)p} y^{p-1} z^{p+2-i}}.$$

This is nonzero in $\mathrm{H}^2(\mathbf{P}^2, \mathscr{O}_{\mathbf{P}^2}(-(i+1)(p+1)))$ if $2 \leq i \leq p$, so $\phi_{C,*}(\theta^i)(\xi) \neq 0$. ■

The following three statements now determine the crucial components of $\mathrm{H}^0(C, \mathscr{F})$:

6.8. Proposition. — $\mathrm{H}^0(C, \mathscr{F}_i) \cong \begin{cases} \mathrm{Div}^{p+i-2}_{\mathrm{red}}(W) & \textit{if } 0 \leq i \leq p-1, \textit{ and} \\ \mathrm{Div}^{p-3}(W) & \textit{if } i = p. \end{cases}$

Proof. Applying 6.3 with $a = b = p - i - 1$ when $i \neq p$, and $a + 1 = b = p$ when $i = p$ shows that $\mathrm{H}^0(C, \mathscr{F}_i)$ contains the representation L appearing on the right side of the purported isomorphism. The exact sequence

$$0 \to \mathrm{H}^0(C, \mathscr{F}_i) \to \mathrm{H}^1(C, \mathscr{D}_i) \xrightarrow{\nu_i^\#} \mathrm{H}^1(C, \mathscr{D}_i^\nu) \to \mathrm{H}^1(C, \mathscr{F}_i) \to 0$$

then implies the statement in the case $0 \leq i \leq 1$ since $\mathrm{H}^1(C, \mathscr{D}_i) = L$ by 5.11 and A.11(i). When $2 \leq i \leq p$, 5.11 and A.11(ii)–(iv) together show that $\mathrm{H}^1(C, \mathscr{D}_i)/L$ is a simple $\mathrm{SU}_3(p)$-representation, so to conclude, it suffices to show that $\nu_i^\#$ is nonzero. As explained in 6.6, there is a factorization

$$\phi_{C,*}(\theta^i) \circ \phi_C^\# : \mathrm{H}^0(C, \mathscr{D}_1^{\otimes i}) \xrightarrow{\mu_i} \mathrm{H}^0(C, \mathscr{D}_i) \xrightarrow{\nu_i^\#} \mathrm{H}^0(C, \mathscr{D}_i^\nu).$$

Observe that μ_i is surjective on global sections: When $i < p$, this is because $\mu_i : \mathscr{D}^{\otimes i} \to \mathscr{D}_i$ is already an isomorphism; when $i = p$, this follows from $\mathrm{coker}(\mu_p : \mathscr{O}_C(-p) \to \Omega^1_{\mathbf{P}W}(1)|_C) = \mathscr{O}_C(p-1)$. Since $\phi_{C,*}(\theta^i) \circ \phi_C^\#$ is nonzero by 6.7, ϕ is also nonzero. ■

6.9. Corollary. — $\mathrm{H}^0(C, \mathscr{F}_{bp+p-1}) \cong \mathrm{Div}^{2p-3-b}_{\mathrm{red}}(W)$ *for* $0 \leq b \leq 2p-3$.

Proof. When $p - 2 \leq b \leq 2p-3$, $\mathscr{F}_{bp+p-1} \cong \mathrm{Div}^{2p-3-b}_{\mathrm{red}}(W) \otimes \mathscr{O}_C$ by the sequence in 5.1 yielding the conclusion in this case. When $0 \leq b \leq p-3$, 6.2, 6.3, and 6.8 together give a sequence of inclusions

$$\mathrm{Div}^{2p-3-b}_{\mathrm{red}}(W) \subseteq \mathrm{H}^0(C, \mathscr{F}_{bp+p-1}) \overset{\partial^b}{\hookrightarrow} \mathrm{H}^0(C, \mathscr{F}_{p-1-b}) = \mathrm{Div}^{2p-3-b}_{\mathrm{red}}(W).$$

Therefore equality holds throughout. ■

6.10. Proposition. — $\mathrm{H}^0(C, \mathscr{F}_{bp}) \cong \begin{cases} \mathrm{Div}^{p-2-b}(W) & \textit{if } 0 \leq b \leq p-2, \textit{ and} \\ 0 & \textit{if } p-1 \leq b \leq 2p-2. \end{cases}$

Proof. When $0 \leq b \leq 1$, this is 6.8. Assume that $2 \leq b \leq 2p-2$. The final statement of 5.1 implies that there is an exact sequence

$$0 \to \mathrm{Fil}_0 \mathscr{F}_{bp} \to \mathscr{F}_{bp} \xrightarrow{\partial} \mathscr{F}_{(b-2)p+p-1} \to \mathrm{gr}_{p-1} \mathscr{F}_{(b-2)p+p-1} \to 0.$$

Since $\mathrm{H}^0(C, \mathrm{Fil}_0 \mathscr{F}_{bp}) = \mathrm{Div}^{p-2-b}(W)$ by 6.1, where negative divided powers are taken to be zero, it suffices to show that ∂ vanishes on global sections. Exactness of the sequence means this is equivalent to injectivity of $\mathscr{F}_{(b-2)p+p-1} \to \mathrm{gr}_{p-1} \mathscr{F}_{(b-2)p+p-1}$ on global sections. So consider the composite

$$\mathrm{Div}^{2p-1-b}_{\mathrm{red}}(W) \otimes \mathscr{O}_C \subset \mathscr{F}_{(b-2)p+p-1} \twoheadrightarrow \mathrm{gr}_{p-1} \mathscr{F}_{(b-2)p+p-1}$$

where the first map is the inclusion of the subbundle from the sequence of 5.1. The first map is an isomorphism on global sections by 6.9. Comparing with A.7 shows that

$$\mathrm{Div}^{2p-1-b}_{\mathrm{red}}(W) = \begin{cases} L(p-b, b-1) & \text{if } 2 \leq b \leq p-1 \text{ and} \\ L(0, 2p-1-b) & \text{if } p \leq b \leq 2p-1, \end{cases}$$

so it is a simple $\mathrm{SU}_3(p)$ representation. Thus it suffices to see that the composite is a nonzero map of sheaves. Since the maps respect filtrations, it suffices to observe that

$$\mathrm{gr}_{p-1}\left(\mathrm{Div}^{2p-1-b}_{\mathrm{red}}(W) \otimes \mathscr{O}_C\right) = \mathrm{Fil}_0\left(\mathrm{Sym}^{2p-1-b}_{\mathrm{red}}(W^\vee) \otimes \mathscr{O}_C\right)^\vee = \mathrm{Div}^{2p-1-b}_{\mathrm{red}}(\mathscr{T})$$

is nonzero. The result now follows. ■

6.11. Theorem. — $\mathrm{H}^0(C, \mathscr{F}) \cong \Lambda_1 \oplus \Lambda_2$ *as a representation of* G, *where*

$$\Lambda_1 := \bigoplus\nolimits_{b=0}^{p-2} \mathrm{Div}^{p-2-b}(W) \otimes \mathrm{Sym}^{p-1}(U) \otimes L_+^{\otimes b} \otimes L_-^{\vee, \otimes p-1}, \;\; \textit{and}$$
$$\Lambda_2 := \bigoplus\nolimits_{a=0}^{p-2} \mathrm{Div}^{p+a-1}_{\mathrm{red}}(W) \otimes \mathrm{Sym}^{p-2-a}(U) \otimes L_-^{\vee, \otimes p-1}.$$

Proof. Begin by identifying each $\mathrm{H}^0(C, \mathscr{F}_i)$ as a representation for $\mathrm{SU}_3(p)$. The claim is that the inclusions from 6.3 are equalities: that

$$\mathrm{H}^0(C, \mathscr{F}_{bp+p-1-a(p+1)}) = \mathrm{Div}^{2p-3-b}_{\mathrm{red}}(W)$$

if $0 \leq b \leq 2p-3$ and $0 \leq a \leq \min(b, p-1)$, and that the group vanishes otherwise. Choose $0 \leq b \leq 3p-3$. Starting from $\mathrm{H}^0(C, \mathscr{F}_{bp+p-1})$ and successively applying ∂ a total of $\min(b, p-1)$ times produces, thanks to 6.2, a chain of inclusions

$$\mathrm{H}^0(C, \mathscr{F}_{bp+p-1}) \subseteq \cdots \subseteq \begin{cases} \mathrm{H}^0(C, \mathscr{F}_{p-1-b}) & \text{if } 0 \leq b \leq p-2, \text{ and} \\ \mathrm{H}^0(C, \mathscr{F}_{(b-p+1)p}) & \text{if } p-1 \leq b \leq 3p-3. \end{cases}$$

By convention, set $\mathrm{H}^0(C, \mathscr{F}_i) = 0$ whenever $i > \delta$. The spaces on the left are given by 6.9 whereas the spaces on the right are given by 6.8 and 6.10, and for each fixed a, the lower and upper bounds match. Therefore equality holds throughout. The $\mathbf{Aut}(L_- \subset U, \beta_U)$ factor of G of the representation is obtained by matching weights and using 6.2 to identify the action of the unipotent radical. ■

6.12. Corollary. — $\dim_{\mathbf{k}} \mathrm{H}^0(C, \mathscr{F}) = (p^2+1)\binom{p}{2} + \binom{p}{3}$.

Proof. Sum the cohomology groups in 6.11 column-wise, summing over residue classes modulo p:

$$\begin{aligned} \dim_{\mathbf{k}} \mathrm{H}^0(C, \mathscr{F}) &= \sum\nolimits_{a=0}^{p-1} \sum\nolimits_{b=0}^{a+p-2} \dim_{\mathbf{k}} \mathrm{H}^0(C, \mathscr{F}_{(a+p-2-b)p+a}) \\ &= \sum\nolimits_{a=0}^{p-1} \sum\nolimits_{b=0}^{a+p-2} \dim_{\mathbf{k}} \mathrm{Div}^b_{\mathrm{red}}(W) \\ &= \sum\nolimits_{a=0}^{p-1} \sum\nolimits_{b=0}^{a+p-2} \left(\dim_{\mathbf{k}} \mathrm{Div}^b(W) - \dim_{\mathbf{k}}(W^{[1]} \otimes \mathrm{Div}^{b-p}(W))\right). \end{aligned}$$

28	36	42	46	51	48	55	42
24	36	39	42	56	72	60	46
18	30	36	36	51	73	75	48
10	18	24	28	36	51	56	48
6	10	15	21	28	36	42	46
3	6	10	15	24	36	39	42
1	3	6	10	18	30	36	36
0	1	3	6	10	18	24	28
0	0	1	3	6	10	15	21
0	0	0	1	3	6	10	15
0	0	0	0	1	3	6	10
0	0	0	0	0	1	3	6
0	0	0	0	0	0	1	3
0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	

36	45	52	58	60	61	71	60	52
31	45	45	58	75	60	77	90	57
21	31	36	45	58	57	63	71	60
15	21	28	36	45	52	58	60	61
11	18	21	31	45	45	58	75	60
6	11	15	21	31	36	45	58	57
3	6	10	15	21	28	36	45	52
1	3	6	11	18	21	31	45	45
0	1	3	6	11	15	21	31	36
0	0	1	3	6	10	15	21	28
0	0	0	1	3	6	11	18	21
0	0	0	0	1	3	6	11	15
0	0	0	0	0	1	3	6	10
0	0	0	0	0	0	1	3	6
0	0	0	0	0	0	0	1	3
0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	

FIGURE 1. The dimensions of the $\mathrm{H}^0(C, \mathscr{F}_i)$ are displayed with $q = 8$ on the left, and $q = 9$ on the right. The numbers are arranged so that the first row displays the dimensions of $\mathrm{H}^0(C, \mathscr{F}_i)$ for $0 \leq i \leq q-1$. These were obtained from computer calculations done with *Macaulay2* [GS].

Since W is a 3-dimensional vector space, $\mathrm{Div}^b(W)$ is $\binom{b+2}{2}$ dimensional for all $b \geq 0$, so using standard binomial coefficient identities gives

$$\begin{aligned}\dim_{\mathbf{k}} \mathrm{H}^0(C, \mathscr{F}) &= \sum\nolimits_{a=0}^{p-1} \sum\nolimits_{b=0}^{a+p-2} \binom{b+2}{2} - 3 \sum\nolimits_{a=2}^{p-1} \sum\nolimits_{b=0}^{b-2} \binom{b+2}{2} \\ &= \sum\nolimits_{a=0}^{p-1} \binom{a+p+1}{3} - 3 \sum\nolimits_{a=2}^{p-1} \binom{a+1}{3} = \binom{2p+1}{4} - 4\binom{p+1}{4}.\end{aligned}$$

It can now be directly verified that $\binom{2p+1}{4} - 4\binom{p+1}{4} = (p^2+1)\binom{p}{2} + \binom{p}{3}$. ■

6.13. Remarks toward general q. — The assumption that $q = p$ was used in at least three ways: First, to to apply the Borel–Weil–Bott Theorem in A.4 and to identify divided powers with symmetric powers in 6.1; Second, to reduce the action of $\boldsymbol{\alpha}_q$ to the action of the single operator ∂ which enjoys the injectivity property 6.3; Third, to show in A.11 that the $\mathrm{SU}_3(q)$ representations appearing are either simple or have very short composition series.

In any case, part of the difficulty to extending the computation past the prime case is that the formula 6.12 does not hold for general q, and there may be some dependence on the exponent of q. A computer computation shows that

$$\dim_{\mathbf{k}} \mathrm{H}^0(C, \mathscr{F}) = \begin{cases} 106 \\ 2096 \\ 3231 \end{cases} \quad \text{whereas} \quad (q^2+1)\binom{q}{2} + \binom{q}{3} = \begin{cases} 106 & \text{if } q = 4, \\ 1876 & \text{if } q = 8, \text{ and} \\ 3036 & \text{if } q = 9. \end{cases}$$

The dimensions of $\mathrm{H}^0(C, \mathscr{F}_i)$ in the cases $q = 8$ and $q = 9$ are given in Figure 1. Certain general features from the prime case hold true—for instance, the action of $\boldsymbol{\alpha}_q$ still relates graded components which differ in weight by $q-1$—but are jumps in certain entries, related to jumps in cohomology of homogeneous bundles on $\mathbf{P}W$.

7. SMOOTH q-BIC THREEFOLDS

Finally, return to a smooth q-bic threefold X in $\mathbf{P}V$, and S its Fano surface of lines. The aim of this section is to compute the cohomology of $\mathcal{O}_S$ when $q = p$, thereby proving Theorem B. The proof is contained in 7.7 at the end of the section, and is achieved through a careful degeneration argument. Specifically, the cohomology of S is related to the cohomology of the Fano scheme S_0 of a q-bic threefold of type $\mathbf{1}^{\oplus 3} \oplus \mathbf{N}_2$ via a special 1-parameter degeneration of the q-bic threefold X; this family carries additional symmetries that allows one to bootstrap the results of §§4–6 to complete the computation in the smooth case. The construction is as follows:

7.1. Proposition. — *Let $x_-, x_+ \in X$ be Hermitian points such that $\langle x_-, x_+ \rangle \not\subset X$. Then there exists a q-bic threefold $\mathfrak{X} \subset \mathbf{P}V \times \mathbf{A}^1$ over $\mathbf{A}^1$ such that*

(i) *the constant sections $x_\pm \colon \mathbf{A}^1 \to \mathbf{P}V \times \mathbf{A}^1$ factor through $\mathfrak{X}$;*
(ii) *$(X_t, x_-, \mathbf{T}_{X_t,x_-})$ is a smooth cone situation for all $t \in \mathbf{A}^1$;*
(iii) *the projection $\mathfrak{X} \to \mathbf{A}^1$ is smooth away from $0 \in \mathbf{A}^1$ and $X = \mathfrak{X}_1$; and*
(iv) *X_0 is of type $\mathbf{1}^{\oplus 3} \oplus \mathbf{N}_2$ with singular point x_+.*

Moreover, there exists a choice of coordinates $(x_0 : x_1 : x_2 : x_3 : x_4)$ on $\mathbf{P}V \cong \mathbf{P}^4$ such that

$$\mathfrak{X} = \mathrm{V}(x_0^q x_1 + t x_0 x_1^q + x_2^{q+1} + x_3^{q+1} + x_4^{q+1}) \subset \mathbf{P}^4 \times \mathbf{A}^1,$$

$x_- = (1:0:0:0:0)$, and $x_+ = (0:1:0:0:0)$.

Property 7.1(iii) means that $\mathfrak{X}$ together with the section x_- defines a family of smooth cone situations, degenerating the situation 2.2(i) to 2.2(ii). It is straightforward to see that the explicit q-bic threefold $\mathfrak{X}$ over $\mathbf{A}^1$ has the advertised properties. However, to pin down the dependencies and automorphisms of the situation, it is useful to give an invariant construction of $\mathfrak{X}$.

Proof. Let (V, β) be a q-bic form defining X and write $x_\pm = \mathbf{P}L_\pm$. The assumption that $\langle x_-, x_+ \rangle = \mathbf{P}U$ is not contained in X is means that the restriction β_U of β to U is nondegenerate. Since U is Hermitian, it has a unique orthogonal complement W.

Set $V[t] := V \otimes_{\mathbf{k}} \mathbf{k}[t]$, and similarly for $U[t]$ and $W[t]$. Let $\beta_W \otimes \mathrm{id}_{\mathbf{k}[t]}$ be the constant extension of β_W to a q-bic form over $\mathbf{k}[t]$ on $W[t]$, and let

$$\beta_U^{L_\pm} \colon U[t]^{[1]} \otimes_{\mathbf{k}[t]} U[t] \to \mathbf{k}[t]$$

be the unique q-bic form over $\mathbf{k}[t]$ with $\mathrm{Gram}(\beta_U^{L_\pm}; e_- \otimes 1, e_+ \otimes 1) = \left(\begin{smallmatrix} 0 & 1 \\ t & 0 \end{smallmatrix}\right)$ where e_- and e_+ are a basis of U satisfying $\beta_U(e_-^{[1]}, e_+) = \beta_U(e_+^{[1]}, e_-) = 1$. Thus its restriction to $t = 1$ is β_U, and its restriction to $t = 0$ is of type $\mathbf{N}_2$. Let $\beta^{L_\pm}$ be q-bic form on $V[t]$ given by the orthogonal sum of $\beta_W[t]$ and $\beta_U^{L_\pm}$. Then the q-bic $\mathfrak{X} \subset \mathbf{P}V \times \mathbf{A}^1$ over $\mathbf{A}^1 := \operatorname{Spec} \mathbf{k}[t]$ defined by $(V[t], \beta^{L_\pm})$ is the desired threefold. ■

7.2. Group scheme actions. — Two group schemes act on the situation $\mathfrak{X} \to \mathbf{A}^1$ from 7.1: First, $\mathfrak{X}$ admits an action over $\mathbf{A}^1$ by the automorphism group scheme $\mathbf{Aut}(V[t], \beta^{L_\pm})$ of the q-bic form over $\mathbf{k}[t]$, as defined in [Che23a, 5.1]. The finite flat subgroup scheme $\mathfrak{G}$ that respects the orthogonal decomposition $\beta^{L_\pm} = \beta_W[t] \oplus \beta_U^{L_\pm}$ and leaves the section $x_- \colon \mathbf{A}^1 \to \mathfrak{X}$ invariant furthermore acts on the family of smooth cone situations, and may be presented as

$$\mathfrak{G} \cong \mathrm{U}_3(q) \times \left\{ \begin{pmatrix} \lambda & \epsilon \\ 0 & \lambda^{-q} \end{pmatrix} \in \mathbf{GL}_2(\mathbf{k}[t]) : \lambda \in \boldsymbol{\mu}_{q^2-1}, \epsilon^q + t\lambda^{q-1}\epsilon = 0 \right\}.$$

Second, consider the $\mathbf{G}_m$-action on $\mathbf{P}V \times \mathbf{A}^1$ with weights

$$\mathrm{wt}(W) = 0, \quad \mathrm{wt}(L_-) = -1, \quad \mathrm{wt}(L_+) = q, \quad \mathrm{wt}(t) = q^2 - 1.$$

This leaves the q-bic form $\beta^{L_\pm}$ defining $\mathfrak{X}$ invariant, and so it induces an action on both $\mathfrak{X}$ and $\mathfrak{G}$ over $\mathbf{A}^1$ such that the action map $\mathfrak{G} \times_{\mathbf{A}^1} \mathfrak{X} \to \mathfrak{X}$ is $\mathbf{G}_m$-equivariant over $\mathbf{A}^1$.

7.3. Family of Fano schemes. — Let $\mathfrak{S} \to \mathbf{A}^1$ be the relative Fano scheme of lines associated with the family of q-bic threefolds $\mathfrak{X} \to \mathbf{A}^1$. The projective geometry constructions of §§2–3 work in families and, applied to the family of smooth cone situations in 7.1(ii), yields a commutative diagram

$$\begin{tikzcd} & \widetilde{\mathfrak{S}} \arrow[dl] \arrow[dr, "\mathrm{P}"] & \\ \mathfrak{S} \arrow[dr, "\Phi"'] & & \mathfrak{T} \arrow[dl, "\Pi"] \\ & C \times \mathbf{A}^1 & \end{tikzcd}$$

of morphisms of schemes over $\mathbf{A}^1$ where C is the smooth q-bic curve in $\mathbf{P}W$; $\mathfrak{T}$ is the degeneracy locus in $\mathbf{P} \times \mathbf{A}^1$ as in 2.6 and 2.10; and $\widetilde{\mathfrak{S}}$ is q-fold covering of $\mathfrak{T}$ as in 2.14. The key properties are: $\widetilde{\mathfrak{S}} \to \mathfrak{S}$ is a blowup along q^3+1 smooth sections of $\Phi \colon \mathfrak{S} \to C \times \mathbf{A}^1$ by 3.7; and $\mathrm{P} \colon \widetilde{\mathfrak{S}} \to \mathfrak{T}$ is a quotient by a unipotent group scheme of order q as in 3.5. In particular, this implies that there is an isomorphism

$$\mathbf{R}^1\Phi_*\mathcal{O}_{\mathfrak{S}} \cong \mathbf{R}^1\Pi_*\mathrm{P}_*\mathcal{O}_{\widetilde{\mathfrak{S}}}$$

of locally free $\mathcal{O}_{C\times\mathbf{A}^1}$-modules. The unipotent group quotient induces a q-step filtration on the right, and so this isomorphism puts a q-step filtration on $\mathbf{R}^1\Phi_*\mathcal{O}_{\mathfrak{S}}$, globalizing that from 3.10.

The family $\Phi \colon \mathfrak{S} \to C \times \mathbf{A}^1$ relates the cohomology of the singular $\varphi_0 \colon S_0 \to C$ and smooth $\varphi \colon S \to C$ fibres above $0 \in \mathbf{A}^1$ and $1 \in \mathbf{A}^1$, respectively, in a rather subtle way: On the one hand, recall that $\mathbf{R}^1\varphi_{0,*}\mathcal{O}_{S_0}$ is a graded $\mathcal{O}_C$-module which, as shown in 4.12, coincides with the positively graded parts of the $\mathcal{O}_C$-module $\mathcal{F}$ introduced in 4.11. Comparing 4.2 with 7.2 shows that this grading coincides with the grading induced by the $\mathbf{G}_m$ action on the family $\mathfrak{X}$. On the other hand, taking the fibre at 1 of the group scheme $\mathfrak{G}$ shows that $\varphi \colon S \to C$ is equivariant only for the finite étale group scheme $\boldsymbol{\mu}_{q^2-1}$, meaning that $\mathbf{R}^1\varphi_*\mathcal{O}_S$ admits only a weight decomposition by $\mathbf{Z}/(q^2-1)\mathbf{Z}$. That these fit into one family gives the following relation between the decompositions:

7.4. Proposition. — *The choice of $x_-, x_+ \in X$ as in 7.1 induces a canonical weight decomposition*

$$\mathbf{R}^1\varphi_*\mathcal{O}_S = \bigoplus\nolimits_{\alpha \in \mathbf{Z}/(q^2-1)\mathbf{Z}} (\mathbf{R}^1\varphi_*\mathcal{O}_S)_\alpha$$

into subbundles, each of which fits into a short exact sequence

$$0 \to \mathcal{F}_\alpha \to (\mathbf{R}^1\varphi_*\mathcal{O}_S)_\alpha \to \mathcal{F}_{\alpha+q^2-1} \to 0.$$

Proof. It remains to produce the short exact sequences. Since $\Phi \colon \mathfrak{S} \to C \times \mathbf{A}^1$ is equivariant for the action of $\mathbf{G}_m$ as described in 7.2, $\mathbf{R}^1\Phi_*\mathcal{O}_{\mathfrak{S}}$ is $\mathbf{G}_m$-equivariant. The Rees construction, as in [Sim91, Lemma 19], endows the unit fibre $\mathbf{R}^1\varphi_*\mathcal{O}_S$ with a filtration whose graded pieces are weight components of the central fibre $\mathbf{R}^1\varphi_{0,*}\mathcal{O}_{S_0}$ which successively increase by the weight q^2-1 of the $\mathbf{G}_m$ action on the base $\mathbf{A}^1$. Identifying $\mathbf{R}^1\varphi_{0,*}\mathcal{O}_{S_0}$ with the positively graded parts of $\mathcal{F}$ as in the proof of 4.12, and noting that the weights appearing in $\mathcal{F}_{>0}$ lie in $[1, 2q^2-q-2]$ by 4.10 and 4.14, it follows that this filtration has only two steps, and so reduces to a short exact sequence

$$0 \to \bigoplus\nolimits_{\alpha=1}^{q^2-1} \mathcal{F}_\alpha \to \mathbf{R}^1\varphi_*\mathcal{O}_S \to \bigoplus\nolimits_{\alpha=1}^{q^2-q-1} \mathcal{F}_{\alpha+q^2-1} \to 0.$$

Finally, since the action of the group scheme $\mathfrak{G}$ is equivariant for the $\mathbf{G}_m$ action, this sequence furthermore respects the $\mathbf{Z}/(q^2-1)\mathbf{Z}$ weight decomposition described above, yielding the short exact sequences in the statement. ■

For indices $\alpha = bq$ with $1 \leq b \leq q-2$, 5.1 identifies the quotient in this sequence as

$$\mathcal{F}_{q^2+bq-1} \cong \mathrm{Div}^{q-2-b}(W) \otimes \mathcal{O}_C.$$

When $q = p$, this makes it easy to show its corresponding exact sequence is not split:

7.5. Lemma. — *If $q = p$, then for each $1 \leq b \leq p-2$, the sequence*

$$0 \to \mathscr{F}_{bp} \to (\mathbf{R}^1\varphi_*\mathscr{O}_S)_{bp} \to \mathrm{Div}^{p-2-b}(W) \otimes \mathscr{O}_C \to 0$$

is not split and $\mathrm{H}^0(C, \mathscr{F}_{bp}) \cong \mathrm{H}^0(C, (\mathbf{R}^1\varphi_*\mathscr{O}_S)_{bp})$.

Proof. That the diagram in 7.3 is equivariant for the $\mathbf{G}_m$ action together with the isomorphism $\mathbf{R}^1\Phi_*\mathscr{O}_{\mathfrak{S}} \cong \mathbf{R}^1\Pi_*\mathrm{P}_*\mathscr{O}_{\tilde{\mathfrak{S}}}$ means that the sequences in 7.4 are compatible with the p-step filtrations from 3.10. Therefore projection to the top graded piece gives a commutative square

$$\begin{tikzcd}
(\mathbf{R}^1\varphi_*\mathscr{O}_S)_{bp} \arrow[r] \arrow[d] & \mathrm{Div}^{p-2-b}(W) \otimes \mathscr{O}_C \arrow[d] \\
\mathrm{gr}_{p-1}(\mathbf{R}^1\varphi_*\mathscr{O}_S)_{bp} \arrow[r] & \mathrm{Div}^{p-2-b}(\mathscr{T})
\end{tikzcd}$$

where the bottom right term is computed as in 6.10.

Suppose now that the sequence in the statement were split. Global sections would then lift on the $(p-1)$-st graded pieces. However, combined with the Borel–Weil–Bott computation in A.4, 7.6 below shows that the bottom left term has no sections:

$$\mathrm{H}^0(C, \mathrm{gr}_{p-1}(\mathbf{R}^1\varphi_*\mathscr{O}_S)_{bp}) = \mathrm{H}^0(C, \mathrm{Div}^{2q-2-b}(\mathscr{T}) \otimes \mathscr{O}_C(-1)) = 0.$$

This gives a contradiction since, either by Borel–Weil–Bott or simplicity of $\mathrm{Div}^{p-2-b}(W)$ as a $\mathrm{U}_3(p)$-representation, the right hand map is an isomorphism on global sections. Therefore the sequence is not split, and the map $\mathscr{F}_{bp} \to (\mathbf{R}^1\varphi_*\mathscr{O}_S)_{bp}$ is an isomorphism on global sections. ∎

It remains to determine the top graded piece of the bq weight component of $\varphi_*\mathscr{O}_S$ with respect to the filtration from 3.10:

7.6. Lemma. — $\mathrm{gr}_{q-1}(\mathbf{R}^1\varphi_*\mathscr{O}_S)_{bq} \cong \mathrm{Div}^{2q-2-b}(\mathscr{T}) \otimes \mathscr{O}_C(-1)$ *for each* $1 \leq b \leq q-2$.

Proof. Taking $i = q-1$ in 3.10 shows

$$\mathrm{gr}_{q-1}(\mathbf{R}^1\varphi_*\mathscr{O}_S) = \mathbf{R}^1\pi_*(\mathrm{gr}_{q-1}(\rho_*\mathscr{O}_{\tilde{S}})) = \mathbf{R}^1\pi_*(\mathscr{O}_T(1,-q) \otimes \pi^*\mathscr{O}_C(-1) \otimes L_-).$$

By 2.10, $\mathscr{O}_T(1,-q)$ is resolved by a complex $[\mathscr{E}_2' \to \mathscr{E}_1']$ of $\mathscr{O}_{\mathbf{P}}$-modules with

$$\begin{aligned}
\mathscr{E}_2' &= \mathscr{O}_{\mathbf{P}}(-q+1,-2q-1) \otimes \pi^*\mathscr{O}_C(-1) \oplus \mathscr{O}_{\mathbf{P}}(-q,-2q) \otimes L_+, \text{ and} \\
\mathscr{E}_1' &= \mathscr{O}_{\mathbf{P}}(0,-2q) \oplus \mathscr{O}_{\mathbf{P}}(-q+1,-q-1) \oplus \mathscr{O}_{\mathbf{P}}(-q+1,-2q) \otimes \pi^*\mathscr{O}_C(-1) \otimes L_+.
\end{aligned}$$

The resolution provides a spectral sequence computing $\mathbf{R}^1\pi_*\mathscr{O}_T(1,-q)$ with E_1 page given by

$$\begin{matrix}
E_1^{-2,3} \xrightarrow{d_1} E_1^{-1,3} \xrightarrow{d_1} E_1^{0,3} \\
\\
E_1^{-2,2} \xrightarrow{d_1} E_1^{-1,2} \xrightarrow{d_1} E_1^{0,2}
\end{matrix}
\quad = \quad
\begin{matrix}
\mathbf{R}^3\pi_*\mathscr{E}_2' \xrightarrow{\phi} \mathbf{R}^3\pi_*\mathscr{E}_1' & & 0 \\
\\
0 & & \mathbf{R}^2\pi_*\mathscr{E}_1' \xrightarrow{\wedge^2\phi^\vee} \mathbf{R}^2\pi_*\mathscr{O}_{\mathbf{P}}(1,-q)
\end{matrix}$$

and with all other terms vanishing. Since $\boldsymbol{\mu}_{q^2-1}$ acts through linear automorphisms of $\mathbf{P}$ over C, the differentials of the spectral sequence are compatible with the $\mathbf{Z}/(q^2-1)\mathbf{Z}$-gradings on each term.

Let $1 \leq b \leq q-2$ and consider the weight bq components of the spectral sequence: Recalling from 2.6 that $\mathbf{P} = \mathbf{P}\mathscr{V}_1 \times_C \mathbf{P}\mathscr{V}_2$ with $\mathscr{V}_1 \cong \mathscr{O}_C(-1) \oplus L_{-,C}$ and $\mathscr{V}_2 \cong \mathscr{T} \oplus L_{+,C}$, it follows that the relative dualizing sheaf of the projective bundle factors are

$$\omega_{\mathbf{P}\mathscr{V}_1/C} \cong \mathscr{O}_{\pi_1}(-2) \otimes \pi_1^*\mathscr{O}_C(1) \otimes L_-^\vee \text{ and } \omega_{\mathbf{P}\mathscr{V}_2/C} \cong \mathscr{O}_{\pi_2}(-3) \otimes \pi_2^*\mathscr{O}_C(-1) \otimes L_+^\vee,$$

and that $\omega_{\mathbf{P}/C} \cong \mathcal{O}_{\mathbf{P}}(-2,-3) \otimes L_-^\vee \otimes L_+^\vee$. Using this, a direct computation gives

$$\begin{aligned}
(\mathbf{R}^3\pi_*(\mathcal{E}_2' \otimes \pi^*\mathcal{O}_C(-1) \otimes L_-))_{bq} &\cong \mathrm{Div}^{2q-2-b}(\mathcal{T}) \otimes \mathcal{O}_C(-1) \otimes L_-^{\otimes q} \otimes L_+^{\otimes b+1}, \\
(\mathbf{R}^3\pi_*(\mathcal{E}_1' \otimes \pi^*\mathcal{O}_C(-1) \otimes L_-))_{bq} &\cong 0, \\
(\mathbf{R}^2\pi_*(\mathcal{E}_1' \otimes \pi^*\mathcal{O}_C(-1) \otimes L_-))_{bq} &\cong \mathrm{Div}^{q-2-b}(\mathcal{T}) \otimes L_- \otimes L_+^{\otimes q+b}, \text{ and} \\
(\mathbf{R}^2\pi_*(\mathcal{O}_{\mathbf{P}}(1,-q) \otimes \pi^*\mathcal{O}_C(-1) \otimes L_-))_{bq} &\cong \mathrm{Div}^{q-2-b}(\mathcal{T}) \otimes L_+^{\otimes b}.
\end{aligned}$$

The differential $\wedge^2\phi^\vee$ between the latter two sheaves is given by $u_1 v_{21}' + u_2 v_{22}'$, where u_1 and u_2 are as in 2.6, and v_{21}' and v_{22}' are the bottom components of v' from 2.9. The section v_{21}' contains $u_2'^q$, where u_2' is constructed in 2.8 as the coordinate function to the subbundle $\mathcal{T} \subset \mathcal{V}_2$. Since the divided powers of $\mathcal{T}$ appearing have exponent strictly less than q, multiplication by $u_2'^q$ is the zero map. A similar analysis then shows that the remaining component $u_2 v_{22}' = u_2^q \cdot \beta_2 \cdot u_1'$ acts via the isomorphism $\beta_2 \colon L_- \to L_+^{\vee,\otimes q}$. This implies that, at least on weight bq components, the spectral sequence degenerates on this page and that

$$\mathrm{gr}_{q-1}(\mathbf{R}^1\varphi_*\mathcal{O}_S)_{bq} \cong (\mathbf{R}^3\pi_*(\mathcal{E}_2' \otimes \pi^*\mathcal{O}_C(-1) \otimes L_-))_{bq} \cong \mathrm{Div}^{2q-2-b}(\mathcal{T}) \otimes \mathcal{O}_C(-1). \qquad \blacksquare$$

7.7. Proof of Theorem B. — Since $\mathrm{H}^1(S,\mathcal{O}_S)$ is canonically the Lie algebra of $\mathbf{Pic}_S$, and, for any prime $\ell \neq p$, $\mathrm{H}^1_{\text{ét}}(S,\mathbf{Z}_\ell)$ is the ℓ-adic Tate module of $\mathbf{Pic}_S$, there is always an inequality

$$\dim_{\mathbf{k}} \mathrm{H}^1(S,\mathcal{O}_S) \geq \dim \mathbf{Pic}_S = \frac{1}{2}\mathrm{rank}_{\mathbf{Z}_\ell} \mathrm{T}_\ell \mathbf{Pic}_S = \frac{1}{2}\dim_{\mathbf{Q}_\ell} \mathrm{H}^1_{\text{ét}}(S,\mathbf{Q}_\ell).$$

By the étale cohomology computation for S in [Che23b, **Theorem B**]—which may also be deduced from the Theorem recalled in the Introduction—the dimension of $\mathrm{H}^1(S,\mathcal{O}_S)$ is always at least $q(q-1)(q^2+1)/2$ with no assumption on q.

Assume $q = p$ is prime. The corresponding upper bound follows by semicontinuity of cohomology, see [Stacks, 0BDN], for the flat family $\Phi \colon \mathfrak{S} \to \mathbf{A}^1$ from 7.3, the cohomology computation for the singular surface S_0 from Theorem C, and the non-splitting result of 7.5. In more detail, and slightly more directly, the Leray spectral sequence for $\varphi \colon S \to C$ yields a short exact sequence

$$0 \to \mathrm{H}^1(C,\varphi_*\mathcal{O}_S) \to \mathrm{H}^1(S,\mathcal{O}_S) \to \mathrm{H}^0(C,\mathbf{R}^1\varphi_*\mathcal{O}_S) \to 0.$$

Since $\varphi_*\mathcal{O}_S = \mathcal{O}_C$ by 3.10, the first term has dimension $p(p-1)/2$. For the second term, consider the $\mathbf{Z}/(p^2-1)\mathbf{Z}$ weight decomposition from the action of $\boldsymbol{\mu}_{p^2-1}$. The short exact sequences in 7.4 yield, for each $\alpha = 1,2,\ldots,p^2-1$, inequalities

$$\dim_{\mathbf{k}} \mathrm{H}^0(C,(\mathbf{R}^1\varphi_*\mathcal{O}_S)_\alpha) \leq \dim_{\mathbf{k}} \mathrm{H}^0(C,\mathcal{F}_\alpha) + \dim_{\mathbf{k}} \mathrm{H}^0(C,\mathcal{F}_{\alpha+p^2-1}).$$

When $\alpha = bp$ with $1 \leq b \leq p-2$, 7.5 refines this to an equality

$$\dim_{\mathbf{k}} \mathrm{H}^0(C,(\mathbf{R}^1\varphi_*\mathcal{O}_S)_\alpha) = \dim_{\mathbf{k}} \mathrm{H}^0(C,\mathcal{F}_{bp}).$$

Summing these over α gives the inequality

$$\dim_{\mathbf{k}} \mathrm{H}^0(C,\mathbf{R}^1\varphi_*\mathcal{O}_S) \leq \dim_{\mathbf{k}} \mathrm{H}^0(C,\mathcal{F}) - \dim_{\mathbf{k}} \mathrm{H}^0(C,\mathcal{F}_0) - \sum\nolimits_{b=1}^{p-2} \dim_{\mathbf{k}} \mathrm{H}^0(C,\mathcal{F}_{p^2+bp-1}).$$

Consider the negative terms on the right. First, 5.1 shows that $\mathcal{F}_{p^2+bp-1} \cong \mathrm{Div}^{p-2-b}(W) \otimes \mathcal{O}_C$. Second, 4.12 implies that $\mathcal{F}_0$ is the cokernel of the map $\mathcal{O}_C \to \phi_{C,*}\mathcal{O}_C$ which, up to an automorphism, is the p^2-power Frobenius morphism. Since the p-power Frobenius already acts by zero on $\mathrm{H}^1(C,\mathcal{O}_C)$ by 1.2, the long exact sequence in cohomology shows

$$\mathrm{H}^0(C,\mathcal{F}_0) \cong \mathrm{H}^1(C,\mathcal{O}_C) \cong \mathrm{Div}^{p-2}(W).$$

Therefore the negative terms in the inequality sum up to

$$\begin{aligned}\dim_{\mathbf{k}} \mathrm{H}^0(C, \mathscr{F}_0) + \sum\nolimits_{b=1}^{p-2} \dim_{\mathbf{k}} \mathrm{H}^0(C, \mathscr{F}_{p^2+bp-1}) &= \sum\nolimits_{b=0}^{p-2} \dim_{\mathbf{k}} \mathrm{Div}^b(W) \\ &= \sum\nolimits_{b=0}^{p-2} \binom{b+2}{2} = \binom{p+1}{3} = \binom{p}{2} + \binom{p}{3}.\end{aligned}$$

Combining this with 6.11 then shows that

$$\dim_{\mathbf{k}} \mathrm{H}^0(C, \mathbf{R}^1\varphi_*\mathscr{O}_S) \leq (p^2+1)\binom{p}{2} + \binom{p}{3} - \binom{p}{2} - \binom{p}{3} = p^2\binom{p}{2}.$$

The short exact sequence for $\mathrm{H}^1(S, \mathscr{O}_S)$ then gives

$$\dim_{\mathbf{k}} \mathrm{H}^1(S, \mathscr{O}_S) \leq \binom{p}{2} + p^2\binom{p}{2} = \frac{1}{2}p(p-1)(p^2+1),$$

completing the proof. ■

APPENDIX A. REPRESENTATION THEORY COMPUTATIONS

This appendix collects some facts and computations pertaining to the modular representation theory of the algebraic group $\mathbf{SL}_3$ and the finite unitary group $\mathrm{SU}_3(q)$, acting linearly via automorphisms on a 3-dimensional $\mathbf{k}$-vector space W.

A.1. Root data. — Choose a maximal torus and Borel subgroup $\mathbf{T} \subset \mathbf{B} \subset \mathbf{SL}_3$, and let

$$\begin{aligned}\mathrm{X}(\mathbf{T}) &:= \mathrm{Hom}(\mathbf{T}, \mathbf{G}_m) \cong \mathbf{Z}\{\epsilon_1, \epsilon_2, \epsilon_3\}/(\epsilon_1+\epsilon_2+\epsilon_3) \\ \mathrm{X}^\vee(\mathbf{T}) &:= \mathrm{Hom}(\mathbf{G}_m, \mathbf{T}) \cong \left\{ a_1\epsilon_1^\vee + a_2\epsilon_2^\vee + a_3\epsilon_3^\vee \in \mathbf{Z}\{\epsilon_1^\vee, \epsilon_2^\vee, \epsilon_3^\vee\} : a_1+a_2+a_3 = 0 \right\}\end{aligned}$$

be the lattices of characters and cocharacters of $\mathbf{T}$; here, upon conjugating $\mathbf{T}$ to the diagonal matrices in $\mathbf{SL}_3$, the characters ϵ_i extract the i-th diagonal entry, whereas the cocharacters $\epsilon_i^\vee$ include into the i-th diagonal entry. Let $\langle -,- \rangle : \mathrm{X}(\mathbf{T}) \times \mathrm{X}^\vee(\mathbf{T}) \to \mathrm{Hom}(\mathbf{G}_m, \mathbf{G}_m) \cong \mathbf{Z}$ be the natural root pairing, so that $\langle \epsilon_i, \epsilon_j^\vee \rangle = \delta_{ij}$. The simple roots, simple coroots, and positive roots corresponding to $\mathbf{B}$ are

$$\alpha_1 := \epsilon_1 - \epsilon_2,\ \ \alpha_2 := \epsilon_2 - \epsilon_3,\quad \alpha_1^\vee := \epsilon_1^\vee - \epsilon_2^\vee,\ \ \alpha_2^\vee := \epsilon_2^\vee - \epsilon_3^\vee,\quad \Phi^+ := \{\alpha_1, \alpha_2, \alpha_1+\alpha_2\}.$$

The fundamental weights are then $\varpi_1 := \epsilon_1$ and $\varpi_2 := \epsilon_1 + \epsilon_2$; the half sum of all the positive roots is then $\rho = \varpi_1 + \varpi_2$; and the set of dominant weights is

$$\mathrm{X}_+(\mathbf{T}) = \left\{ a\varpi_1 + b\varpi_2 \in \mathrm{X}(\mathbf{T}) : a, b \in \mathbf{Z}_{\geq 0} \right\}.$$

Highest weight theory puts the set of simple representations of $\mathbf{SL}_3$ in bijection with the set of dominant weights $\mathrm{X}_+(\mathbf{T})$; write $L(a,b)$ for the simple module with highest weight $a\varpi_1 + b\varpi_2$.

A.2. Flag varieties. — Let $\mathbf{Flag}\,W \cong \mathbf{SL}_3/\mathbf{B}$ be the full flag variety of W. This is the $(1,1)$-divisor in $\mathbf{P}W \times \mathbf{P}W^\vee$ cut out by the trace section, and so

$$\mathrm{Pic}(\mathbf{Flag}\,W) = \left\{ \mathscr{O}_{\mathbf{Flag}\,W}(a,b) := \mathscr{O}_{\mathbf{P}W}(a) \boxtimes \mathscr{O}_{\mathbf{P}W^\vee}(b)|_{\mathbf{Flag}\,W} : a, b \in \mathbf{Z} \right\}.$$

The map $a\varpi_1 + b\varpi_2 \mapsto \mathscr{O}_{\mathbf{Flag}\,W}(a,b)$ gives an isomorphism $\mathrm{X}(\mathbf{T}) \to \mathrm{Pic}(\mathbf{Flag}\,W)$ of abelian groups. Following the conventions of [Jan03, II.2.13(1)], the *Weyl module* corresponding to a dominant weight $a\varpi_1 + b\varpi_2 \in \mathrm{X}_+(\mathbf{T})$ is

$$\Delta(a,b) := \mathrm{H}^0(\mathbf{Flag}\,W, \mathscr{O}_{\mathbf{Flag}\,W}(b,a))^\vee \cong \mathrm{H}^0(\mathbf{P}W, \mathrm{Sym}^a(\mathscr{T}_{\mathbf{P}W}(-1)) \otimes \mathscr{O}_{\mathbf{P}W}(b))^\vee$$

where the isomorphism follows from the simple computation that, for $a, b \in \mathbf{Z}$,

$$\mathrm{pr}_{\mathbf{P}W,*}(\mathscr{O}_{\mathbf{Flag}\,W}(a,b)) = \begin{cases} \mathrm{Sym}^b(\mathscr{T}_{\mathbf{P}W}(-1)) \otimes \mathscr{O}_{\mathbf{P}W}(a) & \text{if } b \geq 0, \\ 0 & \text{if } b < 0. \end{cases}$$

For example, $\Delta(a,0) = \mathrm{Sym}^a(W)^\vee$ and $\Delta(0,b) = \mathrm{Div}^b(W) \cong \mathrm{Sym}^b(W^\vee)^\vee$ are the spaces of a-th symmetric and b-th divided powers, respectively.

Cohomology of line bundles on $\mathbf{Flag}\, W$ is classically determined via the Borel–Weil–Bott Theorem of [Bot57, Dem68]. This, however, is rather subtle in positive characteristic, see [Jan03, II.5.5]. In general, Kempf's Theorem [Kem76, Theorem 1 on p.586] shows that higher cohomology always vanishes when the corresponding weight λ is dominant. In the present case of $\mathbf{SL}_3$, Griffith gave a complete answer in [Gri80, Theorem 1.3]. Using this, a straightforward computation gives:

A.3. Lemma. — *Let $0 \leq b \leq p-1$. Then*

(i) $\mathrm{H}^0(\mathbf{P}^2, \mathrm{Sym}^b(\mathscr{T}_{\mathbf{P}^2}(-1))(a)) = 0$ *whenever $a < 0$, and*

(ii) $\mathrm{H}^1(\mathbf{P}^2, \mathrm{Sym}^b(\mathscr{T}_{\mathbf{P}^2}(-1))(a)) = 0$ *whenever $a < p$.* ■

This leads to the following computation for smooth plane curves $C \subset \mathbf{P}^2$ of degree $p+1$:

A.4. Corollary. — *For integers $0 \leq b \leq p-1$ and $a \leq 0$,*

$$\mathrm{H}^0(C, \mathrm{Sym}^b(\mathscr{T}_{\mathbf{P}W}(-1))(a)|_C) = \begin{cases} \mathrm{Sym}^b(W) & \text{if } a = 0, \text{ and} \\ 0 & \text{if } a < 0. \end{cases}$$

Proof. The restriction sequence

$$0 \to \mathrm{Sym}^b(\mathscr{T}_{\mathbf{P}W}(-1))(a-p-1) \to \mathrm{Sym}^b(\mathscr{T}_{\mathbf{P}W}(-1))(a) \to \mathrm{Sym}^b(\mathscr{T}_{\mathbf{P}W}(-1))(a)|_C \to 0$$

implies it suffices to show that $\mathrm{H}^0(\mathbf{P}W, \mathrm{Sym}^b(\mathscr{T}_{\mathbf{P}W}(-1))) = \mathrm{Sym}^b(W)$ and

$$\mathrm{H}^0(\mathbf{P}W, \mathrm{Sym}^b(\mathscr{T}_{\mathbf{P}W}(-1))(a)) = \mathrm{H}^1(\mathbf{P}W, \mathrm{Sym}^b(\mathscr{T}_{\mathbf{P}W}(-1))(a-p)) = 0 \quad \text{when } a < 0.$$

The identification of global sections follows the Euler sequence; since $0 \leq b \leq p-1$, the vanishing follows from the Borel–Weil–Bott Theorem *á la* Griffith, see A.3. ■

A.5. Jantzen filtration and sum formula. — The Weyl modules $\Delta(\lambda)$ from A.2 are generally not irreducible in positive characteristic. Their simple composition factors can sometimes be described using Jantzen's filtration and sum formula, as described in [Jan03, II.8.19]. In the situation at hand, this means the following: given a dominant weight $\lambda \in \mathrm{X}_+(\mathbf{T})$, there is a decreasing filtration

$$\Delta(\lambda) = \Delta(\lambda)^0 \supseteq \Delta(\lambda)^1 \supseteq \Delta(\lambda)^2 \supseteq \cdots$$

such that $L(\lambda) = \Delta(\lambda)/\Delta(\lambda)^1$. Furthermore, there is the *sum formula*:

$$\sum\nolimits_{i>0} \mathrm{ch}(\Delta(\lambda)^i) = \sum\nolimits_{\alpha \in \Phi^+} \sum\nolimits_{m : 0 < mp < \langle \lambda + \rho, \alpha^\vee \rangle} \nu_p(mp) \chi(s_{\alpha,mp} \cdot \lambda)$$

where ch extracts the $\mathbf{T}$-character of a module, $\nu_p \colon \mathbf{Z} \to \mathbf{Z}$ is the p-adic valuation, $s_{\alpha,mp}$ is the affine reflection $\lambda \mapsto \lambda + (mp - \langle \lambda, \alpha^\vee \rangle)\alpha$ on $\mathrm{X}(\mathbf{T})$, $s_{\alpha,mp} \cdot \lambda := s_{\alpha,mp}(\lambda + \rho) - \rho$ is the dot action, and

$$\chi(\lambda) := \sum\nolimits_{i \geq 0} (-1)^i [\mathrm{H}^i(\mathbf{Flag}\, W, \mathscr{O}_{\mathbf{Flag}\, W}(\lambda))]$$

is the Euler characteristic of the line bundle corresponding to λ with values in the representation ring of $\mathbf{T}$. As a simple application, consider a weight $\lambda = a\varpi_1 + b\varpi_2$ in which all the root pairings

$$\langle \lambda + \rho, \alpha_1^\vee \rangle = a+1, \quad \langle \lambda + \rho, \alpha_2^\vee \rangle = b+1, \quad \langle \lambda + \rho, \alpha_1^\vee + \alpha_2^\vee \rangle = a+b+2$$

are at most p. Then the right hand side of the sum formula is empty, implying the following:

A.6. Lemma. — *If $a, b \in \mathbf{Z}_{\geq 0}$ satisfy $a + b \leq p-2$, then $\Delta(a,b)$ is simple.* ■

The next two statements describe the structure of the Weyl modules of weights $b\varpi_2$ and $\varpi_1 + b\varpi_2$:

A.7. Lemma. — *The Weyl module $\Delta(0,b)=\mathrm{Div}^b(V)$ of highest weight $b\varpi_2$ satisfies:*

(i) *If $0\leq b\leq p-1$, then $L(0,b)=\Delta(0,b)$ is simple.*
(ii) *If $p\leq b\leq 2p-3$, then $L(0,b)=W^{[1]}\otimes\mathrm{Div}^{b-p}(W)$ and there is a short exact sequence*

$$0\to L(b-p+1,2p-2-b)\to\Delta(0,b)\to L(0,b)\to 0.$$

Proof. That $\Delta(0,b)$ is simple when $0\leq b\leq p-2$ follows from A.6. When $b\geq p-1$, the following term appears in the sum formula:

$$\begin{aligned}\chi(s_{\alpha_1+\alpha_2,p}\cdot b\varpi_2)&=\chi((p-b-2)\varpi_1+(p-2)\varpi_2)\\&=\chi(s_{\alpha_1}\cdot((b-p)\varpi_1+(2p-3-b)\varpi_2))=-\chi((b-p)\varpi_1+(2p-3-b)\varpi_2)\end{aligned}$$

with the final inequality is due to [Jan03, II.5.9]. This, in particular, vanishes when $b=p-1$ because the line bundle $\mathcal{O}_{\mathbf{Flag}\,W}(-1,a)$ never has cohomology, giving (i).

If $p\leq b\leq 2p-3$, the identification of $L(0,b)$ follows from (i) together with the Steinberg Tensor Product Theorem, [Jan03, II.3.17]. For the short exact sequence, note that the sum formula has two terms, indexed by (α_2,p) and $(\alpha_1+\alpha_2,p)$. The latter is as above, and the former is given by

$$\begin{aligned}\chi(s_{\alpha_2,p}\cdot b\varpi_2)=\chi((b-p+1)\varpi_1+(2p-2-b)\varpi_2)&=\mathrm{ch}(\Delta(b-p+1,2p-2-b))\\&=\mathrm{ch}(L(b-p+1,2p-2-b))+\mathrm{ch}(L(b-p,2p-3-b)).\end{aligned}$$

where the final equality arises from the sum formula applied to this Weyl module. Putting this together with the calculation above shows that

$$\sum\nolimits_{i>0}\mathrm{ch}(\Delta(0,b)^i)=\chi(s_{\alpha_2,p}\cdot b\varpi_2)+\chi(s_{\alpha_1+\alpha_2,p}\cdot b\varpi_2)=\mathrm{ch}(L(b-p+1,2p-2-b))$$

which means $\Delta(0,b)^1=L(b-p+1,2p-2-b)$, whence the exact sequence in (ii). ■

A.8. Lemma. — *The Weyl module $\Delta(1,b)\cong\ker(\mathrm{ev}\colon W^\vee\otimes\mathrm{Div}^b(W)\to\mathrm{Div}^{b-1}(W))$ is simple if $0\leq b\leq p-3$, and, if $b=p-2$, it fits into a short exact sequence*

$$0\to L(0,p-3)\to\Delta(1,p-2)\to L(1,p-2)\to 0.$$

Proof. Simplicity when $0\leq b\leq p-3$ follows from A.6. When $b=p-2$, the sum formula reads

$$\sum\nolimits_{i>0}\mathrm{ch}(\Delta(1,p-2)^i)=\chi(s_{\alpha_1+\alpha_2,p}\cdot(\varpi_1+(p-2)\varpi_2))=\mathrm{ch}(L(0,p-3)).$$

Thus $L(0,p-3)$ is the only composition factor in $\Delta(1,p-2)^1$, giving the result. ■

Let $\mathrm{SU}_3(p)$ be the étale subgroup scheme of $\mathbf{SL}_3$ which preserves a nondegenerate q-bic form (W,β). Steinberg's Restriction Theorem [Ste63], see also [Hum06, Theorem 2.11], implies that the irreducible representations of $\mathrm{SU}_3(p)$ arise via restriction from $\mathbf{SL}_3$. Namely:

A.9. Theorem. — *Let $0\leq a,b\leq p-1$. The restriction of the $\mathbf{SL}_3$-modules $L(a,b)$ to $\mathrm{SU}_3(p)$ remain simple, are pairwise nonisomorphic, and give all isomorphism classes of simple $\mathrm{SU}_3(p)$-modules.* ■

Abusing notation, write $L(a,b)$ and $\Delta(a,b)$ for the $\mathrm{SU}_3(p)$-modules obtained via restriction of the corresponding $\mathbf{SL}_3$-modules. The next statement gives an alternate construction of the Weyl module $\Delta(1,b)$ from A.8 as a $\mathrm{SU}_3(p)$-module:

A.10. Lemma. — *For each $0\leq b\leq p-1$, there is an isomorphism of $\mathrm{SU}_3(p)$-modules:*

$$\Delta(1,b)\cong\ker(f:W^{[1]}\otimes\mathrm{Div}^b(W)\xrightarrow{\beta}W^\vee\otimes\mathrm{Div}^b(W)\xrightarrow{\mathrm{ev}}\mathrm{Div}^{b-1}(W)).$$

Proof. By construction, there is a $\mathrm{SU}_3(p)$-equivariant commutative diagram

$$\begin{tikzcd} W^{[1]} \otimes \mathrm{Div}^b(W) \arrow[r, "f"'] \arrow[d, "\beta^\vee"'] & \mathrm{Div}^{b-1}(W) \arrow[d, equal] \\ V^\vee \otimes \mathrm{Div}^b(W) \arrow[r, "\mathrm{ev}"] & \mathrm{Div}^{b-1}(W). \end{tikzcd}$$

Thus the kernels of the two rows are isomorphic as representations of $\mathrm{SU}_3(p)$. Since the bottom kernel is $\Delta(1,b)$ from **A.8**, the result follows. ■

Let $C \subset \mathbf{P}^2$ be the smooth q-bic curve associated with the nonsingular q-bic form (W,β). Then $\mathrm{SU}_3(p)$ acts through linear automorphisms on C, and so acts on various cohomology groups of C. The following identifies a few of these representations that are particularly useful in §**6**:

A.11. Lemma. — *The* $\mathrm{SU}_3(p)$ *representations* $\mathrm{H}^1(C,\mathcal{O}_C(-i))$ *for* $0 \le i \le p$ *are:*

(i) *If* $0 \le i \le 1$*, then* $\mathrm{H}^1(C,\mathcal{O}_C(-i)) \cong \mathrm{Div}^{p+i-2}(W)$ *is simple.*

(ii) *If* $2 \le i \le p$*, then there is a short exact sequence*

$$0 \to \mathrm{Div}^{p+i-2}_{\mathrm{red}}(W) \to \mathrm{H}^1(C,\mathcal{O}_C(-i)) \to \Delta(1,i-2) \to 0.$$

(iii) *If* $i \neq p$*, then the quotient* $\Delta(1,i-2) = L(1,i-2)$ *is simple.*

(iv) *If* $i = p$*, then* $\Delta(1,p-2) \cong \mathrm{H}^0(C,\Omega^1_{\mathbf{P}W}(1)|_C)$ *and a short exact sequence*

$$0 \to \mathrm{Div}^{p-3}(W) \to \mathrm{H}^0(C,\Omega^1_{\mathbf{P}W}(1)|_C) \to L(1,p-2) \to 0.$$

Proof. Cohomology of the ideal sheaf sequence of C in $\mathbf{P}W$ twisted by $\mathcal{O}_{\mathbf{P}W}(-i)$ shows that

$$\mathrm{H}^1(C,\mathcal{O}_C(-i)) \cong \ker(\cdot f : \mathrm{Div}^{p+i-2}(W) \to \mathrm{Div}^{i-3}(W)),$$

with which (i) then follows from **A.7**(i). For $2 \le i \le p$, consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Div}^{i+p-2}_{\mathrm{red}}(W) \arrow[r] & \mathrm{Div}^{i+p-2}(W) \arrow[r] \arrow[d, "f"'] & W^{[1]} \otimes \mathrm{Div}^{i-2}(W) \arrow[r] \arrow[d, "f"] & 0 \\ & & \mathrm{Div}^{i-3}(W) \arrow[r, equal] & \mathrm{Div}^{i-3}(W) & \end{tikzcd}$$

in which the top row is exact and vertical maps surjective. Since, by **A.10**,

$$\Delta(1,i-2) \cong \ker(\cdot f : W^{[1]} \otimes \mathrm{Div}^{i-2}(W) \to \mathrm{Div}^{i-3}(W)),$$

taking kernels of the vertical maps thus yields the exact sequence of (ii). When $2 \le i \le p-1$, the first statement of **A.8** shows that $\Delta(1,i-2) = L(1,i-2)$ is simple, proving (iii). When $i = p$, the Euler sequence yields an isomorphism

$$\mathrm{H}^1(C,\Omega^1_{\mathbf{P}W}(1)|_C) \cong \ker(W^\vee \otimes \mathrm{H}^1(C,\mathcal{O}_C) \to \mathrm{H}^1(C,\mathcal{O}_C(-1))) \cong \Delta(1,p-2),$$

upon which the second statement of **A.8** yields the exact sequence of (iv). ■

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