PROFILES, LINEAR SPACES, AND UNIRATIONALITY OF COMPLETE INTERSECTIONS

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ABSTRACT. Complete intersections may be unexpectedly simple over fields of positive characteristic: for instance, they may be unirational despite being of general type. One explanation is given by *profiles*, structure that tracks the special shape of polynomials, refining the degree. The aim of this work is to show that complete intersections with small profile should be considered simple by generalizing two classical results on low degree complete intersections: First, the basic geometry of Fano schemes associated with complete intersections depends only on the profile, so that complete intersections with small profile contain many linear spaces. Second, a general complete intersection is unirational once its dimension is sufficiently large compared to its profile.

INTRODUCTION

High degree hypersurfaces and complete intersections in projective space are, by most measures, geometrically complicated. For instance, unlike the simplest of algebraic varieties [Kol01], high degree hypersurfaces over the complex numbers contain very few—if any!—rational subvarieties: see [Ein88, Voi96, Pac04, RY20]. New phenomena appear, however, in positive characteristic $p > 0$, complicating the correlation between degree and geometric complexity. The Fermat hypersurface

$$X := \{x_0^{q+1} + \cdots + x_n^{q+1} = 0\} \subset \mathbf{P}^n$$

of degree $q+1$, with $q = p^e$ an integer power, is an exemplar: Many properties of X are reminiscent of those typically expected in quadric hypersurfaces: it is projectively self-dual and its Gauss map is a homeomorphism [Wal56, KP91]; its smooth hyperplane sections have constant moduli [Bea90]; it has many linear spaces [Che25b]; and, if $n \geq 3$, it is even unirational [Shi74]! Traditionally, these curiosities are explained *ad hoc* by identifying the equation of X with a Hermitian form for the quadratic extension $\mathbf{F}_{q^2} \mid \mathbf{F}_q$, as in [BC66, Hef85, Shi01].

In contrast, I view the quadratic nature of X as the confluence between the almost linear nature of q-powers in characteristic p and the special form of the equation of X. This perspective places this Fermat hypersurface, a historically isolated example, in a broad context and suggests a systematic way of identifying related phenomena. The approach taken here is as follows:

A *profile* with respect to q is an integer polynomial $a(t) = a_0 + a_1 t + \cdots + a_m t^m \in \mathbf{Z}_{\geq 0}[t]$ subject to a technical condition, given in 1.1, which is satisfied, for example, if $a_j \leq q-1$ for each $j = 0, \ldots, m$. A polynomial with coefficients a field $\mathbf{k}$ of characteristic p is said to have profile $a(t)$ with respect to q—or simply is called a $(q;a)$-*tic polynomial*—if it is of the form

$$f(x_0, \ldots, x_n) = \sum\nolimits_{i \in I} \prod\nolimits_{j=0}^{m} f_{i,j}(x_0, \ldots, x_n)^{q^j} \in \mathbf{k}[x_0, \ldots, x_n]$$

where each $f_{i,j}$ is a homogeneous polynomial of degree a_j. The form of this expression is invariant under linear changes of variables, so $(q;a)$-tic polynomials span a canonical linear subspace within the space of polynomials of degree $a(q) = a_0 + a_1 q + \cdots + a_m q^m$. The vanishing locus in $\mathbf{P}^n$ of a $(q;a)$-tic polynomial is a $(q;a)$-*tic hypersurface*. The Fermat hypersurface above is an example with profile $a(t) = 1 + t$ with respect to q. The purpose of this works is to develop the following principle: *$(q;a)$-tic hypersurfaces behave as if they were hypersurfaces of degree $a(1+\varepsilon) \approx a_0 + a_1 + \cdots + a_m$ for some small real number $\varepsilon > 0$.*

One aspect of this principle is that hypersurfaces sharing the same profile should be treated as a single family. Concretely, one looks for properties of $(q;a)$-tic hypersurfaces which may be expressed in terms of the profile a, ideally independently of, or at least uniformly in, the prime power q. A simple illustration of this is found in the geometry of linear spaces:

Theorem A. — *Let $a = \sum_{j\geq 0} a_j t^j$ be a profile such that $a \neq 2t^m$. The Fano scheme $\mathbf{F}_r(X)$ of r-planes in a $(q;a)$-tic hypersurface $X \subset \mathbf{P}^n$ is cut out of the Grassmannian by a section of a vector bundle of rank*

$$\delta(n,a,r) := (r+1)(n-r) - \prod\nolimits_{j\geq 0} \binom{a_j + r}{r}.$$

If $\delta(n,a,r) < 0$, then $\mathbf{F}_r(X)$ is empty for X general. If $\delta(n,a,r) \geq 0$, then $\mathbf{F}_r(X)$ is

(i) *nonempty for every X;*
(ii) *irreducible of dimension $\delta(n,a,r)$ for X general;*
(iii) *smooth of dimension $\delta(n,a,r)$ for X general when the profile has constant term $a_0 \neq 0$; and*
(iv) *connected for every X when $\delta(n,a,r) > 0$.*

If $\mathbf{F}_r(X)$ is of dimension $\delta(n,a,r)$, then its dualizing sheaf is given by a power of the Plücker line bundle:

$$\omega_{\mathbf{F}_r(X)} \cong \mathcal{O}_{\mathbf{F}_r(X)}(\gamma(a,r,q) - n - 1) \text{ where } \gamma(a,r,q) := \frac{a(q)}{r+1} \cdot \prod\nolimits_{j\geq 0} \binom{a_j + r}{r}.$$

This is a summary of 3.1, 3.2, and 3.12 in the case of a hypersurface, all of which are formulated and proven without the restriction $a \neq 2t^m$ and more generally for $(q;\mathbf{a})$-tic complete intersections, vanishing loci of a regular sequence of $(q;a)$-tic polynomials with a ranging over a multi-set of profiles $\mathbf{a}$. These generalize classical results on Fano schemes of complete intersections and, once the techniques developed for handling $(q;a)$-tic equations are developed in §§1–2, their proofs essentially follow the strategy from the classical case found in [DM98, §2] and [Kol96, §V.4].

A second aspect of this principle is, of course, that $(q;a)$-tic hypersurfaces whose profile is small compared to its dimension should be geometrically simple, regardless of the prime power q. Theorem A is some evidence in this direction since the dimension of the Fano scheme depends only on a and not on q. To explain a second result in this direction, recall classical results of Morin and Predonzan from [Mor42, Pre49] which show that a general complete intersection in $\mathbf{P}^n$ is unirational whenever its total degree d is much smaller than n; see [Rot55, pp.44–46] for a classical source, but also [PS92] for a succinct exposition in modern language and [Ram90] for an improved bound. An analogue of this for $(q;\mathbf{a})$-tic complete intersections is:

Theorem B. — *Given a multi-profile $\mathbf{a}$, there exists an integer $n_0 := n_0(\mathbf{a})$, depending only on $\mathbf{a}$, such that for all $n \geq n_0$, a general $(q;\mathbf{a})$-tic complete intersection in $\mathbf{P}^n$ is unirational.*

This is proven in §7 via an intricate inductive argument, using the constructions developed in §§4–6. The integer $n_0(\mathbf{a})$ can be computed for small $\mathbf{a}$: see 7.8. A similar result, formulated and proven in a different language, seems to appear in [Shi95]; see also the related result in [Shi92].

Briefly, the unirationality construction of Morin and Predonzan goes as follows: given a general complete intersection $X \subset \mathbf{P}^n$ of multi-degree $\mathbf{d} = (d_1, \ldots, d_c)$, projection away from a general r-plane in X yields a fibration $\widetilde{X} \to \mathbf{P}^{n-r-1}$ whose generic fibre X' is itself a complete intersection of multi-degree $\mathbf{d}' = (d_1 - 1, \ldots, d_c - 1)$ in a $\mathbf{P}^{r+1}$. With an appropriate choice of r and n depending on $\mathbf{d}$, it is possible to find a base change of X' that carries a large linear space, allowing the argument to proceed inductively. This strategy does not adapt in a straightforward manner for $(q;\mathbf{a})$-tic complete intersections because the generic fibre of the projection $\widetilde{X} \to \mathbf{P}^{n-r-1}$ does not seem to have structure that is captured by the theory of profiles: see 4.5 for an explicit example.

Instead, Theorem B is established with a generalization of an old unirationality construction for cubic hypersurfaces found in [CG72, Appendix B] and [Mur72, §2] which is based on the following observation: Given a hypersurface $X \subset \mathbf{P}^n$ of degree d, the space of *penultimate tangents*

$$X' = \{(x,[\ell]) : \ell \subset \mathbf{P}^n \text{ a line intersecting } X \text{ at } x \text{ with multiplicity} \geq d-1\}$$

is generically a family of complete intersections of multi-degree $\mathbf{d}' = (d-2, d-3, \ldots, 1)$ over X. Moreover, for X general, there is a dominant rational map $X' \dashrightarrow X$ sending $(x,[\ell])$ to the residual point of intersection $x' = X \cap \ell - (d-1)x$. By restricting X' to a sufficiently large and general linear space in X, unirationality can be established by induction on a poset consisting of all multi-degrees. In particular, it is essential to prove the result for all complete intersections.

Much effort is made to perform the constructions in §§4–7 globally, so that they work well in families, albeit introducing additional technicalities. The hope is to eventually make the generality conditions on X in Theorems A and B effective n sufficiently large. In characteristic 0, the works [HMP98, BR21] show that the Fano schemes $\mathbf{F}_r(X)$ of *every* smooth hypersurface $X \subset \mathbf{P}^n$ of degree d is irreducible of its expected dimension once $n \geq 2\binom{d+r-1}{r} + r$. This may then be used in a refinement of the Morin construction to show that *every* smooth hypersurface of degree d is unirational once $n \geq 2^{d!}$. While it is possible that these results may be extended to characteristics $p > d$, Theorem A shows that these statements cannot hold verbatim when $p \leq d$ wherein there exists a nontrivial profile a with $a(p) = d$: see 1.3(i). An optimistic guess would be that the appearance of these canonical sub-linear systems is the only issue which arises, suggesting the following:

Conjecture C. — *Let $a(t) = a_0 + \cdots + a_m t^m \in \mathbf{Z}_{\geq 0}[t]$ with $a_0 \neq 0$ and $r \in \mathbf{Z}_{\geq 1}$. Then there exists an integer $n_0(a,r)$ such that for every prime $p \geq \max_i\{a_i\}$, every $n \geq n_0(a,r)$, and every smooth $(p;a)$-tic hypersurface $X \subset \mathbf{P}^n$, the Fano scheme $\mathbf{F}_r(X)$ is irreducible of dimension $\delta(n,a,r)$.*

Note that $n_0(a,r)$ does not depend on p, so that Conjecture C predicts a uniform phenomenon for varying characteristic. Unfortunately, this statement cannot be quite right for higher prime powers q, since there will be a profile with respect to p with larger expected dimension. A result in this direction would be a first step toward an effective statement for unirationality.

Further questions. — This work is primarily focussed on the linear projective geometry of $(q;a)$-tic hypersurfaces, and so many basic questions and properties remain to be explored: automorphisms, moduli, singularities, and so forth. Three main directions that seem the most interesting are:

First, the principle that $(q;a)$-tic hypersurfaces behave as if they were of degree $a(1+\varepsilon)$ is taken to be a rough qualitative heuristic, and it would be interesting to make this more precise and quantitative. For instance, the work in [Che25b, Che24] shows that the geometry of lines in hypersurfaces of profile $a(t) = 1 + t$, or *q-bic hypersurfaces*, is reminiscent of that of cubics.

Second, the theory developed here is *extrinsic* in that $(q;a)$-tic-ness is a structure with which an object may be equipped with. Might there be an intrinsic characterization? One instance occurs with q-bic hypersurfaces, wherein [KKP+22] show that they are characterized amongst hypersurfaces of the same degree as those having the smallest F-pure threshold. From a different direction, many examples suggest that non-F-splitness is related to the special form of the defining equations: see [Sai17, BLRT23, KKP+21, MW25] for a few examples.

Third, the definition in §2 of $(q;a)$-tic polynomials generalize to provide canonical linear systems in $\Gamma(Y, \mathscr{L}^{\otimes a(q)})$ for any scheme Y and $\mathscr{L} \in \operatorname{Pic} Y$. The situation here is the case $(Y,\mathscr{L}) = (\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))$, and other cases that have occurred in the literature include Deligne–Lusztig varieties [DL76] and the Frobenius incidence correspondence of [Shi12]. It would be interesting to study the special properties of the divisors and linear systems constructed in this fashion.

Outline. — §1 develops the combinatorics of profiles and §2 gives a definition of $(q;a)$-tic hypersurfaces and studies their basic properties. Fano schemes of $(q;\mathbf{a})$-tic complete intersections are studied in §3. Families of $(q;\mathbf{a})$-tic complete intersections are defined in §4. The penultimate tangent construction for families is made in §5 and the corresponding residual point map is studied in §6. Finally, the unirationality result is proven in §7.

Notation. — Throughout, $\mathbf{k}$ denotes an algebraically closed field of characteristic $p > 0$ and $q := p^e$ is a positive integer power of p. Unless otherwise stated, V denotes a $\mathbf{k}$-vector space of dimension $n+1$. Write $\mathrm{Fr}\colon \mathbf{k} \to \mathbf{k}$ for the q-power Frobenius morphism and, for any $\mathbf{k}$-vector space W, let $W^{[1]} := \mathbf{k} \otimes_{\mathrm{Fr},\mathbf{k}} W$ be its Frobenius twist. Set $W^{[0]} := W$ and, for each integer $i \geq 1$, inductively define $W^{[i+1]} := (W^{[i]})^{[1]}$. Schemes are all taken to be over $\mathbf{k}$ and $\mathbf{P}V$ is the projective space of lines in V.

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1. PROFILES

A profile is a discrete invariant associated with a polynomial in positive characteristic $p > 0$ which records its special shape with respect to a positive integer power $q = p^e$ of the characteristic. Profiles are subject to a technical condition which ensures a certain uniqueness in writing polynomials in their distinguished form: see 1.5. While this hypothesis is not strictly necessary at this point, it often becomes important in applications. This section defines profiles and studies their combinatorics, especially various order relations amongst profiles.

1.1. Profiles. — Given an integer polynomial $a(t) := a_0 + a_1 t + \cdots + a_m t^m$ with nonnegative coefficients, consider the tensor functor defined by

$$\mathrm{S}^a(V^\vee) := \bigotimes\nolimits_{j=0}^m \mathrm{Sym}^{a_j}(V^\vee)^{[j]}.$$

Identifying $\mathrm{Sym}^{a_j}(V^\vee)^{[j]}$ with the $\mathbf{k}$-linear subspace of $\mathrm{Sym}^{a_j q^j}(V^\vee)$ consisting of q^j-powers provides a multiplication map

$$\mathrm{mult}\colon \mathrm{S}^a(V^\vee) \to \mathrm{Sym}^{a(q)}(V^\vee).$$

Call $a(t)$ a *profile* with respect to the prime power q if this multiplication map is injective for all finite-dimensional $\mathbf{k}$-vector spaces V. Denote by

$$\mathbf{Prfl} := \{a \in \mathbf{Z}_{\geq 0}[t] : \mathrm{mult}\colon \mathrm{S}^a(V^\vee) \to \mathrm{Sym}^{a(q)}(V^\vee) \text{ is injective for all finite-dimensional } V\}$$

the set of profiles with respect to q. Given $a \in \mathbf{Prfl}$ and a vector space V, $\mathrm{S}^a(V^\vee)$ will often tacitly be identified with its image in $\mathrm{Sym}^{a(q)}(V^\vee)$ under the injective multiplication map.

Whether or not a given element of $\mathbf{Z}_{\geq 0}[t]$ is a profile depends on q: For instance, $a(t) := t + q$ is not a profile with respect to q since, for a 2-dimensional vector space $V^\vee = \mathbf{k}\cdot u \oplus \mathbf{k}\cdot v$, the elements $u^{[1]} \otimes v^q$ and $v^{[1]} \otimes u^q$ are distinct in $\mathrm{S}^a(V^\vee)$, but have the same image in $\mathrm{Sym}^{2q}(V^\vee)$. The same polynomial $t+q$ is, however, a profile with respect to q^k for any $k \geq 2$. In what follows, however, q remains fixed, the dependence of $\mathbf{Prfl}$ on q will be suppressed.

1.2. Partial orderings on profiles. — Endow $\mathbf{Prfl}$ with two partial orderings as follows: First, viewing a profile as a sequence of nonnegative integers provides the product ordering $\preceq$, where for profiles $a := \sum_{j \geq 0} a_j t^j$ and $b := \sum_{j\geq 0} b_j t^j$,

$$a \preceq b \iff a_j \leq b_j \text{ for all } j \geq 0.$$

Second, identifying a $a \in \mathbf{Prfl}$ with the space $\mathrm{S}^a(V^\vee)$ provides the containment ordering $\sqsubseteq$, where $a \sqsubseteq b$ if and only if $a(q) = b(q) =: d$ and

$$\mathrm{S}^a(V^\vee) \subseteq \mathrm{S}^b(V^\vee) \subseteq \mathrm{Sym}^d(V^\vee) \text{ for all finite-dimensional } \mathbf{k}\text{-vector spaces } V,$$

where the tensor functors are identified with their image under the multiplication map.

1.3. Examples and properties. — Regarding the partial orderings $\preceq$ and $\sqsubseteq$ on $\mathbf{Prfl}$:

(i) The product ordering is strictly monotonic for *numerical degrees*: If $a \prec b$, then $a(q) < b(q)$. The containment ordering, on the other hand, splits $\mathbf{Prfl}$ into connected components

$$\mathbf{Prfl}_d := \{a \in \mathbf{Prfl} : a(q) = d\}$$

indexed by numerical degrees. Each $\mathbf{Prfl}_d$ has a unique $\sqsubseteq$-maximal element given by the constant profile $a_{d\text{-max}}(t) = d$, and a unique $\sqsubseteq$-minimal element given by $a_{d\text{-min}}(t) = a_0 + a_1 t + \cdots + a_m t^m$ with coefficients $0 \leq a_j \leq q-1$ arising from the base q expansion of d.

(ii) More interestingly, both orderings are strictly monotonous for coefficient sums. This is clear for $\preceq$. To see that $a \sqsubset b$ implies $a(1) < b(1)$, observe that $\mathrm{S}^a(V^\vee) \subset \mathrm{S}^b(V^\vee)$ for varying vector spaces V gives the dimensional inequality

$$\prod\nolimits_{j\geq 0} \binom{a_j + n}{a_j} < \prod\nolimits_{j \geq 0} \binom{b_j+n}{b_j} \text{ for all } n \geq 0.$$

Using that $\binom{n}{k} \sim \frac{n^k}{k!}$ for large n and fixed k, taking logarithms, and doing away with constants shows that for all $n \gg 0$,

$$\sum\nolimits_{j\geq 0} a_j \cdot \log(a_j + n) < \sum\nolimits_{j \geq 0} b_j \cdot \log(b_j + n).$$

By taking n even larger, $\log(n)$ may be made arbitrarily close to $\log(c+n)$ for any constant c. Since there are only finitely many constants a_j and b_j, this implies that $a(1) < b(1)$.

(iii) Neither $\preceq$ nor $\sqsubseteq$ give total orderings on $\mathbf{Prfl}$. This remains the case for $\sqsubseteq$ even upon restriction to a connected component $\mathbf{Prfl}_d$: for instance,

$$a(t) := t^2 + (q+1) \text{ and } b(t) := (q+1)t + 1$$

are profiles of numerical degree $d = q^2 + q + 1$ which are $\sqsubseteq$-incomparable.

(iv) Call $b \in \mathbf{Prfl}$ *nonreduced* if its constant term is zero: $b(0) = 0$. Any profile a preceding a nonreduced profile b in either ordering is also nonreduced.

(v) Both orderings are compatible with addition in the following sense: For $a, b, c \in \mathbf{Prfl}$ such that $a + c$ and $b + c$ are also profiles,

$$\text{if } a \sqsubseteq b, \text{ then } a + c \sqsubseteq b + c,$$

and similarly for $\preceq$. This is straightforward to see for $\preceq$. For $\sqsubseteq$, let V be a $\mathbf{k}$-vector space and let $f \in \mathrm{S}^{a+c}(V^\vee)$. Then there exists an expansion of the form

$$f = \sum\nolimits_{i \in I} g_i h_i \text{ with } g_i \in \mathrm{S}^a(V^\vee) \text{ and } h_i \in \mathrm{S}^c(V^\vee).$$

Viewing the g_i as elements of $\mathrm{S}^b(V^\vee)$ shows that $f \in \mathrm{S}^{b+c}(V^\vee)$.

A third ordering $\rightsquigarrow$ on $\mathbf{Prfl}$ obtained by combining the two will be also be useful: For $a, b \in \mathbf{Prfl}$,

$$a \rightsquigarrow b \iff \text{there exists } a' \in \mathbf{Z}_{\geq 0}[t] \text{ such that } a \preceq a' \sqsubseteq b.$$

Here, the relations $\preceq$ and $\sqsubseteq$ are extended to $\mathbf{Z}_{\geq 0}[t]$ in the natural way: $a \preceq a'$ means that each coefficient of a' is at least that of a, and $a' \sqsubseteq b$ means that the image of $\mathrm{S}^{a'}(V^\vee)$ under the multiplication map is contained in $\mathrm{S}^b(V^\vee)$ for all finite-dimensional $\mathbf{k}$-vector spaces V.

1.4. Lemma. — *The relation $\rightsquigarrow$ is a partial ordering on* **Prfl**.

Proof. Reflexivity and antisymmetry follow directly from the corresponding properties of $\preceq$ and $\sqsubseteq$. For transitivity, consider $a, b, c \in \mathbf{Prfl}$ such that $a \rightsquigarrow b$ and $b \rightsquigarrow c$. By definition, this means that there are $a', b' \in \mathbf{Z}_{\geq 0}[t]$ satisfying

$$a \preceq a' \sqsubseteq b \;\text{ and }\; b \preceq b' \sqsubseteq c.$$

Consider $a'' := a' + (b' - b)$. Since $b \preceq b'$, each coefficient of $b' - b$ is nonnegative, and so $a \preceq a''$. Compatibility with addition from 1.3(v) holds more generally for $\sqsubseteq$ on $\mathbf{Z}_{\geq 0}[t]$, and it implies that $a'' \sqsubseteq b'$. Transitivity of $\sqsubseteq$ then gives $a'' \sqsubseteq c$, showing that a'' witnesses the relation $a \rightsquigarrow c$. ∎

These partial orderings, $\preceq$ in particular, make it simple to formulate a criterion for when a nonnegative integer polynomial $a \in \mathbf{Z}_{\geq 0}[t]$ is a profile with respect to q:

1.5. Lemma. — *Given $a \in \mathbf{Z}_{\geq 0}[t]$, the following conditions are equivalent:*

(i) $\mathrm{mult}\colon \mathrm{S}^a(V) \to \mathrm{Sym}^{a(q)}(V)$ *is injective for every finite-dimensional* $\mathbf{k}$*-vector space* V*;*
(ii) $\mathrm{mult}\colon \mathrm{S}^a(V) \to \mathrm{Sym}^{a(q)}(V)$ *is injective for a* 2*-dimensional* $\mathbf{k}$*-vector space* V*; and*
(iii) *the function* $\{b \in \mathbf{Z}_{\geq 0}[t] : 0 \preceq b \preceq a\} \to \mathbf{Z}$ *given by* $b \mapsto b(q)$ *is injective.*

Proof. (i) ⇒ (ii) is clear. For (ii) ⇒ (iii), choose a basis $V = \mathbf{k}\cdot u \oplus \mathbf{k}\cdot v$. Writing $a = a_0 + a_1 t + \cdots + a_m t^m$, a basis for $\mathrm{S}^a(V)$ is given by

$$\begin{aligned}\mathrm{S}^a(\mathbf{k}\cdot u \oplus \mathbf{k}\cdot v) &= \bigotimes\nolimits_{j=0}^{m} \mathrm{Sym}^{a_j}(\mathbf{k}\cdot u \oplus \mathbf{k}\cdot v)^{[j]} = \bigotimes\nolimits_{j=0}^{m}\Big(\bigoplus\nolimits_{b_j=0}^{a_j} \mathbf{k}\cdot (u^{a_j - b_j} v^{b_j})^{[j]}\Big) \\ &= \bigoplus\nolimits_{b_0=0}^{a_0}\cdots\bigoplus\nolimits_{b_m=0}^{a_m} \mathbf{k}\cdot (u^{a_0-b_0}v^{b_0}) \otimes (u^{a_1-b_1}v^{b_1})^{[1]} \otimes \cdots \otimes (u^{a_m-b_m}v^{b_m})^{[m]}.\end{aligned}$$

Multiplication sends the displayed basis element to $u^{a(q)-b(q)}v^{b(q)}$ where $b := b_0 + b_1 t + \cdots + b_m t^m$, so injectivity of $\mathrm{mult}\colon \mathrm{S}^a(V) \to \mathrm{Sym}^{a(q)}(V)$ is, in fact, equivalent to injectivity of the function $b \mapsto b(q)$.

Finally, for (iii) ⇒ (i), suppose for sake of contradiction that there is a V for which $\mathrm{mult}\colon \mathrm{S}^a(V) \to \mathrm{Sym}^{a(q)}(V)$ is not injective; choose such a V of minimal dimension and a nonzero element $\alpha \in \mathrm{S}^a(V)$ in the kernel. The previous argument shows that $\dim_{\mathbf{k}} V \geq 2$, making it possible to choose a nonzero vector $v \in V$, a complementary subspace $0 \neq U \subseteq V$, and a splitting $V = U \oplus \mathbf{k}\cdot v$. Writing

$$\mathrm{Sym}^{a_j}(V) = \mathrm{Sym}^{a_j}(U \oplus \mathbf{k}\cdot v) = \bigoplus\nolimits_{b_j=0}^{a_j} \mathrm{Sym}^{a_j - b_j}(U)\cdot v^{b_j}$$

provides α with a unique expansion of the form

$$\alpha = C\cdot v^{a(q)} + \sum\nolimits_{0 \preceq b \prec a}\Big(\sum\nolimits_{i \in I_b} (\beta_{i,0}\cdot v^{b_0}) \otimes (\beta_{i,1}\cdot v^{b_1})^{[1]} \otimes \cdots \otimes (\beta_{i,m}\cdot v^{b_m})^{[m]}\Big)$$

for some $C \in \mathbf{k}$ and some $\beta_{i,j} \in \mathrm{Sym}^{a_j - b_j}(U)$ for each $i \in I_b$ and $0 \leq j \leq m$. Since $\alpha \in \ker(\mathrm{mult})$,

$$0 = \mathrm{mult}(\alpha) = C\cdot v^{a(q)} + \sum\nolimits_{0 \preceq b \prec a}\Big(\sum\nolimits_{i \in I_b} \beta_{i,0}\beta_{i,1}^{[1]}\cdots\beta_{i,m}^{[m]}\Big)\cdot v^{b(q)}.$$

The hypothesis that $\{b \in \mathbf{Z}_{\geq 0}[t] : 0 \preceq b \preceq a\} \to \mathbf{Z}\colon b \mapsto b(q)$ is injective implies that $C = 0$ and that each parenthesized term must vanish. Since α is nonzero, some $\beta_{i,j}$ is nonzero, and so $\mathrm{mult}\colon \mathrm{S}^{a-b}(U) \to \mathrm{Sym}^{a(q)-b(q)}(U)$ is not injective for some $b \prec a$. Multiplying by any element in $\mathrm{S}^b(U)$ with nonzero image in $\mathrm{Sym}^{b(q)}(U)$ then implies that $\mathrm{mult}\colon \mathrm{S}^a(U) \to \mathrm{Sym}^{a(q)}(U)$ is also not injective. This contradicts the minimality of V, completing the proof. ∎

A simple but very useful consequence of this characterization used tacitly throughout—in 3.14 for instance—is the following

1.6. Corollary. — *Let $a, b \in \mathbf{Z}_{\geq 0}[t]$ with $a \preceq b$. If $b \in$* **Prfl***, then also $a \in$* **Prfl**. ∎

2. $(q;\mathbf{a})$-TIC SCHEMES

Each $a \in \mathbf{Prfl}$ determines a canonical linear system

$$\Gamma(Y,\mathscr{L}^{\otimes a}) := \mathrm{image}(\mathrm{mult}\colon \mathrm{S}^a\Gamma(Y,\mathscr{L}) \to \Gamma(Y,\mathscr{L}^{\otimes a(q)}))$$

for every $\mathbf{k}$-scheme Y and $\mathscr{L} \in \mathrm{Pic}\,Y$; note that for general $(Y,\mathscr{L})$, the displayed multiplication map may not be injective. Divisors in the corresponding linear series tend to be special compared to the general divisor in the complete linear series associated with $\mathscr{L}^{\otimes a(q)}$ due to the special form of its equations. One way to access these special properties is by carrying as additional structure a lift of its defining section to the tensor product space $\mathrm{S}^a\Gamma(Y,\mathscr{L})$. This article is concerned primarily with the case of projective n-space $Y = \mathbf{P}V$ and $\mathscr{L} = \mathscr{O}_{\mathbf{P}V}(1)$, and this section describes some of the basic properties of complete intersections of such hypersurfaces.

2.1. Definitions. — Let $a \in \mathbf{Prfl}$ be a profile.

- A $(q;a)$-*tic tensor* is an element $\alpha \in \mathrm{S}^a(V^\vee)$.
- A $(q;a)$-*tic polynomial* is the image $f_\alpha := \mathrm{mult}(\alpha)$ of a $(q;a)$-tic tensor under multiplication.
- A $(q;a)$-*tic hypersurface* is the zero locus $X_\alpha := \mathrm{V}(f_\alpha)$ in $\mathbf{P}V$ of a nonzero $(q;a)$-tic polynomial.

More generally, given a *multi-profile* $\mathbf{a}$—a finite multi-set consisting of elements of $\mathbf{Prfl}$—let

$$\mathrm{S}^{\mathbf{a}}(V^\vee) := \bigoplus\nolimits_{a\in\mathbf{a}} \mathrm{S}^a(V^\vee)$$

be the space of $(q;\mathbf{a})$-*tic tensors*. Given such a tensor $\boldsymbol{\alpha}$, the vanishing locus $X_{\boldsymbol{\alpha}} := \mathrm{V}(f_\alpha : \alpha \in \boldsymbol{\alpha}) \subseteq \mathbf{P}V$ of its associated polynomials is called a $(q;\mathbf{a})$-*tic scheme*; when the sequence of defining polynomials form a regular sequence, $X_{\boldsymbol{\alpha}}$ is called a $(q;\mathbf{a})$-*tic complete intersection*.

2.2. Examples. — Given a profile $a = a_0 + a_1t + \cdots + a_mt^m$, a $(q;a)$-tic polynomial is of the form

$$f = \sum\nolimits_{i\in I} g_{i,0}\cdot(g_{i,1})^q\cdots(g_{i,m})^{q^m}$$

where $g_{i,j}$ is a homogeneous polynomial of degree a_j. Some simple instances include:

(i) The constant polynomial $a(t) = d$ is a profile for any prime power q, and $(q;a)$-tic polynomial is synonymous for degree d polynomial. In particular, if $a(t) = 1$ is a *linear* profile, then $(q;a)$-tic polynomials are linear polynomials.

(ii) If a profile $a(t) = a_1t + \cdots + a_mt^m$ has zero constant term, so it is *nonreduced*, then any $(q;a)$-tic polynomial is a $\mathbf{k}$-linear combination of q-power monomials, and its associated hypersurface is geometrically nonreduced.

(iii) $a(t) = 1 + t + \cdots + t^m$ is a profile for any q, and $(q;a)$-tic polynomials are of the form

$$f(x_0,\ldots,x_n) = \sum\nolimits_{i_0,\ldots,i_m=0}^n c_{i_0,i_1,\ldots,i_m}\cdot\prod\nolimits_{j=0}^m (x_{i_j})^{q^j} \text{ for scalars } c_{i_0,i_1,\ldots,i_m} \in \mathbf{k}.$$

In particular, when $a(t) = 1 + t$, these are the q-bic hypersurfaces from [Che25b, 1.6], and the underlying $(q;a)$-tic tensor is but a q-bic form in the sense of [Che25a, 1.1].

(iv) $a(t) = 2 + t$ is a profile whenever $q \neq 2$, and $(q;a)$-tic polynomials are of the form

$$f(x_0,\ldots,x_n) = \sum\nolimits_{0\le i\le j\le n}\sum\nolimits_{k=0}^n c_{ijk}\cdot x_ix_jx_k^q \text{ for scalars } c_{ijk} \in \mathbf{k}.$$

(v) The Fermat polynomial $x_0^d + \cdots + x_n^d$ of degree d is a $(q;a)$-tic polynomial for any profile a of numerical degree $a(q) = d$. This provides a cheap proof of the fact that the general $(q;a)$-tic hypersurface is smooth whenever its numerical degree is coprime to the characteristic p; see 2.8 for another argument in the general case.

A simple but extremely useful observation is that a $(q;\mathbf{a})$-tic structure is preserved upon passing to linear sections. The following statement follows easily from functoriality of tensor functors:

2.3. Lemma. — *Let $X \subseteq \mathbf{P}V$ be a $(q;\mathbf{a})$-tic scheme associated to a tensor $\boldsymbol{\alpha}$. If $\mathbf{P}U \subseteq \mathbf{P}V$ is a linear subspace, then $X \cap \mathbf{P}U$ is the $(q;\mathbf{a})$-tic scheme in $\mathbf{P}U$ associated with the tensor $\boldsymbol{\alpha}|_U$.* ■

2.4. $(q;a)$-tic Veronese. — The linear system $\mathrm{S}^a(V^\vee)$ of $(q;a)$-tic polynomials, being a tensor product of Frobenius twists of the complete linear systems $\Gamma(\mathbf{P}V, \mathcal{O}_{\mathbf{P}V}(a_j))$, is base point free and so defines a morphism

$$\mathrm{Ver}_a \colon \mathbf{P}V \to \mathbf{P}(\mathrm{S}^a(V^\vee)^\vee)$$

called the $(q;a)$-*tic Veronese morphism*. This description shows that Ver_a canonically factors as

$$\mathrm{Ver}_a \colon \mathbf{P}V \xrightarrow{(\mathrm{Ver}_{a_j})} \prod\nolimits_{j\geq 0} \mathbf{P}(\mathrm{Sym}^{a_j}(V^\vee)^\vee) \xrightarrow{\prod \mathrm{Fr}^j} \prod\nolimits_{j\geq 0} \mathbf{P}(\mathrm{Sym}^{a_j}(V^\vee)^{\vee,[j]}) \xrightarrow{\mathrm{Seg}} \mathbf{P}(\mathrm{S}^a(V^\vee)^\vee)$$

where the first map is the tuple whose j-th factor is the a_j-th Veronese embedding; the second map is a product of powers of the q-power $\mathbf{k}$-linear Frobenius morphism, with Fr^j acting on the j-th factors; and the third map is the Segre embedding. This gives the first statement of:

2.5. Lemma. — *The $(q;a)$-tic Veronese morphism $\mathrm{Ver}_a \colon \mathbf{P}V \to \mathbf{P}(\mathrm{S}^a(V^\vee)^\vee)$ is universally injective. Furthermore, if a is reduced, then Ver_a is a closed immersion.*

Proof. Assume a is reduced, meaning its constant term satisfies $a_0 \neq 0$, and consider the factorization from 2.4. Since the Segre embedding is a closed immersion, it suffices to show that

$$\Big(\prod\nolimits_{j\geq 0} \mathrm{Fr}^j\Big) \circ \big(\mathrm{Ver}_{a_j}\big)_{j\geq 0} \colon \mathbf{P}V \to \prod\nolimits_{j\geq 0} \mathbf{P}(\mathrm{Sym}^{a_j}(V^\vee)^{\vee,[j]})$$

is an isomorphism onto its image. But this clear since projection onto the 0-th factor provides the a_0-th Veronese embedding of $\mathbf{P}V$, which is an isomorphism onto its image whenever $a_0 \neq 0$. ■

2.6. Example. — Consider $a = t^r(1+t^m)$ for integers $r \geq 0$ and $m > 0$, and $\mathbf{P}V \cong \mathbf{P}^1$. Then the a-th Veronese is the morphism $\mathbf{P}^1 \to \mathbf{P}^3$ given by

$$\mathrm{Ver}_{t^r(1+t^m)}(x:y) = (x^{q^r(1+q^m)} : (x^{q^m}y)^{q^r} : (xy^{q^m})^{q^r} : y^{q^r(1+q^m)}).$$

When $r = 0$, then $\mathrm{Ver}_{1+t^m} \colon \mathbf{P}^1 \to \mathbf{P}^3$ is an isomorphism onto its image, providing what might be viewed as a generalization of the twisted cubic, which may be obtained by taking $q = 2$ and $m = 1$.

2.7. Parameter space. — Let $\mathbf{a}$ be a multi-profile. A parameter space for $(q;\mathbf{a})$-tic schemes in $\mathbf{P}V$ is given by the multi-projective space

$$(q;\mathbf{a})\text{-}\mathbf{tics}_{\mathbf{P}V} := \prod\nolimits_{a\in\mathbf{a}} \mathbf{P}\mathrm{S}^a(V^\vee).$$

The tautological line subbundles come together to form a tautological $(q;\mathbf{a})$-tic tensor

$$\boldsymbol{\alpha}_{\mathrm{taut}} \colon \bigoplus\nolimits_{a\in\mathbf{a}} \mathrm{pr}_a^* \mathcal{O}_{\mathbf{P}\mathrm{S}^a(V^\vee)}(-1) \to \mathcal{O}_{(q;\mathbf{a})\text{-}\mathbf{tics}_{\mathbf{P}V}} \otimes \mathrm{S}^{\mathbf{a}}(V^\vee)$$

which cuts out a tautological family $\mathscr{X}_{\mathrm{taut}}$ in the product $\mathbf{P}V \times (q;\mathbf{a})\text{-}\mathbf{tics}_{\mathbf{P}V}$.

Standard arguments show that the second projection $\mathscr{X}_{\mathrm{taut}} \to (q;\mathbf{a})\text{-}\mathbf{tics}_{\mathbf{P}V}$ is dominant if and only if $\#\mathbf{a} \leq n$. In this case, as usual, a property is said to hold for a *general* $(q;\mathbf{a})$-tic scheme in $\mathbf{P}V$ if it holds for each fibre of $\mathscr{X}_{\mathrm{taut}}$ over a nonempty open subset of $(q;\mathbf{a})\text{-}\mathbf{tics}_{\mathbf{P}V}$. In the following, call the multi-profile $\mathbf{a}$ *reduced* if every $a \in \mathbf{a}$ is reduced:

2.8. Proposition. — *If the multi-profile $\mathbf{a}$ is reduced and $\#\mathbf{a} \leq n$, then a general $(q;\mathbf{a})$-tic scheme in $\mathbf{P}V$ is smooth of dimension $n - \#\mathbf{a}$.*

Proof. Let $\Delta \subseteq (q;\mathbf{a})\text{-}\mathbf{tics}_{\mathbf{P}V}$ be the subset parameterizing $(q;\mathbf{a})$-tic schemes which are not smooth of dimension $n-\#\mathbf{a}$. This is closed subset as it is supported on the image of the nonsmooth locus of the proper morphism $\mathscr{X}_{\mathrm{taut}} \to (q;\mathbf{a})\text{-}\mathbf{tics}_{\mathbf{P}V}$. Thus it suffices to show that there exists a single smooth $(q;\mathbf{a})$-tic complete intersection. This follows from Bertini's theorem as given in [Stacks, 0FD6], where the hypotheses are satisfied thanks to 2.4 and 2.5. ■

It would be interesting to study the discriminant locus Δ. For q-bic hypersurfaces as in 2.2(iii), Δ is cut out by a determinantal equation, similar to the case of quadrics, and carries an intricate stratification corresponding to singularity types of the corresponding hypersurfaces: see [Che25a, §6] for details. It would also be interesting to compute numerical invariants of Δ, such as its multi-degree, as is done in [Ben12] in the classical case.

3. Linear spaces

One of the most apparent special properties of a $(q;\mathbf{a})$-tic complete intersection $X \subseteq \mathbf{P}V$ is that it contains many more linear spaces than would be expected given its numerical degree. From the point of view of this article, this is because its *Fano schemes* $\mathbf{F}_r(X)$—the Hilbert schemes parameterizing r-planes contained in X—inherit a $(q;\mathbf{a})$-tic structure, resulting in a smaller set of defining equations in the Grassmannian $\mathbf{G} := \mathbf{G}(r+1,V)$ of r-planes in $\mathbf{P}V$. Writing $\mathscr{S}$ for the tautological subbundle of rank $r+1$ on $\mathbf{G}$, this explicitly means the following:

3.1. Lemma. — *Let $X \subseteq \mathbf{P}V$ be a $(q;\mathbf{a})$-tic scheme. Its Fano scheme $\mathbf{F}_r(X)$ of r-planes is cut out of the Grassmannian $\mathbf{G}$ by a section of $\mathrm{S}^{\mathbf{a}}(\mathscr{S}^\vee) := \bigoplus_{a\in\mathbf{a}} \mathrm{S}^a(\mathscr{S}^\vee)$.*

Proof. Injectivity of the multiplication maps in the definition of profiles in 1.1 implies that the polynomials defining X vanish on an r-plane $\mathbf{P}U \subseteq \mathbf{P}V$ if and only if the corresponding $(q;\mathbf{a})$-tic tensor $\boldsymbol{\alpha} \in \mathrm{S}^{\mathbf{a}}(V^\vee)$ vanishes along the restriction $V^\vee \to U^\vee$. Thus $\mathbf{F}_r(X)$ is cut out by the section

$$\boldsymbol{\alpha}|_{\mathscr{S}} \colon \mathscr{O}_{\mathbf{G}} \to \mathscr{O}_{\mathbf{G}} \otimes \mathrm{S}^{\mathbf{a}}(V^\vee) \to \mathrm{S}^{\mathbf{a}}(\mathscr{S}^\vee).$$ ■

This provides a lower bound on the dimension of the Fano scheme $\mathbf{F}_r(X)$ of r-planes associated with a $(q;\mathbf{a})$-tic scheme $X \subseteq \mathbf{P}V$ in an n-dimensional projective space: If $\mathbf{F}_r(X) \neq \varnothing$, then

$$\dim \mathbf{F}_r(X) \geq \delta(n,\mathbf{a},r) := (r+1)(n-r) - \sum\nolimits_{a\in\mathbf{a}} \prod\nolimits_{j\geq 0} \binom{a_j + r}{r} \quad \text{where } a = \sum\nolimits_{j\geq 0} a_j t^j.$$

Notably, the *expected dimension* $\delta(n,\mathbf{a},r)$ of the Fano scheme, depends on $\mathbf{a}$ but not on q; equivalently, this means that the expected dimension of $\mathbf{F}_r(X)$ does not directly depend on the degree of X!

The main result of this section is that $\mathbf{F}_r(X)$ is of its expected dimension for a general $(q;\mathbf{a})$-tic scheme X, and will be furthermore smooth provided that the multi-profile $\mathbf{a}$ is reduced. As is well-known, the statement requires a slight modification when X is a quadric, in which case the dimension estimate $\delta(n,\mathbf{a},r)$ is too big when $r \approx \frac{1}{2}n$. To give a uniform statement, define

$$\delta_-(n,\mathbf{a},r) := \min\{\delta(n,\mathbf{a},r), n-2r-\#\mathbf{a}\}.$$

One may verify that, other than in the case $\mathbf{a} = (2t^k) \cup \mathbf{a}'$ where each $a \in \mathbf{a}'$ is of the form t^m, $\delta(n,\mathbf{a},r) \geq 0$ if and only if $\delta_-(n,\mathbf{a},r) \geq 0$, and similarly for > 0. The statement is now the following:

3.2. Theorem. — *Let $X \subseteq \mathbf{P}V$ be a $(q;\mathbf{a})$-tic scheme. If*

(i) *$\delta_-(n,\mathbf{a},r) < 0$, then $\mathbf{F}_r(X)$ is empty for general X;*
(ii) *$\delta_-(n,\mathbf{a},r) \geq 0$, then $\mathbf{F}_r(X)$ is nonempty and has dimension $\delta(n,\mathbf{a},r)$ for general X; and*
(iii) *$\delta_-(n,\mathbf{a},r) > 0$, then $\mathbf{F}_r(X)$ is connected for all X.*

Furthermore, if $\mathbf{a}$ is reduced, then $\mathbf{F}_r(X)$ is also smooth for general X when $\delta_-(n,\mathbf{a},r) \geq 0$.

The proof of 3.2 occupies the bulk of this section: see, in particular, 3.11 where the intervening statements are put together to complete the argument. Begin with two reductions:

3.3. Reductions. — First, it suffices to treat the case $\mathbf{a}$ is reduced. If $\mathbf{a}$ is nonreduced, let $\mathbf{a}'$ be the reduced multi-profile obtained by maximally dividing out powers of t from each profile in $\mathbf{a}$. Given a $(q;\mathbf{a})$-tic scheme X, there is a canonical $(q;\mathbf{a}')$-tic scheme X' obtained by taking q-power roots of the appropriate equations. This comes with a closed immersion $X' \to X$ which is furthermore a universal homeomorphism. This provides a universal homeomorphism $\mathbf{F}_r(X') \to \mathbf{F}_r(X)$ and so topological properties of the two are the same.

Second, we may assume that $1 \notin \mathbf{a}$. Otherwise, amongst the defining equations of a $(q;\mathbf{a})$-tic scheme $X \subseteq \mathbf{P}V = \mathbf{P}^n$ is a linear one. Eliminating that allows us to view $X \subseteq \mathbf{P}^{n-1}$. Writing $\mathbf{a} = (1) \cup \mathbf{a}'$, a direct computation shows that $\delta_-(n,\mathbf{a},r) = \delta_-(n-1,\mathbf{a}',r)$, and so it suffices to prove 3.2 for X viewed as a $(q;\mathbf{a}')$-tic scheme in $\mathbf{P}^{n-1}$.

From now on, assume that the profile $\mathbf{a}$ is reduced and $1 \notin \mathbf{a}$. The argument looks to lower bound the codimension of the singular locus Z_r of the second projection from the incidence correspondence

$$\mathbf{Inc}_r := \mathbf{Inc}_{V,r,\mathbf{a}} := \left\{([U],[\boldsymbol{\alpha}]) \in \mathbf{G}(r+1,V) \times (q;\mathbf{a})\text{-}\mathbf{tics}_{\mathbf{P}V} : \mathbf{P}U \subseteq X_{\boldsymbol{\alpha}}\right\}$$

to the parameter space of $(q;\mathbf{a})$-tic tensors from 2.7. Observe that $\mathrm{pr}_1 : \mathbf{Inc}_r \to \mathbf{G}(r+1,V)$ is surjective, with the fibre over a point $[U]$ a multi-projective space given by

$$\mathbf{Inc}_{r,[U]} = \prod\nolimits_{a \in \mathbf{a}} \mathbf{P}\mathrm{S}^a(V^\vee)_U$$

where $\mathrm{S}^a(V^\vee)_U := \ker(\mathrm{S}^a(V^\vee) \to \mathrm{S}^a(U^\vee))$ is the kernel of the restriction map. This has codimension $\sum_{a \in \mathbf{a}} \prod_{j \geq 0} \binom{r+a_j}{r}$ in the product and so gives:

3.4. Lemma. — *The incidence correspondence* $\mathbf{Inc}_r$ *is irreducible, proper, and smooth of dimension*

$$\dim \mathbf{Inc}_r = \delta(n,\mathbf{a},r) + \dim (q;\mathbf{a})\text{-}\mathbf{tics}_{\mathbf{P}V}\,. \qquad \blacksquare$$

The first task is to explicitly describe the closed subset $Z_r \subseteq \mathbf{Inc}_r$ on which $\mathrm{pr}_2 : \mathbf{Inc}_r \to (q;\mathbf{a})\text{-}\mathbf{tics}_{\mathbf{P}V}$ is not smooth of expected dimension $\delta(n,\mathbf{a},r)$: Let $\mathbf{P}U$ be any r-plane contained in $X := X_{\boldsymbol{\alpha}}$. Write $\mathbf{a} = (a_1,\ldots,a_c)$ as a c-tuple of profiles $a_i = \sum_{j \geq 0} a_{i,j} t^j$, and let $\alpha_i \in \mathrm{S}^{a_i}(V^\vee)$ be the components of the $(q;\mathbf{a})$-tic tensor $\boldsymbol{\alpha}$ defining X. Consider the map

$$\begin{aligned} \rho_{U,f_{\boldsymbol{\alpha}}} : U^\vee \otimes (V/U) &\to \bigoplus\nolimits_{i=1}^{c} \mathrm{H}^0(\mathbf{P}U, \mathcal{O}_{\mathbf{P}U}(a_i(q))) \\ \xi \otimes \bar{v} &\mapsto (\xi \cdot \partial_v f_{\alpha_1}|_{\mathbf{P}U}, \ldots, \xi \cdot \partial_v f_{\alpha_c}|_{\mathbf{P}U}) \end{aligned}$$

which takes a pure tensor $\xi \otimes \bar{v}$ to the c-tuple of polynomials whose i-th entry is ξ times the directional derivative $\partial_v f_{\alpha_i}$ of the i-th equation of X along any lift $v \in V$ of $\bar{v}$, all then restricted to $\mathbf{P}U$. Since first-order derivatives act linearly through q-powers, $\rho_{U,f_{\boldsymbol{\alpha}}}$ in fact takes values within the subspace of $(q;\mathbf{a})$-tic polynomials. More precisely:

3.5. Lemma. — *There exists a linear map* $\rho_{U,\boldsymbol{\alpha}} : U^\vee \otimes (V/U) \to \mathrm{S}^{\mathbf{a}}(U^\vee)$ *which factors* $\rho_{U,f_{\boldsymbol{\alpha}}}$ *through the multipication map* $\mathrm{mult} : \mathrm{S}^{\mathbf{a}}(U^\vee) \to \bigoplus_{i=1}^{c} \mathrm{H}^0(\mathbf{P}U, \mathcal{O}_{\mathbf{P}U}(a_i(q)))$.

Proof. The map $\rho_{U,\boldsymbol{\alpha}}$ is that which simply acts on the first component of each tensor α_i. Explicitly, the pure tensor $\xi \otimes \bar{v}$ is mapped to the c-tuple with i-th term

$$\xi \cdot \partial_v \alpha_i|_{\mathbf{P}U} = \sum\nolimits_k (\xi \cdot \partial_v \alpha_{i,0,k}) \otimes (\alpha_{i,1,k})^{[1]} \otimes \cdots \otimes (\alpha_{i,m_i,k})^{[m_i]}|_{\mathbf{P}U}$$

where each $\alpha_{i,j,k} \in \mathrm{Sym}^{a_{i,j}}(V^\vee)$. The preceding comments imply that $\rho_{U,f_{\boldsymbol{\alpha}}} = \mathrm{mult} \circ \rho_{U,\boldsymbol{\alpha}}$. $\blacksquare$

If $\mathbf{P}U$ were contained in the smooth locus of X, then the tangent space to $\mathbf{F}_r(X)$ at the point $[\mathbf{P}U]$ is given by the space of sections the normal bundle of $\mathbf{P}U \subseteq X$. The normal bundle sequence, 3.5, and injectivity of mult together given canonical identifications

$$\mathrm{H}^0(\mathbf{P}U, \mathcal{N}_{\mathbf{P}U/X}) \cong \ker \rho_{U,f_{\boldsymbol{\alpha}}} \cong \ker \rho_{U,\boldsymbol{\alpha}}.$$

Therefore $\mathbf{F}_r(X)$ is smooth of dimension $\delta(n,\mathbf{a},r)$ at the point $[\mathbf{P}U]$ if and only if the map $\rho_{U,\boldsymbol{\alpha}}$ is surjective. This property characterizes the complement of Z_r in general:

3.6. Lemma. — $\mathrm{pr}_2 \colon \mathbf{Inc}_r \to (q;\mathbf{a})\text{-}\mathbf{tics}_{\mathbf{P}V}$ *is smooth of dimension* $\delta(n,\mathbf{a},r)$ *at a point* $([U],[\boldsymbol{\alpha}])$ *if and only if* $\rho_{U,\boldsymbol{\alpha}} \colon U^\vee \otimes (V/U) \to \mathrm{S}^{\mathbf{a}}(U^\vee)$ *is surjective. In particular,*

$$Z_r = \big\{([U],[\boldsymbol{\alpha}]) \in \mathbf{Inc}_r : \rho_{U,\boldsymbol{\alpha}} \colon U^\vee \otimes (V/U) \to \mathrm{S}^{\mathbf{a}}(U^\vee) \text{ is not } \textit{surjective}\big\}.$$

Proof. Both $\mathbf{Inc}_r$ and $(q;\mathbf{a})\text{-}\mathbf{tics}_{\mathbf{P}V}$ are smooth, so pr_2 is smooth of dimension $\delta(n,\mathbf{a},r)$ at $([U],[\boldsymbol{\alpha}])$ if and only if the map $\mathcal{T}_{\mathbf{Inc}_r} \to \mathrm{pr}_2^*\mathcal{T}_{(q;\mathbf{a})\text{-}\mathbf{tics}_{\mathbf{P}V}}$ on tangent bundles is surjective there. The discussion above 3.4 shows that the fibre of pr_1 over $[U]$ maps isomorphically via pr_2 to a multi-projective subspace in the parameter space corresponding to the vector space $\mathrm{S}^{\mathbf{a}}(V^\vee)_U$:

$$\mathrm{pr}_1^{-1}([U]) \cong \prod\nolimits_{i=1}^c \mathbf{P}\big(\ker(\mathrm{S}^{a_i}(V^\vee) \to \mathrm{S}^{a_i}(U^\vee))\big) \subseteq \prod\nolimits_{i=1}^c \mathbf{P}\mathrm{S}^{a_i}(V^\vee) = (q;\mathbf{a})\text{-}\mathbf{tics}_{\mathbf{P}V}$$

Restricting the tangent map to the subbundle $\mathcal{T}_{\mathrm{pr}_1} \subseteq \mathcal{T}_{\mathbf{Inc}_r}$ and taking fibres thus gives a sequence

$$0 \to \mathcal{T}_{\mathrm{pr}_1,([U],[\boldsymbol{\alpha}])} \to \mathcal{T}_{(q;\mathbf{a})\text{-}\mathbf{tics}_{\mathbf{P}V},[\boldsymbol{\alpha}]} \to \mathrm{S}^{\mathbf{a}}(U^\vee) \to 0.$$

Combined with the isomorphism $\mathrm{pr}_1^*\mathcal{T}_{\mathbf{G}} \cong \mathcal{T}_{\mathbf{Inc}_r}/\mathcal{T}_{\mathrm{pr}_1}$ from the tangent bundle sequence of pr_1, this means that the tangent map of pr_2 at $([U],[\boldsymbol{\alpha}])$ induces a map

$$(\star) \qquad U^\vee \otimes (V/U) \cong \mathcal{T}_{\mathbf{Inc}_r,([U],[\boldsymbol{\alpha}])}/\mathcal{T}_{\mathrm{pr}_1,([U],[\boldsymbol{\alpha}])} \to \mathcal{T}_{(q;\mathbf{a})\text{-}\mathbf{tics}_{\mathbf{P}V},[\boldsymbol{\alpha}]}/\mathcal{T}_{\mathrm{pr}_1,([U],[\boldsymbol{\alpha}])} \cong \mathrm{S}^{\mathbf{a}}(U^\vee)$$

and pr_2 is smooth of the expected dimension at $([U],[\boldsymbol{\alpha}])$ if and only if this map is surjective.

Identify this map via deformation theory: A pure tensor $\xi \otimes \bar{v}$ in $U^\vee \otimes (V/U) \cong \mathcal{T}_{\mathbf{G},[U]}$ corresponds to the first-order deformation of $U \subseteq V$ given by the $\mathbf{k}[\epsilon]$-submodule

$$U[\epsilon\xi \cdot \bar{v}] := \langle u + \epsilon\xi(u)\cdot v : u \in U\rangle = \{u_1 + \epsilon(\xi(u_1)\cdot v + u_2) : u_1, u_2 \in U\} \subseteq V \otimes_{\mathbf{k}} \mathbf{k}[\epsilon].$$

Its preimage in $\mathcal{T}_{\mathbf{Inc}_r,([U],[\boldsymbol{\alpha}])}$ classifies first-order deformations $X_{\boldsymbol{\alpha}+\epsilon\boldsymbol{\beta}}$, where $\boldsymbol{\beta} \in \mathrm{S}^{\mathbf{a}}(V^\vee)$, which contain $\mathbf{P}(U[\epsilon\xi\cdot\bar{v}])$. The tangent map $\mathcal{T}_{\mathbf{Inc}_r} \to \mathcal{T}_{(q;\mathbf{a})\text{-}\mathbf{tics}_{\mathbf{P}V}}$ is the forgetful map which extracts the $(q;\mathbf{a})$-tic tensor $\boldsymbol{\beta}$ parameterizing the first-order deformation of $X_{\boldsymbol{\alpha}}$. The map $(\star)$ thus acts as $\xi \otimes \bar{v} \mapsto \boldsymbol{\beta}|_U$ for any choice of such $\boldsymbol{\beta}$. To express $\boldsymbol{\beta}$ in terms of $\boldsymbol{\alpha}$, observe that the condition that $X_{\boldsymbol{\alpha}+\epsilon\boldsymbol{\beta}}$ contains $\mathbf{P}(U[\epsilon\xi\cdot\bar{v}])$ means that, for all $u_1, u_2 \in U$,

$$(\boldsymbol{\alpha}+\epsilon\boldsymbol{\beta})(u_1 + \epsilon(\xi(u_1)\cdot v + u_2)) = \boldsymbol{\alpha}(u_1 + \epsilon(\xi(u_1)\cdot v + u_2)) + \epsilon\boldsymbol{\beta}(u_1) = 0.$$

Writing $\alpha_i = \sum_k \alpha_{i,0,k} \otimes (\alpha_{i,1,k})^{[1]} \otimes \cdots \otimes (\alpha_{i,m_i,k})^{[m_i]}$ for each component of $\boldsymbol{\alpha}$,

$$\alpha_{i,j,k}^{[j]}(u_1 + \epsilon(\xi(u_1)\cdot v + u_2)) = \begin{cases} \alpha_{i,j,k}(u_1)^{q^j} & \text{if } j > 0, \text{ and} \\ \alpha_{i,0,k}(u_1) + \epsilon(\xi(u)\cdot\partial_v\alpha_{i,0,k}(u) + \partial_{u_2}\alpha_{i,0,k}(u)) & \text{if } j = 0. \end{cases}$$

Expanding the tensor then gives

$$\alpha_i(u_1 + \epsilon(\xi(u_1)\cdot v + u_2)) = \alpha_i(u_1) + \epsilon(\xi(u_1)\cdot\partial_v\alpha_i(u_1) + \partial_{u_2}\alpha_i(u_1)).$$

That $\boldsymbol{\alpha}|_U = 0$ means that the first term vanishes: $\alpha_i(u_1) = 0$ for all $u_1 \in U$. A directional derivative of a polynomial vanishing on U in a direction in U remains vanishing on U, so $\partial_{u_2}\alpha_i(u_1) = 0$ for all $u_1, u_2 \in U$. Put together, this gives the result since

$$\boldsymbol{\beta}(u) = -\xi(u)\cdot\partial_v\boldsymbol{\alpha}(u) = -\rho_{U,\boldsymbol{\alpha}}(\xi\otimes\bar{v})(u) \ \text{ for all } u \in U. \qquad \blacksquare$$

This description of Z_r is homogeneous in $[U]$, making it possible to fix a subspace U and to simply estimate the codimension of the inclusion of fibres $Z_{r,[U]} \subset \mathbf{Inc}_{r,[U]}$. Proceed by additionally parameterizing a bound on the image of $\rho_{U,\boldsymbol{\alpha}}$: Let $\mathbf{H}$ be the projective space of hyperplanes in $\mathrm{S}^{\mathbf{a}}(U^\vee)$, and consider the correspondence

$$Z'_{r,[U]} := \big\{([\boldsymbol{\alpha}],[\varphi]) \in \mathbf{Inc}_{r,[U]} \times \mathbf{H} : \varphi \circ \rho_{U,\boldsymbol{\alpha}} \colon U^\vee \otimes (V/U) \to \mathbf{k} \text{ is zero}\big\}$$

parameterizing $X_{\boldsymbol{\alpha}}$ containing $\mathbf{P}U$ and a hyperplane in $\mathrm{S}^{\mathbf{a}}(U^\vee)$ containing the image of $\rho_{U,\boldsymbol{\alpha}}$. Projection to $\mathbf{Inc}_{r,[U]}$ maps this onto $Z_{r,[U]}$. The fibre of projection to $\mathbf{H}$ over a point $\varphi \colon \mathrm{S}^{\mathbf{a}}(U^\vee) \to \mathbf{k}$ is a product of projective spaces on the kernel of the map

$$\Phi \colon \mathrm{S}^{\mathbf{a}}(V^\vee)_U \to \mathrm{Hom}(U^\vee \otimes (V/U), \mathbf{k}) \qquad \boldsymbol{\alpha} \mapsto \varphi \circ \rho_{U,\boldsymbol{\alpha}}.$$

Writing $\mathbf{a} - 1 = (a_1 - 1, \ldots, a_c - 1)$, the following gives a stratification of the image $Z'_{r,[U]}$ in $\mathbf{H}$ over which the fibres have the same dimension:

3.7. Lemma. — *Let $\mu \colon \mathrm{S}^{\mathbf{a}-1}(U^\vee) \to \mathrm{Hom}(U^\vee, \mathrm{S}^{\mathbf{a}}(U^\vee))$ be the map adjoint to multiplication, and set*

$$\mathbf{H}_k := \big\{[\varphi] \in \mathbf{H} : \mathrm{rank}\big(\varphi_* \circ \mu \colon \mathrm{S}^{\mathbf{a}-1}(U^\vee) \to \mathrm{Hom}(U^\vee, \mathbf{k})\big) = k+1\big\} \textit{ for each } 0 \le k \le r.$$

Then $Z'_{r,[U]} \times_{\mathbf{H}} \mathbf{H}_k$ is of codimension $(k+1)(n-r)$ in $\mathbf{Inc}_{r,[U]} \times \mathbf{H}_k$.

Proof. Identify the assignment $\boldsymbol{\alpha} \mapsto \rho_{U,\boldsymbol{\alpha}}$ as the composition of linear maps

$$\rho_{U,-} \colon \mathrm{S}^{\mathbf{a}}(V^\vee)_U \longrightarrow \mathrm{Hom}(V/U, \mathrm{S}^{\mathbf{a}-1}(U^\vee)) \xrightarrow{\mu_*} \mathrm{Hom}(V/U, \mathrm{Hom}(U^\vee, \mathrm{S}^{\mathbf{a}}(U^\vee)))$$

where the first arrow sends $\boldsymbol{\alpha}$ to the linear map $\bar{v} \mapsto \partial_v \boldsymbol{\alpha}|_U$, notation as in the proof of 3.5. Observe also that the first map here is surjective, since a lift of a given $\bar{\boldsymbol{\beta}} \in \mathrm{Hom}(V/U, \mathrm{S}^{\mathbf{a}-1}(U^\vee))$ is

$$\sum\nolimits_{j=r+1}^{n} \xi_i \cdot \boldsymbol{\beta}(\bar{v}_i) \in \mathrm{S}^{\mathbf{a}}(V^\vee)_U$$

where the $\bar{v}_i$ form a basis of V/U with dual coordinate ξ_i, and $\boldsymbol{\beta}(\bar{v}_i) \in \mathrm{S}^{\mathbf{a}-1}(V^\vee)$ is any lift of its barred counterpart. Together, these observations imply that

$$\begin{aligned}\mathrm{rank}(\Phi) &= \mathrm{rank}(\varphi_* \circ \mu_* \colon \mathrm{Hom}(V/U, \mathrm{S}^{\mathbf{a}-1}(U^\vee)) \to \mathrm{Hom}(V/U, \mathrm{Hom}(U^\vee, \mathbf{k}))) \\ &= \mathrm{rank}(\varphi_* \circ \mu \colon \mathrm{S}^{\mathbf{a}-1}(U^\vee) \to \mathrm{Hom}(U^\vee, \mathbf{k})) \cdot \dim_{\mathbf{k}}(V/U).\end{aligned}$$

This means that the fibre of $Z'_{r,[U]}$ over $[\varphi] \in \mathbf{H}_k$ is of codimension $(k+1)(n-r)$ in $\mathbf{Inc}_{r,[U]} \times \{[\varphi]\}$, and this yields the statement. ■

To relate this with the codimension of $Z_{r,[U]}$ in $\mathbf{Inc}_{r,[U]}$, it remains to bound the dimension of $\mathbf{H}_k$:

3.8. Lemma. — $\dim \mathbf{H}_k \le (k+1)(r-k) + \sum\nolimits_{i=1}^{c} \prod\nolimits_{j=0}^{m_i} \binom{k+a_{i,j}}{k} - 1.$

Proof. The image of $\varphi_* \circ \mu$ is contained in the $(k+1)$-dimensional subspace $\mathrm{Hom}(U_0^\vee, \mathbf{k})$ if and only if $\varphi \colon \mathrm{S}^{\mathbf{a}}(U^\vee) \to \mathbf{k}$ vanishes on the image of the multiplication map

$$(U/U_0)^\vee \otimes \mathrm{S}^{\mathbf{a}-1}(U^\vee) \to \mathrm{S}^{\mathbf{a}}(U^\vee).$$

The cokernel of this map is isomorphic to $\mathrm{S}^{\mathbf{a}}(U_0^\vee)$, and any such φ is determined by its values thereon. In other words, the closure $\mathbf{H}_k$ admits a surjection from the projective bundle on $\mathrm{S}^{\mathbf{a}}(-)$ applied to the dual tautological bundle on the Grassmannian $\mathbf{G}(k+1, U)$, yielding the dimension bound. ■

3.9. Proposition. — $\mathrm{codim}(Z_r \subset \mathbf{Inc}_r) \ge \delta_-(n, \mathbf{a}, r) + 1.$

Proof. It suffices to show that the corresponding codimension estimate holds for each fibre $Z_{r,[U]}$ and $\mathbf{Inc}_{r,[U]}$ over points $[U] \in \mathbf{G}(r+1,V)$. Since $Z_{r,[U]}$ is the image of $Z'_{r,[U]}$ under the first projection,

$$\begin{aligned}\operatorname{codim}(Z_{r,[U]} \subset \mathbf{Inc}_{r,[U]}) &\geq \min_{0\leq k\leq r} \operatorname{codim}(Z'_{r,[U]} \times_{\mathbf{H}} \mathbf{H}_k \subset \mathbf{Inc}_{r,[U]} \times \mathbf{H}_k) - \dim \mathbf{H}_k \\ &\geq \min_{0\leq k\leq r} (k+1)(n+k-2r) - \sum\nolimits_{i=1}^{c} \prod\nolimits_{j=0}^{m_i} \binom{k+a_{i,j}}{k} + 1\end{aligned}$$

where the second inequality follows from 3.7 and 3.8. Now view the rightmost quantity as a polynomial k; it is the difference between a quadratic polynomial and one of degree

$$\|\mathbf{a}\|_\infty := \max\{|a_i| := a_{i,0} + a_{i,1} + \cdots + a_{i,m_i} : 1 \leq i \leq c\},$$

the maximal coefficient sum of the profiles $a_i \in \mathbf{a}$. The identity $\binom{k+d+1}{k} = \frac{k+d+1}{d+1}\binom{k+d}{k}$ easily implies that the all derivatives of $\sum_{i=1}^{c}\prod_{j=0}^{m_i}\binom{k+a_{i,j}}{k}$ with respect to k are increasing in the parameters $a_{i,j} \geq 0$. Explicitly computing second derivatives for the multi-profiles (3), $(1+t)$, and $(2,2)$ implies that whenever $\|\mathbf{a}\|_\infty \geq 3$ or $\|\mathbf{a}\|_\infty = 2$ and $\mathbf{a} \neq (2)$, the function in the minimum is concave for $k \geq 0$. When $\mathbf{a} = (2)$, the function is an increasing. In all cases, this means that the minimum is achieved at the endpoints, either when $k = 0$ or $k = r$, so

$$\operatorname{codim}(Z_{r,[U]} \subset \mathbf{Inc}_{r,[U]}) \geq \min\{n-2r-c, \delta(n,\mathbf{a},r)\} + 1 = \delta_-(n,\mathbf{a},r) + 1. \qquad \blacksquare$$

Non-smooth points of $\mathbf{F}_{r-1}(X)$ often contribute to non-smooth points of $\mathbf{F}_r(X)$, producing components of Z_r that are too large. Writing Δ_r for the image of Z_r under $\mathrm{pr}_2 \colon \mathbf{Inc}_r \to (q;\mathbf{a})\text{-}\mathbf{tics}_{\mathbf{P}V}$, the following statement says that members of $Z_r^\circ := Z_r \setminus \mathrm{pr}_2^{-1}(\Delta_{r-1})$ are parameterized by the piece of highest codimension from 3.7:

3.10. Lemma. — *Let $([U],[\boldsymbol{\alpha}]) \in Z_r^\circ$. If $[\varphi] \in \mathbf{H}$ is such that $\varphi \circ \rho_{U,\boldsymbol{\alpha}} = 0$, then $[\varphi] \in \mathbf{H}_r$.*

Proof. Let $([U],[\boldsymbol{\alpha}]) \in Z_r$ and choose $[\varphi] \in \mathbf{H}_k$ such that $\varphi \circ \rho_{U,\boldsymbol{\alpha}} = 0$. If $k < r$, then by its definition from 3.7, this means that $\varphi_* \circ \mu \colon \mathrm{S}^{\mathbf{a}-1}(U^\vee) \to \mathrm{Hom}(U^\vee,\mathbf{k})$ is not surjective; choose a hyperplane $\mathrm{Hom}(U_0^\vee,\mathbf{k})$ containing the image. As in 3.8, this means that φ vanishes on the image of the multiplication map

$$(U/U_0)^\vee \otimes \mathrm{S}^{\mathbf{a}-1}(U^\vee) \to \mathrm{S}^{\mathbf{a}}(U^\vee)$$

and that φ descends to a nonzero linear functional $\varphi_0 \colon \mathrm{S}^{\mathbf{a}}(U_0^\vee) \to \mathbf{k}$ on the cokernel. Consider now the point $([U_0],[\boldsymbol{\alpha}]) \in \mathbf{Inc}_{r-1}$ and the corresponding tangent map $\rho_{U_0,\boldsymbol{\alpha}} \colon U_0^\vee \otimes (V/U_0) \to \mathrm{S}^{\boldsymbol{a}}(U_0^\vee)$. Since $X_{\boldsymbol{\alpha}}$ contains $\mathbf{P}U$, the tensor $\partial_u \boldsymbol{\alpha}$ for $u \in U$ lifting a basis of U/U_0 vanishes on $\mathbf{P}U \supset \mathbf{P}U_0$, and so $\rho_{U_0,\boldsymbol{\alpha}}$ factors through the map

$$U_0^\vee \otimes (V/U) \to \mathrm{S}^{\boldsymbol{a}}(U_0^\vee).$$

But this map is but a restriction of $\rho_{U,\boldsymbol{\alpha}}$, and so its image is contained in the kernel of φ_0. Thus $\rho_{U_0,\boldsymbol{\alpha}}$ is not surjective, so $([U_0],[\boldsymbol{\alpha}]) \in Z_{r-1}$ by 3.6, meaning that $([U],[\boldsymbol{\alpha}]) \in \mathrm{pr}_2^{-1}(\Delta_{r-1})$. $\blacksquare$

3.11. — It remains to put everything together to prove 3.2:

For (i), if $\delta_-(n,\mathbf{a},r) < 0$ and $\mathbf{a} \neq (2)$, then $\mathrm{pr}_2 \colon \mathbf{Inc}_r \to (q;\mathbf{a})\text{-}\mathbf{tics}_{\mathbf{P}V}$ cannot be dominant by the dimension computation in 3.4, and so $\mathbf{F}_r(X) = \varnothing$ for general X. The case $\mathbf{a} = (2)$ is well-known.

For (ii), if $\delta_-(n,\mathbf{a},r) \geq 0$, then it follows from 3.10 that the intersection of $Z_r \setminus \mathrm{pr}_2^{-1}(\Delta_{r-1})$ with $\mathbf{Inc}_{r,[U]}$ is contained in the image of $Z'_{r,[U]} \times_{\mathbf{H}} \mathbf{H}_r$ for each $[U] \in \mathbf{G}(r+1,V)$. The argument of 3.9 then implies that the codimension of its closure in $\mathbf{Inc}_r$ is at least $\delta_-(n,\mathbf{a},r)+1$, and so

$$\dim\left(\overline{Z_r \setminus \mathrm{pr}_2^{-1}(\Delta_{r-1})}\right) \leq \dim \mathbf{Inc}_r - \delta_-(n,\mathbf{a},r) - 1 \leq \dim (q;\mathbf{a})\text{-}\mathbf{tics}_{\mathbf{P}V} - 1.$$

Therefore $\Delta_r \setminus \Delta_{r-1}$ is not all of $(q;\mathbf{a})$-$\mathbf{tics}_{\mathbf{P}V}$. Induction on r—the base case with $r = 0$ is the statement 2.8 that the general $(q;\mathbf{a})$-tic scheme is smooth—then shows that Δ_r is a proper closed subset of $(q;\mathbf{a})$-$\mathbf{tics}_{\mathbf{P}V}$, meaning that $\mathbf{F}_r(X)$ is smooth of the expected dimension for general X by 3.6.

For (iii), when $\delta_-(n,\mathbf{a},r) > 0$, consider the Stein factorization

$$\mathrm{pr}_2 \colon \mathbf{Inc}_r \to \mathbf{Inc}'_r \to (q;\mathbf{a})\text{-}\mathbf{tics}_{\mathbf{P}V}$$

of the second projection. Then Z_r contains the preimage of the branch locus of $\mathbf{Inc}'_r \to (q;\mathbf{a})$-$\mathbf{tics}_{\mathbf{P}V}$. Since Z_r has codimension at least 2 in $\mathbf{Inc}_r$ by 3.9, purity of the branch locus, as in [Stacks, 0BMB], implies that $\mathbf{Inc}'_r \to (q;\mathbf{a})$-$\mathbf{tics}_{\mathbf{P}V}$ is finite étale; the target is a multi-projective space and so it is simply connected, thus this is an isomorphism, and properties of the Stein factorization imply that pr_2 has connected fibres. In other words, $\mathbf{F}_r(X)$ is connected for every X. ■

When $\mathbf{F}_r(X)$ is of its expected dimension $\delta(n,\mathbf{a},r)$, 3.1 shows that it is cut out in $\mathbf{G}$ by a regular section of $\mathrm{S}^{\mathbf{a}}(\mathscr{S}^\vee)$. Various simple numerical invariants of the Fano scheme may then be determined via Schubert calculus, as is done in [DM98, §§3–4] for classical complete intersections, and [Che25b, 1.13–1.15] and [Che24, 1.11–1.13] in the q-bic case. For now, record the fact that the dualizing sheaf $\omega_{\mathbf{F}_r(X)}$ is a power of the Plücker line bundle $\mathscr{O}_{\mathbf{F}_r(X)}(1)$:

3.12. Proposition. — *If $X \subseteq \mathbf{P}^n$ is a $(q;\mathbf{a})$-tic scheme such that $\dim \mathbf{F}_r(X) = \delta(n,\mathbf{a},r)$, then*

$$\omega_{\mathbf{F}_r(X)} \cong \mathscr{O}_{\mathbf{F}_r(X)}(\gamma(\mathbf{a},r,q) - n - 1) \;\; \textit{where} \;\; \gamma(\mathbf{a},r,q) := \frac{1}{r+1}\sum\nolimits_{a \in \mathbf{a}} a(q) \cdot \prod\nolimits_{j \geq 0} \binom{a_j + r}{r}.$$

Proof. Duality theory, as in [Stacks, 0AU3], shows that the dualizing sheaf is given in this case by $\omega_{\mathbf{F}_r(X)} \cong \omega_{\mathbf{G}}|_{\mathbf{F}_r(X)} \otimes \det \mathrm{S}^{\mathbf{a}}(\mathscr{S}^\vee)$. Tensor product formulae show that, for a profile $a = \sum_{j \geq 0} a_j t^j$,

$$\det \mathrm{S}^a(\mathscr{S}^\vee) = \det\Big(\bigotimes\nolimits_{j \geq 0} \mathrm{Sym}^{a_j}(\mathscr{S}^\vee)^{[j]}\Big) \cong \bigotimes\nolimits_{j \geq 0} \det\big(\mathrm{Sym}^{a_j}(\mathscr{S}^\vee)\big)^{\otimes q^j \prod_{k \geq 0}\binom{a_k + r}{r}/\binom{a_j+r}{r}}.$$

Combined with fact that $\det \mathrm{Sym}^{a_j}(\mathscr{S}^\vee)$ is the $\binom{a_j+r}{r+1}$-th power of $\mathscr{O}_{\mathbf{F}_r(X)}(1)$ gives the result. ■

3.13. r-planes through a point. — Let $X \subseteq \mathbf{P}V$ be a $(q;\mathbf{a})$-tic scheme and consider the scheme

$$\mathbf{F}_r(X,x) := \{[\mathbf{P}U] \in \mathbf{F}_r(X) : x \in \mathbf{P}U \subseteq X\}$$

parameterizing r-planes in X through a given closed point x. As usual, this may be canonically identified as the Fano scheme of $(r-1)$-planes of a scheme $X_{1,x} \subseteq \mathbf{P}(V/L)$, where L is the 1-dimensional space underlying x and $X_{1,x} := \mathbf{F}_1(X,x)$ is the scheme of lines in X through x. A classical fact, see [HRS04, §2] for example, is that if $X \subseteq \mathbf{P}V$ is a scheme defined by equations of multi-degree $\mathbf{d} = (d_1, d_2, \ldots, d_c)$, then $X_{1,x} \subseteq \mathbf{P}(V/L)$ is defined by equations of multi-degree

$$\mathbf{d}_1 := (d_1, d_1 - 1, \ldots, 2, 1;\ d_2, d_2 - 1, \ldots, 2, 1; \ldots;\ d_c, d_c - 1, \ldots, 2, 1).$$

A generalization of this to $(q;\mathbf{a})$-tic schemes is as follows:

3.14. Proposition. — *Let $X \subseteq \mathbf{P}V \cong \mathbf{P}^n$ be a $(q;\mathbf{a})$-tic scheme. For every closed point $x \in X$, the scheme $X_{1,x} = \mathbf{F}_1(X,x)$ of lines in X through x is a $(q;\mathbf{a}_1)$-tic scheme in $\mathbf{P}^{n-1}$, where*

$$\mathbf{a}_1 := (b \in \mathbf{Prfl} : 0 \prec b \preceq a \textit{ with } a \in \mathbf{a}).$$

Proof. It is illustrative to work slightly more globally and to describe the scheme

$$X_1 := \{(x, [\ell]) \in X \times \mathbf{F}_1(X) : x \in \ell \subseteq X\}$$

of pointed lines locally relative to X. Choose projective coordinates $\mathbf{x} := (x_0 : \cdots : x_n)$ and defining $(q;a)$-tic equations $X = \mathrm{V}(f_a : a \in \mathbf{a})$. Over the standard affine open subscheme $\mathrm{D}(x_n)$, the space of pointed lines in $\mathbf{P}^n$ restricts to a trivial $\mathbf{P}^{n-1}$-bundle, and fibre coordinates $\mathbf{y} := (y_0 : \cdots : y_{n-1})$ may be chosen so that the point $(\mathbf{x},\mathbf{y}) \in \mathrm{D}(x_n) \times \mathbf{P}^{n-1}$ represents the line ℓ parameterized by

$$\varphi : \mathbf{P}^1 \to \mathbf{P}^n : (\xi : \eta) \mapsto \xi\mathbf{x} + \eta\mathbf{y} := (\xi x_0 + \eta y_0 : \cdots : \xi x_{n-1} + \eta y_{n-1} : \xi x_n).$$

Then $\ell \subseteq X$ if and only if $\varphi^*(f_a) = f_a(\xi\mathbf{x}+\eta\mathbf{y}) = 0$ for each $a \in \mathbf{a}$. View $f_a(\xi\mathbf{x}+\eta\mathbf{y})$ as a polynomial in the auxiliary variables $(\xi : \eta)$; its coefficients are polynomials $f_{a,b}(\mathbf{x};\mathbf{y})$ which provide the equations for X_1 in $\mathrm{D}(x_n) \times \mathbf{P}^{n-1}$, and have the following form:

3.15. Lemma. — *Let $f_a(x_0,\ldots,x_n)$ be a $(q;a)$-tic polynomial. Then there is a unique expansion*

$$f_a(\xi x_0 + \eta y_0, \ldots, \xi x_{n-1} + \eta y_{n-1}, \xi x_n) = \sum\nolimits_{0 \preceq b \preceq a} f_{a,b}(x_0,\ldots,x_n; y_0,\ldots,y_{n-1}) \cdot \xi^{a(q)-b(q)}\eta^{b(q)}$$

where the polynomials $f_{a,b}(x_0,\ldots,x_n;y_0,\ldots,y_{n-1})$ are homogeneous of bi-profile $(a-b,b)$.

Proof. When the profile $a = a_0$ is a constant, this is classical: simply group terms with respect to the monomials in ξ and η. For a general profile $a = a_0 + a_1 t + \cdots + a_m t^m$, note that $f_a(\xi\mathbf{x}+\eta\mathbf{y})$ is a sum of polynomials of the form

$$\prod\nolimits_{j=0}^{m} f_{a_j}(\xi\mathbf{x}+\eta\mathbf{y})^{q^j} = \prod\nolimits_{j=0}^{m}\Big(\sum\nolimits_{b_j=0}^{a_j} f_{a_j,b_j}(\mathbf{x};\mathbf{y}) \cdot \xi^{a_j-b_j}\eta^{b_j}\Big)^{q^j}$$

where the f_{a_j} are homogeneous of degree a_j, and f_{a_j,b_j} is bihomogeneous of bidegree $(a_j - b_j, b_j)$. The product expands to a sum of terms of the form

$$\Big(\prod\nolimits_{j=0}^{m} f_{a_j,b_j}(\mathbf{x};\mathbf{y})^{q^j}\Big) \cdot \xi^{a(q)-b(q)}\eta^{b(q)}$$

which is a polynomial with profile $b := b_0 + b_1 t + \cdots + b_m t^m$ in $\mathbf{y}$, and profile $a - b$ in $\mathbf{x}$. Injectivity of the multiplication maps implies via 1.5 that any coefficient of $\xi^{a(q)-b(q)}\eta^{b(q)}$ is of bi-profile $(a-b,b)$, from which the result follows. ■

To complete the proof of 3.14, observe that the bi-profile $(a,0)$ terms in 3.15 are $f_{a,0}(\mathbf{x},\mathbf{y}) = f_a(\mathbf{x})$ simply the equations of X. Therefore, over $X \cap \mathrm{D}(x_n)$, the polynomials $f_{a,b}(\mathbf{x};\mathbf{y})$ with $0 \prec b \preceq a$ and $a \in \mathbf{a}$ present X_1 as a $(q;\mathbf{a}_1)$-tic scheme in a projective $(n-1)$-space. ■

Writing $\mathbf{F}_r(X,x) \cong \mathbf{F}_{r-1}(X_{1,x})$ and combining 3.14 with 3.1 and 3.2 shows that a $(q;\mathbf{a})$-tic scheme is covered by r-planes as soon as $\delta\_(n-1,\mathbf{a}_1,r-1) \geq 0$. A simple criterion for $\delta(n-1,\mathbf{a}_1,r-1) \geq 0$ may be obtained by using the identity

$$1 + \sum\nolimits_{0 \prec b \preceq a}\prod\nolimits_{j\geq 0}\binom{b_j + r - 1}{r-1} = \prod\nolimits_{j \geq 0}\Big(\sum\nolimits_{b_j=0}^{a_j}\binom{b_j+r-1}{r-1}\Big) = \prod\nolimits_{j\geq 0}\binom{a_j + r}{r}.$$

3.16. Corollary. — *Let $X \subseteq \mathbf{P}V$ be a $(q;\mathbf{a})$-tic scheme. If*

$$n \geq \max\Big\{2r - 1 + \#\mathbf{a}_1, r + \frac{1}{r}\sum\nolimits_{a\in\mathbf{a}}\prod\nolimits_{j\geq 0}\binom{a_j+r}{r} - \frac{1}{r}\#\mathbf{a}\Big\},$$

then X is covered by r-planes. ■

A more global version of 3.14 will be given in §5 and will feature in the unirationality construction in Theorem B. The next section prepares for this by clarifying what a family of $(q;\mathbf{a})$-tic schemes ought to be and by developing some tools for manipulating such families.

4. Families

Consider a family $\mathcal{X} \to S$ of complete intersections in a projective bundle $\pi\colon \mathbf{P}\mathcal{V} \to S$ over the field $\mathbf{k}$ which is cut out by a regular section $\sigma\colon \mathcal{O}_{\mathbf{P}\mathcal{V}} \to \mathcal{E}$ of a finite locally free $\mathcal{O}_{\mathbf{P}\mathcal{V}}$-module. What additional structure should be required to elevate $\mathcal{X}$ into a family of $(q;\mathbf{a})$-tic complete intersections? As a minimum, each fibre $\mathcal{X}_s$ ought to be a $(q;\mathbf{a})$-tic complete intersection in $\mathbf{P}\mathcal{V}_s$, meaning as in 2.1 that the equations σ_s are induced by a $(q;\mathbf{a})$-tic tensor $\boldsymbol{\alpha}_s$. This tensor is a crucial part of the structure that defines a $(q;\mathbf{a})$-tic complete intersection, so one ought to ask that the $\boldsymbol{\alpha}_s$ vary continuously across the family. The following examples illustrate some of the subtleties involved:

4.1. Example. — Fix a nonconstant profile $a := a_0 + a_1 t + \cdots + a_m t^m \in \mathbf{Prfl}$, let $S := \mathbf{A}^2$ be the affine plane with coordinates (s_1, s_2), and choose homogeneous polynomials $f_1, f_2, g \in \mathcal{O}_S[x_0, \ldots, x_n]$, where f_1 and f_2 are general coprime $(q;a)$-tic polynomials and where g is simply general of degree $d := a(q)$. Consider the closed subscheme of $\mathbf{P}^n_S$ defined by the section

$$\sigma = (f_1 + s_1 g, f_2 + s_2 g)^\vee \colon \mathcal{O}_{\mathbf{P}^n_S} \to \mathcal{O}_{\mathbf{P}^n_S}(d)^{\oplus 2}.$$

Setting $\mathbf{a} := (d, a_0 + a_1 t + \cdots + a_m t^m)$, then $\mathcal{X}$ is a family of codimension 2 complete intersections in $\mathbf{P}^n_S$ with the property that each fibre is a $(q;\mathbf{a})$-tic complete intersection, but there is no continuously varying family of $(q;\mathbf{a})$-tic tensor defining $\mathcal{X}$ in a neighbourhood of the origin of S.

Proof. That each fibre is a $(q;\mathbf{a})$-tic complete intersection is straightforward: At a point s where $s_i \neq 0$, $\mathcal{X}_s$ is cut out by the equations $f_i + s_i g = s_1 f_2 - s_2 f_1 = 0$. Over the origin, $\mathcal{X}_0$ is cut out by $f_1 = f_2 = 0$, and either choice of f_i being considered as a $(q;a)$-tic equation suffices. Suppose now that $U \subseteq S$ is a neighbourhood of 0 over which $\mathcal{X}|_U$ is defined in $\mathbf{P}^n_U$ by a family of $(q;\mathbf{a})$-tic tensors $\boldsymbol{\alpha}(s)$. The $(q;a)$-tic component of the tensor takes the form

$$\boldsymbol{\alpha}(s)_a = x f_1 + y f_2 \text{ where } x, y \in \Gamma(U, \mathcal{O}_S) \text{ satisfies } x s_1 + y s_2 = 0.$$

Thus y is divisible by s_1 and x is divisible by s_2, and so $\boldsymbol{\alpha}(s)_a$ vanishes at the origin, contradicting the assumption that it provides the $(q;a)$-tic equation of $\mathcal{X}$ over all of U. ■

Excising the origin from the base ensures that a tensor does glue together to become a section of a non-split vector bundle:

4.2. Example. — Take the base $S := \mathbf{A}^2 \setminus \{0\}$ and continue with the example $\mathcal{X} \subset \mathbf{P}^n_S$ of 4.1. Consider the standard affine open cover given by the complement $U_i := \mathrm{D}(s_i)$ of the s_i-axis for $i = 1, 2$. Then

$$\mathcal{X}|_{U_i} = \mathrm{V}(f_i + s_i g, s_1 f_2 - s_2 f_1) \subset \mathbf{P}^n_{U_i}$$

is a presentation of $\mathcal{X}$ as a $(q;\mathbf{a})$-tic complete intersection over U_i. This presentation globalizes over S in the following sense: Write $\mathbf{P}^n_S = \mathbf{P}(\mathcal{O}_S \otimes V)$ for a vector space V, and consider for $i = 1, 2$ the map of locally free $\mathcal{O}_{U_i}$-modules

$$\boldsymbol{\alpha}_i = (f_i + s_i g, s_1 f_2 - s_2 f_1)^\vee \colon \mathcal{O}_{U_i} \to \big(\mathcal{O}_{U_i} \otimes \mathrm{Sym}^d(V^\vee)\big) \oplus \big(\mathcal{O}_{U_i} \otimes \mathrm{S}^a(V^\vee)\big).$$

These glue via the automorphism on the intersection $U_{1,2} := U_1 \cap U_2$ given by

$$\varphi_{1,2} := \begin{pmatrix} s_2/s_1 & 1/s_1 \\ 0 & 1 \end{pmatrix} \in \mathrm{Aut}\Big(\big(\mathcal{O}_{U_{1,2}} \otimes \mathrm{Sym}^d(V^\vee)\big) \oplus \big(\mathcal{O}_{U_{1,2}} \otimes \mathrm{S}^a(V^\vee)\big)\Big).$$

Thus there is a global section $\boldsymbol{\alpha}\colon \mathcal{O}_S \to \mathscr{A}$ such that $\boldsymbol{\alpha}|_{U_i} = \boldsymbol{\alpha}_i$, where $\mathscr{A}$ fits in an extension

$$0 \to \mathcal{O}_S \otimes \mathrm{Sym}^d(V^\vee) \to \mathscr{A} \to \mathcal{O}_S \otimes \mathrm{S}^a(V^\vee) \to 0.$$

This tensor underlies $\mathscr{X} \subset \mathbf{P}^n_S$ in the sense that the section σ defining $\mathscr{X}$ from 4.1 factors as

$$\sigma = \varepsilon \circ \pi^*\boldsymbol{\alpha} \colon \mathscr{O}_{\mathbf{P}^n_S} \to \pi^*\mathscr{A} \to \mathscr{O}_{\mathbf{P}^n_S}(d)^{\oplus 2}$$

where ε is the surjective morphism of $\mathscr{O}_{\mathbf{P}^n_S}$-modules locally induced by the maps

$$\pi_*\varepsilon|_{U_1} = \begin{pmatrix} 1 & 0 \\ s_2/s_1 & 1/s_1 \end{pmatrix} \text{ and } \pi_*\varepsilon|_{U_2} = \begin{pmatrix} s_1/s_2 & -1/s_2 \\ 1 & 0 \end{pmatrix}$$

between $(\mathscr{O}_{U_i} \otimes_{\mathbf{k}} \mathrm{Sym}^d(V^\vee)) \oplus (\mathscr{O}_{U_i} \otimes_{\mathbf{k}} \mathrm{S}^a(V^\vee)) \to \mathscr{O}_{U_i} \otimes_{\mathbf{k}} \mathrm{Sym}^d(V^\vee)^{\oplus 2}$. Moreover, there is a change of fibre coordinates of $\mathbf{P}^n_S$ in which ε is in fact locally induced by the evaluation map of $\pi^*\pi_*\mathscr{O}_{\mathbf{P}^n_S}(d)$. Despite this, the extension in which $\mathscr{A}$ fits is nontrivial: Otherwise, projecting $\boldsymbol{\alpha}$ to the Sym^d-component provides a polynomial h cutting out $\mathscr{X}$ along with the equation $s_1f_2 - s_2f_1$. Comparing generators of the ideal of $\mathscr{X}$ shows that $h = x(f_1 + s_1g) + y(f_2 + s_2g)$ for some

$$x, y \in \Gamma(S, \mathscr{O}_S) = \mathbf{k}[s_1, s_2] \text{ satisfying } xs_1 + ys_2 \in \Gamma(S, \mathscr{O}_S)^\times = \mathbf{k}^\times.$$

This is impossible, and so the extension is non-split. ■

Toward a definition of a family of $(q;\mathbf{a})$-tic schemes, suppose one is given a section of the form

$$\boldsymbol{\alpha} \colon \mathscr{O}_S \to \bigoplus\nolimits_{a \in \mathbf{a}} \mathrm{S}^a(\mathscr{V}^\vee).$$

Adjunction along $\pi \colon \mathbf{P}\mathscr{V} \to S$ together with the relative evaluation maps induce a canonical map

$$\sigma := \varepsilon \circ \pi^*\boldsymbol{\alpha} \colon \mathscr{O}_{\mathbf{P}\mathscr{V}} \to \bigoplus\nolimits_{a \in \mathbf{a}} \pi^*\mathrm{S}^a(\mathscr{V}^\vee) \to \bigoplus\nolimits_{a \in \mathbf{a}} \mathscr{O}_\pi(a(q))$$

whose zero locus $\mathscr{X}$ ought to be called a family of $(q;\mathbf{a})$-tic schemes over S. However, there are families like those in 4.2 whose defining section takes values in a vector bundle which is only locally of this form. Additional data is therefore necessary to globalize this construction. The solution taken here is to require a map ε globalizing the evaluation map.

4.3. Definitions. — A *family of* $(q;\mathbf{a})$*-tic tensors* valued in a finite locally free $\mathscr{O}_S$-module $\mathscr{V}$ is a section $\boldsymbol{\alpha} \colon \mathscr{O}_S \to \mathscr{A}$, where $\mathscr{A}$ is locally on S isomorphic to

$$\bigoplus\nolimits_{a \in \mathbf{a}} \mathrm{S}^a(\mathscr{V}^\vee).$$

A *family of* $(q;\mathbf{a})$*-tic schemes* in a projective bundle $\pi \colon \mathbf{P}\mathscr{V} \to S$ consists of the data of

- a closed subscheme $\mathscr{X} \subseteq \mathbf{P}\mathscr{V}$ cut out by a section $\sigma \colon \mathscr{O}_{\mathbf{P}\mathscr{V}} \to \mathscr{E}$ of a vector bundle;
- a family of $(q;\mathbf{a})$-tic tensors $\boldsymbol{\alpha} \colon \mathscr{O}_S \to \mathscr{A}$; and
- a surjective morphism $\varepsilon \colon \pi^*\mathscr{A} \to \mathscr{E}$ of $\mathscr{O}_{\mathbf{P}\mathscr{V}}$-modules.

These data are subject to the conditions that:

(i) there is a factorization $\sigma = \varepsilon \circ \pi^*\boldsymbol{\alpha} \colon \mathscr{O}_{\mathbf{P}\mathscr{V}} \to \pi^*\mathscr{A} \to \mathscr{E}$; and
(ii) there exists an open cover $S = \bigcup_{i \in I} U_i$ on which the morphism $\varepsilon \colon \pi^*\mathscr{A} \to \mathscr{E}$ is isomorphic to the canonical map induced by evaluation along π:

$$\mathrm{ev}_\pi \colon \bigoplus\nolimits_{a \in \mathbf{a}} \pi^*\mathrm{S}^a(\mathscr{V}^\vee) \to \bigoplus\nolimits_{a \in \mathbf{a}} \mathscr{O}_\pi(a(q)).$$

When σ is a regular section making $\mathscr{X} \to S$ fibrewise a complete intersection, furthermore call the triple $(\mathscr{X}, \boldsymbol{\alpha}, \varepsilon)$ a *family of* $(q;\mathbf{a})$*-tic complete intersections*. The family $\mathscr{X} \to S$ is often referred to as the family of $(q;\mathbf{a})$-tic schemes or complete intersections, leaving the $\boldsymbol{\alpha}$ and ε implicit.

4.4. Example. — Since $(q;\mathbf{a})$-tic structures are inherited upon passing to linear sections by 2.3, families of linear sections of $(q;\mathbf{a})$-tic schemes provide a source of non-trivial examples. To give a simple illustration, consider linear projection of a projective space $\mathbf{P}V$ centred along a subspace $\mathbf{P}U$. This provides a rational map to $\mathbf{P}(V/U)$ which is resolved on the blowup $b\colon \widetilde{\mathbf{P}}V \to \mathbf{P}V$ centred along $\mathbf{P}U$; as usual, b exhibits $\widetilde{\mathbf{P}}V$ as the projective bundle over $\mathbf{P}(V/U)$ whose underlying vector bundle $\mathscr{V}$ canonically arises via the diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{O}_{\mathbf{P}(V/U)} \otimes U \arrow[r] \arrow[d, equal] & \mathscr{V} \arrow[r] \arrow[d] & \mathscr{O}_{\mathbf{P}(V/U)}(-1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{O}_{\mathbf{P}(V/U)} \otimes U \arrow[r] & \mathscr{O}_{\mathbf{P}(V/U)} \otimes V \arrow[r] & \mathscr{O}_{\mathbf{P}(V/U)} \otimes V/U \arrow[r] & 0\,.
\end{tikzcd}$$

If $X \subseteq \mathbf{P}V$ is a $(q;\mathbf{a})$-tic scheme not contained in $\mathbf{P}U$, then its *total transform* $\mathscr{X} := b^{-1}(X)$ in $\widetilde{\mathbf{P}}V$ is a family of $(q;\mathbf{a})$-tic schemes over $S := \mathbf{P}(V/U)$: Dualizing the inclusion $\mathscr{V} \subseteq \mathscr{O}_{\mathbf{P}(V/U)} \otimes V$ and applying the tensor functor $\mathrm{S}^{\mathbf{a}}$ provides a restriction map

$$\mathscr{O}_{\mathbf{P}(V/U)} \otimes \mathrm{S}^{\mathbf{a}}(V^\vee) \to \mathrm{S}^{\mathbf{a}}(\mathscr{V}^\vee).$$

Mapping a $(q;\mathbf{a})$-tic tensor defining X along this provides a family of $(q;\mathbf{a})$-tic tensors $\boldsymbol{\alpha}$ defining $\mathscr{X}$.

4.5. — If, furthermore, $\mathbf{P}U \subseteq X$, then its *strict transform* $\widetilde{X}$ in $\widetilde{\mathbf{P}}V$ generally does *not* appear to carry useful additional structure from the point of view of this article. For instance, consider the smooth q-bic surface

$$X := \{(x_0 : x_1 : x_2 : x_3) \in \mathbf{P}^3 : x_0^q x_1 + x_0 x_1^q + x_2^q x_3 + x_2 x_3^q = 0\}.$$

Projection from the line $\mathbf{P}U = (0 : x_1 : 0 : x_3)$ exhibits the strict transform $\widetilde{X}$ as a family of degree q plane curves, the general fibre of which is isomorphic to

$$C := \{(y_0 : y_1 : y_2) \in \mathbf{P}^2 : y_0^q + y_1 y_2^{q-1} = 0\},$$

see [Che22, 2.5.3] for instance. In general, this equation does not belong to any proper subspace of $\mathrm{Sym}^q(\mathbf{k}^{\oplus 3})$ of the form $\mathrm{S}^a(\mathbf{k}^{\oplus 3})$ for $a \in \mathbf{Prfl}$ of numerical degree q. This perhaps suggests that the class of $(q;\mathbf{a})$-tic schemes is not sufficiently flexible. It would be useful to develop methods to handle a larger class of schemes that includes irreducible components of $(q;\mathbf{a})$-tic schemes.

4.6. Transition functions. — Injectivity of multiplication maps associated with profiles from 1.1 together with condition 4.3(ii) implies that the map $\pi_*\varepsilon\colon \mathscr{A} \to \pi_*\mathscr{E}$ is an injection which locally on S is isomorphic to the inclusion of $(q;a)$-tic polynomials amongst all polynomials of degree $a(q)$, with a ranging over profiles in $\mathbf{a}$. This endows $\mathscr{A}$ with some additional structure: Choose an open covering $S = \bigcup_{i \in I} U_i$ and trivializations of $\mathscr{A}$ and $\pi_*\mathscr{E}$ so that $\pi_*\varepsilon|_{U_i}$ is the identified with the inclusion

$$\textstyle\bigoplus_{a \in \mathbf{a}} \mathrm{S}^a(\mathscr{V}^\vee)|_{U_i} \subseteq \bigoplus_{a \in \mathbf{a}} \mathrm{Sym}^{a(q)}(\mathscr{V}^\vee)|_{U_i}.$$

For each pair of indices $i, j \in I$, let $\psi_{i,j}$ and $\varphi_{i,j}$ be the transition functions of $\mathscr{A}$ and $\pi_*\mathscr{E}$ with these trivializations over $U_{i,j} := U_i \cap U_j$. Since $\mathscr{E}$ is locally a sum of the $\mathscr{O}_\pi(a(q))$, the $\varphi_{i,j}$ are induced by multiplication of polynomials; in particular, for $a, b \in \mathbf{a}$, the (a,b)-component

$$(\varphi_{i,j})_{a,b}\colon \mathrm{Sym}^{a(q)}(\mathscr{V}^\vee)|_{U_{i,j}} \to \mathrm{Sym}^{b(q)}(\mathscr{V}^\vee)|_{U_{i,j}}$$

is induced by multiplication by a polynomial $f_{i,j,a,b}$ of degree $b(q) - a(q)$. Since $\mathscr{A}$ is identified as a subbundle of $\pi_*\mathscr{E}$, this means that the map

$$(\psi_{i,j})_{a,b}\colon \mathrm{S}^a(\mathscr{V}^\vee)|_{U_{i,j}} \to \mathrm{S}^b(\mathscr{V}^\vee)|_{U_{i,j}}$$

is also induced by multiplication by the same polynomial $f_{i,j,a,b}$. Keeping track of profiles shows that whether or not $(\psi_{i,j})_{a,b}$ must vanish is related to the composite partial ordering $\rightsquigarrow$ from 1.4:

4.7. Lemma. — *If $(\psi_{i,j})_{a,b} \neq 0$, then $a \rightsquigarrow b$.* ■

As a simple though useful consequence, this provides a $(q;\mathbf{a})$-tic tensor $\boldsymbol{\alpha}\colon \mathcal{O}_S \to \mathcal{A}$ with a canonical decomposition into types. Namely, partition the collection $\mathbf{a}$ of profiles into

$$\mathbf{a}_{\mathrm{lin}} := \{a \in \mathbf{a} : a(t) = 1\},\ \ \mathbf{a}_{\mathrm{pow}} := \{a \in \mathbf{a} : a(0) = 0\},\ \ \mathbf{a}_{\mathrm{nlr}} := \{a \in \mathbf{a} : a(t) \neq 1 \text{ and } a(0) = 0\}$$

those that are linear, those that are nonreduced, and those that are non-linear and reduced. The linear and nonreduced components of $\mathcal{A}$ give two canonical quotients:

4.8. Lemma. — *Let $\mathcal{X}$ be a family of $(q;\mathbf{a})$-tic schemes in a projective bundle $\pi\colon \mathbf{P}\mathcal{V} \to S$. There is a canonical short exact sequence of locally free $\mathcal{O}_S$-modules*

$$0 \to \mathcal{A}_{\mathrm{nlr}} \to \mathcal{A} \to \mathcal{A}_{\mathrm{lin}} \oplus \mathcal{A}_{\mathrm{pow}} \to 0$$

where, for each type $\in \{\mathrm{nlr}, \mathrm{lin}, \mathrm{pow}\}$*, $\mathcal{A}_{\mathrm{type}}$ is locally on S isomorphic to $\bigoplus_{a \in \mathbf{a}_{\mathrm{type}}} \mathrm{S}^a(\mathcal{V}^\vee)$. An analogous and compatible exact sequence exists for $\mathcal{E}$.*

Proof. From 4.7 together with 1.3(i), it follows that $(\psi_{i,j})_{a,1} = 0$ for each $a \in \mathbf{a} \setminus \mathbf{a}_{\mathrm{lin}}$, meaning that the local summands of the form $\mathcal{V}^\vee$ fit together as a quotient bundle $\mathcal{A} \to \mathcal{A}_{\mathrm{lin}}$. Similarly, if $a \in \mathbf{a} \setminus \mathbf{a}_{\mathrm{pow}}$ and $b \in \mathbf{a}_{\mathrm{pow}}$, then $(\psi_{i,j})_{a,b} = 0$ since profiles preceding a nonreduced profile must also be nonreduced as observed in 1.3(iv). Hence the local summands indexed by $\mathbf{a}_{\mathrm{pow}}$ also fit together to form a quotient $\mathcal{A} \to \mathcal{A}_{\mathrm{pow}}$. The latter argument also implies that all possible transition functions between $\mathcal{A}_{\mathrm{lin}}$ and $\mathcal{A}_{\mathrm{pow}}$ must vanish, from which the remaining conclusions follow. ■

Dually, 4.7 implies that equations whose profile is maximal for $\rightsquigarrow$ in $\mathbf{a}$ fit together into a subbundle of $\mathcal{A}$. One such class of maximal profiles are those $a \in \mathbf{a}$ with maximal coefficient sum $a(1)$: see 1.3(ii). This is most useful when restricted to $\mathcal{A}_{\mathrm{nlr}} \neq 0$:

4.9. Lemma. — *Let $\mathcal{X}$ be a family of $(q;\mathbf{a})$-tic schemes in a projective bundle $\pi\colon \mathbf{P}\mathcal{V} \to S$. If $\mathbf{a}_{\mathrm{nlr}} \neq \varnothing$, then there exists a nonzero subbundle of the form*

$$\mathrm{S}^a(\mathcal{V}^\vee) \otimes \mathcal{M} \subseteq \mathcal{A}_{\mathrm{nlr}} \subseteq \mathcal{A}$$

for some locally free $\mathcal{O}_S$-module $\mathcal{M}$ and some $a \in \mathbf{a}_{\mathrm{nlr}}$ with maximal coefficient sum $a(1)$. ■

The canonical *type decomposition* from 4.8 of a $(q;\mathbf{a})$-tic tensor allows one to sometimes perform certain simplifications to the equations defining the family $\mathcal{X}$. For instance—a construction which is of course much more generally applicable to any family of projective schemes—the linear equations may be used to cut out a projective subbundle containing $\mathcal{X}$, thereby reducing the number of equations to keep track of. Precisely:

4.10. Lemma. — *Let $\mathcal{X}$ be a family of $(q;\mathbf{a})$-tic schemes in a projective bundle $\pi\colon \mathbf{P}\mathcal{V} \to S$. There exists a canonical subbundle $\mathbf{P}\mathcal{V}' \subseteq \mathbf{P}\mathcal{V}$ containing $\mathcal{X}$ in which it is a family of $(q;\mathbf{a} \setminus \mathbf{a}_{\mathrm{lin}})$-tic schemes.*

Proof. Let $\lambda\colon \mathcal{E} \to \mathcal{E}_{\mathrm{lin}}$ be the quotient from 4.8 in which the linear equations of $\mathcal{X} \subseteq \mathbf{P}\mathcal{V}$ take values in, so that $\mathcal{E}_{\mathrm{lin}} \cong \mathcal{O}_\pi(1) \otimes \pi^*\mathcal{M}$ for some locally free $\mathcal{O}_S$-module $\mathcal{M}$ of rank $\#\mathbf{a}_{\mathrm{lin}}$. The subbundle $\mathbf{P}\mathcal{V}' \subseteq \mathbf{P}\mathcal{V}$ defined by $\sigma \circ \lambda$ then contains $\mathcal{X}$, in which it is a complete intersection cut out by the induced section $\sigma'\colon \mathcal{O}_{\mathbf{P}\mathcal{V}'} \to \mathcal{E}'$ valued in the restriction of $\ker(\lambda)$ to $\mathbf{P}\mathcal{V}'$.

To provide $\mathcal{X}$ with the structure of a family of $(q;\mathbf{a} \setminus \mathbf{a}_{\mathrm{lin}})$-tic schemes in $\mathbf{P}\mathcal{V}'$, begin with its $(q;\mathbf{a})$-tic tensor $\boldsymbol{\alpha}\colon \mathcal{O}_S \to \mathcal{A}$ and evaluation map $\varepsilon\colon \pi^*\mathcal{A} \to \mathcal{E}$ with respect to $\mathbf{P}\mathcal{V}$. Since ε is locally given by evaluation along π, the kernel

$$\mathcal{A}_0 := \ker(\pi_*(\lambda \circ \varepsilon)\colon \mathcal{A} \to \mathcal{V}^\vee \otimes \mathcal{M})$$

is a subbundle of $\mathscr{A}$ with the property that ε restricted to $\pi^*\mathscr{A}_0$ factors through $\ker(\lambda)$. Compose this with the restriction map $\ker(\lambda) \to \mathscr{E}'$, push along π, and consider the $\mathscr{O}_S$-module

$$\mathscr{A}' := \mathrm{image}(\mathscr{A}_0 \to \pi_* \ker(\lambda) \to \pi_*\mathscr{E}').$$

Locally on S, the morphism defining the image is the composite

$$\bigoplus\nolimits_{a \in \mathbf{a} \setminus \mathbf{a}_{\mathrm{lin}}} \mathrm{S}^a(\mathscr{V}^\vee) \subseteq \bigoplus\nolimits_{a \in \mathbf{a} \setminus \mathbf{a}_{\mathrm{lin}}} \mathrm{Sym}^{a(q)}(\mathscr{V}^\vee) \to \bigoplus\nolimits_{a \in \mathbf{a} \setminus \mathbf{a}_{\mathrm{lin}}} \mathrm{Sym}^{a(q)}(\mathscr{V}'^\vee)$$

of inclusion $(q;a)$-tic subbundles followed by the restriction maps from $\mathscr{V}$ to $\mathscr{V}'$, which implies that $\mathscr{A}'$ is locally a sum of the $(q;a)$-tic bundles $\mathrm{S}^a(\mathscr{V}'^\vee)$. To construct a $(q;\mathbf{a}')$-tic tensor valued in $\mathscr{A}'$, consider the commutative diagram

$$\begin{array}{ccccc} \mathscr{A}_0 & \longrightarrow & \mathscr{A}' & \hookrightarrow & \pi_*\mathscr{E}' \\ \cap\!| & & & & \cap\!| \\ \mathscr{A} & \hookrightarrow & \pi_*\mathscr{E} & \longrightarrow & \pi_*(\mathscr{E}|_{\mathbf{P}\mathscr{V}'}). \end{array}$$

Composing the bottom row with $\boldsymbol{\alpha}\colon \mathscr{O}_S \to \mathscr{A}$ yields the map $\pi_*(\sigma|_{\mathbf{P}\mathscr{V}'})$, which has vanishing linear components, and so it factors through $\pi_*\mathscr{E}'$. Since $\mathscr{A}_0$ and $\mathscr{A}$ have the same image in $\pi_*(\mathscr{E}|_{\mathbf{P}\mathscr{V}'})$, the commutative diagram implies that this then lifts to the required map $\boldsymbol{\alpha}'\colon \mathscr{O}_S \to \mathscr{A}'$. ■

Perhaps a more interesting simplification is possible when the collection of profiles $\mathbf{a} = \mathbf{a}_{\mathrm{pow}}$ consists only of nonreduced profiles, meaning that each equation of $\mathscr{X}$ over S is, geometrically, a q-power. Upon adjoining suitable roots of the coefficients, it is possible to take q-th roots of all the equations of $\mathscr{X}$ to obtain a scheme $\mathscr{X}'$ with multi-profile $\mathbf{a}/t := (a/t : a \in \mathbf{a})$:

4.11. Lemma. — *Let $\mathscr{X}$ be a family of $(q;\mathbf{a})$-tic schemes with $\mathbf{a} = \mathbf{a}_{\mathrm{pow}}$. Then there exists a family of $(q;\mathbf{a}/t)$-tic schemes $\mathscr{X}'$ and a universal homeomorphism $\mathscr{X}' \to \mathscr{X}$ fitting in the commutative square*

$$\begin{array}{ccc} \mathscr{X}' & \longrightarrow & \mathscr{X} \\ \big\downarrow & & \big\downarrow \\ S & \xrightarrow{\mathrm{Fr}} & S. \end{array}$$

Proof. Write $\mathrm{Fr}_{\mathbf{P}\mathscr{V}/S}\colon \mathbf{P}\mathscr{V} \to \mathbf{P}\mathscr{V}^{[1]}$ for the S-linear relative q-power Frobenius morphism: the morphism which takes q-powers of the fibre coordinates and leaves coefficients fixed, induced by the commutative diagram

$$\begin{tikzcd} \mathbf{P}\mathscr{V} \arrow[r, "\mathrm{Fr}_{\mathbf{P}\mathscr{V}/S}"'] \arrow[rr, bend left, "\mathrm{Fr}"] \arrow[dr, "\pi"'] & \mathbf{P}\mathscr{V}^{[1]} \arrow[r] \arrow[d, "\pi^{[1]}"] & \mathbf{P}\mathscr{V} \arrow[d, "\pi"] \\ & S \arrow[r, "\mathrm{Fr}"] & S \end{tikzcd}$$

where the right hand square witnesses $\mathbf{P}\mathscr{V}^{[1]} = \mathbf{P}\mathscr{V} \times_{S,\mathrm{Fr}} S$. The main point now is that $\mathscr{E}$ admits a descent along $\mathrm{Fr}_{\mathbf{P}\mathscr{V}/S}$, meaning there is an $\mathscr{O}_{\mathbf{P}\mathscr{V}^{[1]}}$-module $\mathscr{E}'$ which pulls back to $\mathscr{E}$. To construct $\mathscr{E}'$, adopt the notation from 4.6 so that

$$\mathscr{E}|_{\pi^{-1}(U_i)} \cong \bigoplus\nolimits_{a \in \mathbf{a}} \mathscr{O}_\pi(a(q))|_{\pi^{-1}(U_i)}.$$

That $\mathbf{a} = \mathbf{a}_{\mathrm{pow}}$ means that each $\mathscr{O}_\pi(a(q))$ is the pullback of $\mathscr{O}_{\pi^{[1]}}(a(q)/q)$ along the relative Frobenius, and so $\mathscr{E}$ admits Frobenius descents locally over S. As explained in 4.6, the transition functions of $\mathscr{E}$ over $U_{i,j}$ have (a,b)-components induced by multiplication by a polynomial:

$$f_{i,j,a,b}\colon \mathscr{O}_\pi(a(q))|_{\pi^{-1}(U_{i.j})} \to \mathscr{O}_\pi(b(q))|_{\pi^{-1}(U_{i,j})}.$$

As in the proof of 4.8, multiplication by $f_{i,j,a,b}$ maps $\mathrm{S}^a(\mathscr{V}^\vee)$ to $\mathrm{S}^b(\mathscr{V}^\vee)$. Since a and b both have vanishing constant term, if $f_{i,j,a,b}$ is nonzero, it also must have profile with vanishing constant term;

in other words, each monomial appearing must be a q-power in the fibre coordinates. This means that the $f_{i,j,a,b}$ admit $U_{i,j}$-linear Frobenius descents $g_{i,j,a,b}$ to $\mathbf{P}\mathscr{V}^{[1]}|_{U_{i,j}}$. This glues the local Frobenius descents to give an $\mathscr{O}_{\mathbf{P}\mathscr{V}^{[1]}}$-module $\mathscr{E}'$ descending $\mathscr{E}$ along $\mathrm{Fr}_{\mathbf{P}\mathscr{V}/S}$.

Pushing the canonical adjunction map $\mathscr{E}' \to \mathrm{Fr}_{\mathbf{P}\mathscr{V}/S,*}\mathscr{E}$ along $\pi^{[1]}$ now yields an injection which is locally given by the inclusion

$$\bigoplus\nolimits_{a\in\mathbf{a}} \mathrm{Sym}^{a(q)/q}(\mathscr{V}^{[1],\vee}) \subseteq \bigoplus\nolimits_{a\in\mathbf{a}} \mathrm{Sym}^{a(q)}(\mathscr{V}^{\vee}).$$

Identifying the $\mathscr{O}_S$-module $\mathscr{A}$ in which the $(q;\mathbf{a})$-tic tensor $\boldsymbol{\alpha}$ takes values locally as

$$\bigoplus\nolimits_{a\in\mathbf{a}} \mathrm{S}^{a}(\mathscr{V}^{\vee}) \cong \bigoplus\nolimits_{a\in\mathbf{a}} \mathrm{S}^{a/t}(\mathscr{V}^{[1],\vee})$$

shows that the injection $\pi_*\varepsilon$ factors as

$$\pi_*\varepsilon\colon \mathscr{A} \to \pi^{[1]}_*\mathscr{E}' \to \pi_*\mathscr{E}.$$

Adjunction along $\pi^{[1]}$ thus provides a relative evaluation map $\varepsilon'\colon \pi^{[1],*}\mathscr{A} \to \mathscr{E}'$ and the pullback of $\boldsymbol{\alpha}$ produces a section $\sigma'\colon \mathscr{O}_{\mathbf{P}\mathscr{V}^{[1]}} \to \mathscr{E}'$: its vanishing locus $\mathscr{X}' \subseteq \mathbf{P}\mathscr{V}^{[1]}$ is now the sought-after $(q;\mathbf{a}/t)$-tic scheme. ∎

4.12. Planing. — An r-*planing* of a family $\mathscr{X}$ of $(q;\mathbf{a})$-tic schemes in $\pi\colon \mathbf{P}\mathscr{V} \to S$ is the data of a closed subscheme $\mathscr{P} := \mathbf{P}\mathscr{U} \subseteq \mathscr{X}$ corresponding to a subbundle $\mathscr{U} \subseteq \mathscr{V}$ of rank $r+1$; in other words, $\mathscr{P}$ is a family of r-planes contained in $\mathscr{X}$. The pair $\mathscr{P} \subseteq \mathscr{X}$ is referred to as a *family of r-planed $(q;\mathbf{a})$-tic schemes* over the base scheme S. Observe that an r-planing $\mathscr{P}$ of $\mathscr{X}$ may be transformed into an r'-planing for any $0 \leq r' \leq r$ upon base change to the Grassmannian bundle $\mathbf{G}(r'+1,\mathscr{U}) \to S$ and taking the projective bundle on the tautological subbundle of rank $r'+1$.

4.13. Classifying maps. — Let $\mathscr{P} \subseteq \mathscr{X}$ be an r-planed family of $(q;\mathbf{a})$-tic schemes in a $\mathbf{P}^n$-bundle over a base S. At least on open subsets of the base S, the family $\mathscr{X}$ and the pair $\mathscr{P} \subseteq \mathscr{X}$ induce classifying maps to the parameter spaces

$$(q;\mathbf{a})\textbf{-tics}_{\mathbf{P}^n} := \prod\nolimits_{a\in\mathbf{a}} \mathbf{P}\mathrm{S}^{a}(\mathbf{k}^{\oplus n+1}) \text{ and}$$
$$\mathbf{Inc}_{n,r,\mathbf{a}} := \{\, ([U],[\boldsymbol{\alpha}]) \in \mathbf{G}(r+1,n+1) \times (q;\mathbf{a})\textbf{-tics}_{\mathbf{P}^n} : \mathbf{P}U \subseteq X_{\boldsymbol{\alpha}} \,\},$$

encountered already in 2.7 and 3.2. To describe these, it is convenient to assume that S is integral, so that the classifying maps may be seen as rational maps from S to one of the two parameter spaces; often, such a rational map will stand in for a choice of classifying map constructed below.

Assume henceforth that the base scheme S is integral. For a nonempty open subscheme $S^\circ \subseteq S$, write $\mathscr{V}^\circ$, $\mathscr{P}^\circ$, and $\mathscr{X}^\circ$ for the restrictions of $\mathscr{V}$, $\mathscr{P}$, and $\mathscr{X}$ over S°. Pick S° over which $\mathscr{V}^\circ$ is trivial and ε may be identified with the evaluation map as in 4.3(ii). Fixing suitable trivializations identifies $\mathscr{X}^\circ$ as a $(q;\mathbf{a})$-tic scheme in the projective space $\mathbf{P}\mathscr{V}^\circ \cong \mathbf{P}^n \times S^\circ$ defined by a $(q;\mathbf{a})$-tic tensor

$$\boldsymbol{\alpha}^\circ\colon \mathscr{O}_{S^\circ} \to \bigoplus\nolimits_{a\in\mathbf{a}} \mathrm{S}^{a}(\mathscr{V}^{\circ,\vee}).$$

This tensor defines a morphism $S^\circ \to (q;\mathbf{a})\textbf{-tics}_{\mathbf{P}^n}$ and, together with $\mathscr{P}^\circ$, a morphism $S^\circ \to \mathbf{Inc}_{n,r,\mathbf{a}}$. These are the *classifying morphisms* for the families $\mathscr{X}^\circ$ and $\mathscr{P}^\circ \subseteq \mathscr{X}^\circ$, respectively.

4.14. Generic families. — Different choices of trivialization produce classifying morphisms which differ by automorphisms of the parameter spaces, so dominance of a classifying map does not depend on any of the choices above; after all, dominance means informally that the family contains the general member of the parameter space in question. By way of terminology, call the family $\mathscr{P} \subseteq \mathscr{X}$ *generic* if the associated classifying map $S \dashrightarrow \mathbf{Inc}_{n,r,\mathbf{a}}$ is dominant.

Genericity propagates along many constructions. As a first example, consider a family $\mathscr{P} \subseteq \mathscr{X}$ of r-planed $(q;\mathbf{a})$-tic schemes in which some of the equations of $\mathscr{X}$ are linear: that is, $\mathbf{a}_{\mathrm{lin}} \neq \varnothing$ in the notation of 4.8. Then $\mathscr{P} \subseteq \mathscr{X}$ may be considered as a family in the smaller projective bundle

$\mathbf{P}\mathscr{V}'$ cut out by those linear equations, as in 4.10. Since an equation in a projective subspace may be extended—in many ways!—to an equation in a larger ambient projective space, it is easy to convince oneself that if $\mathscr{P} \subseteq \mathscr{X}$ is generic as a family in $\mathbf{P}\mathscr{V}$, then it remains generic as a family in $\mathbf{P}\mathscr{V}'$. A more careful argument is given in the following:

4.15. Lemma. — *Let $\mathscr{P} \subseteq \mathscr{X}$ be a generic family of r-planed $(q;\mathbf{a})$-tic schemes in $\pi\colon \mathbf{P}\mathscr{V} \to S$ over an integral base. Suppose that there is a subbundle $\pi'\colon \mathbf{P}\mathscr{V}' \to S$ containing $\mathscr{X}$ as a family of $(q;\mathbf{a}')$-tic schemes where $\varnothing \neq \mathbf{a}' \subsetneq \mathbf{a}$ and $\mathbf{a} \setminus \mathbf{a}' \subseteq \mathbf{a}_{\mathrm{lin}}$. Then $\mathscr{P} \subseteq \mathscr{X}$ is also generic viewed as a family in $\mathbf{P}\mathscr{V}'$.*

Proof. It suffices to treat the universal case: Write $\mathbf{P}^n = \mathbf{P}V$, $c := \#\mathbf{a} \setminus \mathbf{a}'$, and let

$$S := \{s \in \mathbf{Inc}_{V,r,\mathbf{a}} : \operatorname{rank} \phi_s \geq c\} \subseteq \mathbf{G}(r+1,V) \times (q;\mathbf{a}')\text{-}\mathbf{tics}_{\mathbf{P}V} \times \prod\nolimits_{i=1}^{c} \mathbf{P}V^\vee$$

be the open subscheme of the incidence correspondence where maximal rank is attained for the map

$$\phi\colon \mathscr{O}_{\mathbf{Inc}_{V,r,\mathbf{a}}} \otimes_{\mathbf{k}} V \to \bigoplus\nolimits_{i=1}^{c} \mathscr{L}_i, \text{ where } \mathscr{L}_i := \mathrm{pr}_i^* \mathscr{O}_{\mathbf{P}V^\vee}(1),$$

obtained by pulling back the evaluation maps $\mathscr{O}_{\mathbf{P}V^\vee} \otimes V \to \mathscr{O}_{\mathbf{P}V^\vee}(1)$ from the c factors in the right-most product. Thus the restriction $\mathscr{P} \subseteq \mathscr{X}$ of the tautological family of r-planed $(q;\mathbf{a})$-tic schemes to S is contained in the projective subbundle on $\mathscr{V}' := \ker(\phi|_S)$ as a family of $(q;\mathbf{a}')$-tic schemes. To construct a classifying map $S \dashrightarrow \mathbf{Inc}_{V',r,\mathbf{a}'}$ as in 4.13 to the parameter space of r-planed $(q;\mathbf{a}')$-tic schemes in $\mathbf{P}^{n-c} = \mathbf{P}V'$, choose an open subscheme $S^\circ \subseteq S$ on which $\mathscr{V}'$ is trivial. Dualizing the composition

$$\mathscr{O}_{S^\circ} \otimes V' \cong \mathscr{V}'|_{S^\circ} \subseteq \mathscr{O}_{S^\circ} \otimes V$$

of a trivialization with the inclusion induces a map which takes the tautological family of $(q;\mathbf{a})$-tic tensors on V to a family of $(q;\mathbf{a}')$-tic tensors on V', and hence a morphism $S^\circ \to \mathbf{Inc}_{V',r,\mathbf{a}'}$. It remains to see that some such morphism is dominant.

Consider an explicit choice of S°: Upon choosing coordinates $\mathbf{P}^n = \mathbf{P}V$, the map ϕ may be viewed as a $c \times (n+1)$ matrix $(a_{i,j})$, where the c linear equations defining $\mathscr{X}$ are given by

$$\ell_i(x_0,\ldots,x_n) = a_{i,0}x_0 + \cdots + a_{i,n}x_n \text{ for } i = 1,\ldots,c.$$

Let $S^\circ \subseteq S$ be the open subscheme on which the rightmost $c \times c$ minor of ϕ is non-vanishing. With a suitable $\mathscr{O}_{S^\circ}$-linear change of coordinates, the linear equations ℓ_i may be transformed to

$$\ell_i(x_0,\ldots,x_n) = x_{n-c+i} - \sum\nolimits_{j=0}^{n-c} b_{i,j}x_j \text{ for } i = 1,\ldots,c$$

where the $b_{i,j} \in \Gamma(S^\circ,\mathscr{O}_{S^\circ})$. This provides a splitting of the surjection $\phi|_{S^\circ}$; the corresponding retraction provides a trivialization of $\mathscr{V}'|_{S^\circ}$ identifying it with the first $n-c+1$ summands of $\mathscr{O}_{S^\circ}^{\oplus n+1}$, and the induced classifying morphism $S^\circ \to \mathbf{Inc}_{V',r,\mathbf{a}'}$ takes a $(q;\mathbf{a})$-tic tensor on $\mathbf{P}^n$ to that on $\mathbf{P}^{n-c}$ obtained by eliminating the last c coordinates as prescribed by the linear equation ℓ_i. Described in this way, it is straightforward that this classifying morphism is surjective: Consider, for example, the closed subscheme T defined by $b_{i,j} = 0$ for all $1 \leq i \leq c$ and $0 \leq j \leq n-c$. Then any point of $\mathbf{Inc}_{V',r,\mathbf{a}'}$ can be lifted to a point of T just by viewing the $(q;\mathbf{a}')$-tic tensor in $n-c+1$ variables as a $(q;\mathbf{a}')$-tic tensor in $n+1$ variables; then augment this to a $(q;\mathbf{a})$-tic tensor by taking into account the linear equations ℓ_i. ■

5. Highly tangent lines

Lines tangent to order $\geq k$ to a projective scheme $X \subseteq \mathbf{P}^n$ may be parameterized by a scheme of the form

$$\mathbf{Tan}_k(X) := \{(x,[\ell]) \in X \times \mathbf{G}(2,n+1) : \mathrm{mult}_x(\ell \cap X) \geq k\}.$$

Familiar cases include: When $k = 0$, this is the restriction to X of the variety of pointed lines in $\mathbf{P}^n$, and is the projective bundle on $\mathscr{T} := \mathscr{T}_{\mathbf{P}^n}(-1)|_X$. When $k = 1$, this is the projectivized tangent bundle of X. When X is a general hypersurface of degree d, it is well-known that the general fibre of the projection $\mathbf{Tan}_k(X) \to X$ to the x-coordinate is a complete intersection of type $(k-1, k-2, \ldots, 2, 1)$ in an $(n-1)$-dimensional projective space.

If X is a general complete intersection of codimension $c \geq 2$, $\mathbf{Tan}_k(X)$ usually consists of several components and will not fibre in complete intersections over X. Nonetheless, upon writing X as an intersection $H_1 \cap \cdots \cap H_c$ of hypersurfaces H_i of degree d_i, each of the schemes

$$\mathbf{Tan}_k(X; H_i) := \{(x, \ell) : \mathrm{mult}_x(\ell \cap H_i) \geq k \text{ and } \ell \subset H_j \text{ for } 1 \leq j \leq c \text{ and } j \neq i\},$$

parameterizing k-fold tangent lines to H_i and pointed lines in the remaining H_j, are distinguished components of $\mathbf{Tan}_k(X)$ which project onto X with complete intersection general fibres. The aim of this section is to make sense of this construction and structure for a family of $(q; \mathbf{a})$-tic schemes.

Rather than discussing $\mathbf{Tan}_k(X)$ for general k, this section focuses on the case of particular relevance, namely, $k = d - 1$ where $d := \deg X$:

$$\mathbf{PenTa}(X) := \mathbf{Tan}_{d-1}(X) := \{(x, [\ell]) \in X \times \mathbf{G}(2, n+1) : \mathrm{mult}_x(\ell \cap X) \geq d - 1\}.$$

Lines parameterized by $\mathbf{PenTa}(X)$ are called *penultimate tangents*. In words, this scheme of penultimate tangents parameterizes pointed lines $x \in \ell \subset \mathbf{P}^n$ such that either ℓ is contained in X or else their intersection, viewed as a Cartier divisor on ℓ, is of the form $\ell \cap X = (d-1)x + x'$ for a *residual point* $x' \in X$. The interest in this case is that extracting the residual point often provides a rational map $\mathrm{res}\colon \mathbf{PenTa}(X) \dashrightarrow X$, which will be studied in §6.

5.1. Local situation. — When $S = \operatorname{Spec} \mathbf{k}$ and $X \subseteq \mathbf{P}^n$ is a $(q; \mathbf{a})$-tic scheme, the situation is easy to describe explicitly. With the notation in the proof of 3.14, the scheme X_1 of pointed lines in X is defined over $\mathrm{D}(x_n) \subseteq \mathbf{P}^n$ as

$$X_1|_{\mathrm{D}(x_n)} = \{(\mathbf{x}, \mathbf{y}) \in \mathrm{D}(x_n) \times \mathbf{P}^{n-1} : f_{a,b}(\mathbf{x}; \mathbf{y}) = 0 \text{ for } a \in \mathbf{a} \text{ and } 0 \preceq b \preceq a\}.$$

A point $(\mathbf{x}, \mathbf{y}) \in \mathrm{D}(x_n) \times \mathbf{P}^{n-1}$ corresponds to a parameterized line $(\xi : \eta) \mapsto \xi\mathbf{x} + \eta\mathbf{y}$, so the point $\mathbf{x}$ is the image of $(1:0)$. Writing $H \subseteq \mathbf{P}^n$ for the hypersurface cut out by a $(q; a)$-tic defining polynomial f_a, the scheme $\mathbf{Tan}_k(H)$ of pointed lines tangent to H to order $\geq k$ is defined by $f_{a,b}(\mathbf{x}; \mathbf{y}) = 0$ for $0 \preceq b \preceq a$ satisfying $b(q) \leq k$. Therefore, over $X \cap \mathrm{D}(x_n)$, the scheme $\mathbf{PenTa}(X; H)$ of penultimate tangents may be obtained from X_1 by omitting the two equations $f_{a,a}(\mathbf{x}; \mathbf{y}) = f_{a,a-1}(\mathbf{x}; \mathbf{y}) = 0$, presenting it as a family of $(q; \mathbf{a}')$-tic where $\mathbf{a}' := \mathbf{a}_1 \setminus (a, a-1)$; this is most useful when $a - 1$ is a nonzero profile, equivalently when the profile a is nonlinear and reduced.

The next few paragraphs make sense of these observations in families.

5.2. Pointed lines. — Given a projective bundle $\pi\colon \mathbf{P}\mathscr{V} \to S$, write $\mathbf{L}\mathscr{V}$ for its space of pointed lines: namely, the incidence correspondence between it and its relative Grassmannian $\mathbf{G}\mathscr{V}$ of lines. Projection to the line exhibits $\mathbf{L}\mathscr{V}$ as the universal line over $\mathbf{G}\mathscr{V}$, whereas projection $\mathrm{pr}_x\colon \mathbf{L}\mathscr{V} \to \mathbf{P}\mathscr{V}$ to the point identifies $\mathbf{L}\mathscr{V}$ as the projective bundle on the relative tangent bundle $\mathscr{T}_\pi \otimes \mathscr{O}_\pi(-1)$ of $\pi\colon \mathbf{P}\mathscr{V} \to S$. These moduli descriptions imply that the tautological bundles of $\mathbf{L}\mathscr{V}$ fit into a canonical short exact sequence

$$0 \to \mathscr{O}_{\mathrm{pr}_x}(1) \to \mathscr{S}^\vee \to \mathrm{pr}_x^* \mathscr{O}_\pi(1) \to 0$$

where $\mathscr{S}$ is the subbundle of rank 2. Pushing along pr_x yields the relative dual Euler sequence

$$0 \to \Omega^1_\pi \otimes \mathscr{O}_\pi(1) \to \pi^*\mathscr{V}^\vee \to \mathscr{O}_\pi(1) \to 0.$$

Consider now a family $\mathscr{X}$ of $(q;\mathbf{a})$-tic schemes in $\pi\colon \mathbf{P}\mathscr{V} \to S$. Its space of pointed lines

$$\mathscr{X}_1 := \{(x,\ell) \in \mathscr{X} \times_S \mathbf{F}_1(\mathscr{X}/S) : x \in \ell\} \hookrightarrow \mathbf{L}\mathscr{V}|_{\mathscr{X}}$$

naturally embeds into the projective bundle on $\mathscr{T} := \mathscr{T}_\pi \otimes \mathscr{O}_\pi(-1)|_{\mathscr{X}}$ in which it has a $(q;\mathbf{a}_1)$-tic structure, globalizing 3.14:

5.3. Proposition. — *Let $\mathscr{X}$ be a family of $(q;\mathbf{a})$-tic schemes in a projective bundle $\pi\colon \mathbf{P}\mathscr{V} \to S$. Then its space of pointed lines $\mathscr{X}_1$ admits a canonical $(q;\mathbf{a}_1)$-tic structure in $\rho\colon \mathbf{P}\mathscr{T} \to \mathscr{X}$, where*

$$\mathscr{T} := \mathscr{T}_\pi \otimes \mathscr{O}_\pi(-1)|_{\mathscr{X}},\ \ \mathbf{a}_1 := (b \in \mathbf{Prfl} : 0 \prec b \preceq a \text{ for } a \in \mathbf{a}),\ \ \text{and}\ \ \mathscr{A}_1 := \ker(\varepsilon\colon \pi^*\mathscr{A} \to \mathscr{E})|_{\mathscr{X}}.$$

Proof. The task is to construct equations of $\mathscr{X}_1$ in $\mathbf{P}\mathscr{T}$ and to provide them with a $(q;\mathbf{a}_1)$-tic structure over $\mathscr{X}$. First, the equations of $\mathscr{X}_1$ in all of $\mathbf{L}\mathscr{V}$ are the pullback of those of the Fano scheme $\mathbf{F}_1(\mathscr{X}/S)$ in $\mathbf{G}\mathscr{V}$, and the latter are given by $\mathrm{pr}_{\ell,*}\mathrm{pr}_x^*\sigma$, where pr_x and pr_ℓ are the projections out of $\mathbf{L}\mathscr{V}$, and $\sigma\colon \mathscr{O}_{\mathbf{P}\mathscr{V}} \to \mathscr{E}$ are the equations of $\mathscr{X}$ in $\mathbf{P}\mathscr{V}$. Writing $\sigma = \varepsilon \circ \pi^*\boldsymbol{\alpha}$ as in 4.3(i) provides a factorization

$$\mathrm{pr}_{\ell,*}\mathrm{pr}_x^*\sigma\colon \mathscr{O}_{\mathbf{G}} \xrightarrow{\gamma^*\boldsymbol{\alpha}} \gamma^*\mathscr{A} \xrightarrow{\varepsilon_0} \mathscr{E}_0 \subseteq \mathrm{pr}_{\ell,*}\mathrm{pr}_x^*\mathscr{E}$$

where $\gamma\colon \mathbf{G}\mathscr{V} \to S$ is the structure map, and $\varepsilon_0\colon \gamma^*\mathscr{A} \to \mathscr{E}_0$ is a canonical map which is locally isomorphic to a direct sum of the evaluation maps $\mathrm{S}^a(\mathscr{V}^\vee) \to \mathrm{S}^a(\mathscr{S}^\vee)$, globalizing 3.1: this uses the fact that $\mathrm{pr}_{\ell,*}\mathrm{pr}_x^*\mathscr{O}_\pi(d) \cong \mathrm{Sym}^d(\mathscr{S}^\vee)$, the local form of ε from 4.3(ii), and the assumption that the multiplication map for $(q;\mathbf{a})$-tic tensors is injective. In summary, $\mathbf{F}_1(\mathscr{X}/S)$ is cut out in $\mathbf{G}\mathscr{V}$ by the section $\sigma_0 := \varepsilon_0 \circ \gamma^*\boldsymbol{\alpha}\colon \mathscr{O}_{\mathbf{G}\mathscr{V}} \to \mathscr{E}_0$, and so $\mathscr{X}_1$ is cut out in $\mathbf{L}\mathscr{V}$ by the pullback $\mathrm{pr}_\ell^*\sigma_0$.

Next, consider the restriction of $\mathrm{pr}_\ell^*\sigma_0$ to $\mathbf{P}\mathscr{T}$: On the one hand, evaluation along pr_ℓ provides a canonical map $\xi\colon \mathrm{pr}_\ell^*\mathscr{E}_0 \subseteq \mathrm{pr}_\ell^*\mathrm{pr}_{\ell,*}\mathrm{pr}_x^*\mathscr{E} \to \mathrm{pr}_x^*\mathscr{E}$ making the square

$$\begin{tikzcd} \mathrm{pr}_\ell^*\gamma^*\mathscr{A} \arrow[r, equal] \arrow[d, "\mathrm{pr}_\ell^*\varepsilon_0"'] & \mathrm{pr}_x^*\pi^*\mathscr{A} \arrow[d, "\mathrm{pr}_x^*\varepsilon"] \\ \mathrm{pr}_\ell^*\mathscr{E}_0 \arrow[r, "\xi"] & \mathrm{pr}_x^*\mathscr{E} \end{tikzcd}$$

commute; in particular, this implies that $\mathrm{pr}_x^*\sigma = \xi \circ \mathrm{pr}_\ell^*\sigma_0$. Locally, this is the sum of the canonical surjections $\mathrm{S}^a(\mathscr{S}^\vee) \to \mathrm{pr}_x^*\mathscr{O}_\pi(a(q))$ induced by applying S^a to the first exact sequence of 5.2. On the other hand, $\mathbf{P}\mathscr{T} = \mathbf{L}\mathscr{V}|_{\mathscr{X}}$ is cut out of $\mathbf{L}\mathscr{V}$ by $\mathrm{pr}_x^*\sigma$. Together, this means that the restricted section $\mathrm{pr}_\ell^*\sigma_0|_{\mathbf{P}\mathscr{T}}$ induces a map

$$\sigma_1\colon \mathscr{O}_{\mathbf{P}\mathscr{T}} \to \mathscr{E}_1 := \ker(\xi\colon \mathrm{pr}_\ell^*\mathscr{E}_0 \to \mathrm{pr}_x^*\mathscr{E})|_{\mathbf{P}\mathscr{T}}.$$

which cuts out $\mathscr{X}_1$ in $\mathbf{P}\mathscr{T}$.

Finally, for the $(q;\mathbf{a}_1)$-tic structure, let $\mathscr{A}_1 := \ker(\varepsilon\colon \pi^*\mathscr{A} \to \mathscr{E})|_{\mathscr{X}}$ and observe that, since $\sigma = \varepsilon \circ \pi^*\boldsymbol{\alpha}$ vanishes on $\mathscr{X}$, the restriction $\pi^*\boldsymbol{\alpha}|_{\mathscr{X}}$ induces a map $\boldsymbol{\alpha}_1\colon \mathscr{O}_{\mathscr{X}} \to \mathscr{A}_1$. Pulling up along $\rho\colon \mathbf{P}\mathscr{T} \to \mathscr{X}$, the composite

$$\rho^*\mathscr{A}_1 \subseteq \mathrm{pr}_x^*\pi^*\mathscr{A}|_{\mathbf{P}\mathscr{T}} = \mathrm{pr}_\ell^*\gamma^*\mathscr{A}|_{\mathbf{P}\mathscr{T}} \xrightarrow{\mathrm{pr}_\ell^*\varepsilon_0} \mathrm{pr}_\ell^*\mathscr{E}_0|_{\mathbf{P}\mathscr{T}} \xrightarrow{\xi} \mathrm{pr}_x^*\mathscr{E}|_{\mathbf{P}\mathscr{T}}$$

vanishes since it is the restriction of $\mathrm{pr}_x^*\varepsilon$, and so it induces a canonical map $\varepsilon_1\colon \rho^*\mathscr{A}_1 \to \mathscr{E}_1$. Tracing through the construction and using $\sigma_0 = \varepsilon_0 \circ \gamma^*\boldsymbol{\alpha}$ then shows that $\sigma_1 = \varepsilon_1 \circ \rho^*\boldsymbol{\alpha}_1$.

It remains to describe the local structure of ε_1. Begin with the open cover $S = \bigcup_{i\in I} U_i$ and trivializations of ε provided by 4.3(ii). Refine the open cover given by the $\pi^{-1}(U_i)$ into an open covering $\mathscr{X} = \bigcup_{j\in J} V_j$ on which, additionally, the dual Euler sequence splits:

$$\pi^*\mathscr{V}^\vee|_{V_j} \cong \mathscr{T}^\vee|_{V_j} \oplus \mathscr{O}_\pi(1)|_{V_j}.$$

Fix such a choice for each $j \in J$. The restriction of $\mathscr{A}_1$ to V_j is then isomorphic to

$$\bigoplus\nolimits_{a \in \mathbf{a}} \ker\big(\mathrm{S}^a(\mathscr{T}^\vee \oplus \mathscr{O}_\pi(1)) \to \mathscr{O}_\pi(a(q))\big)|_{V_j} \cong \bigoplus\nolimits_{a \in \mathbf{a}} \bigoplus\nolimits_{0 \prec b \preceq a} \mathrm{S}^b(\mathscr{T}^\vee) \otimes \mathscr{O}_\pi(a(q) - b(q))|_{V_j}$$

where the maps on the left are projection onto the $\mathrm{S}^a(\mathscr{O}_\pi(1)) = \mathscr{O}_\pi(a(q))$ factor. Now choose over V_j a splitting of the first exact sequence in 5.2 compatible with the evaluation map $\mathrm{pr}_x^* \pi^* \mathscr{V}^\vee \to \mathscr{S}^\vee$. Over each V_j, the map $\varepsilon_1 \colon \rho^* \mathscr{A}_1 \to \mathscr{E}_1$ is isomorphic to the sum of the evaluation maps

$$\mathrm{S}^b(\mathrm{ev}_\rho) \otimes \mathrm{id} \colon \rho^* \mathrm{S}^b(\mathscr{T}^\vee) \otimes \mathscr{O}_\pi(a(q) - b(q)) \to \mathscr{O}_\rho(b(q)) \otimes \rho^* \mathscr{O}_\pi(a(q) - b(q))$$

as a ranges over $\mathbf{a}$ and $0 \prec b \preceq a$. This verifies that $(\mathscr{X}_1, \boldsymbol{\alpha}_1, \varepsilon_1)$ indeed is a family of $(q;\mathbf{a}_1)$-tic schemes in $\rho \colon \mathbf{P}\mathscr{T} \to \mathscr{X}$, as desired. ∎

In light of the discussion in 5.1, it will be useful to describe how to access, within the bundle $\mathscr{E}_1$ of equations for $\mathscr{X}_1$ in $\mathbf{P}\mathscr{T}$ constructed in 5.3, those of degree $a(q)$ and $a(q) - 1$, at least when $\mathscr{X}$ is a family of $(q;a)$-tic hypersurfaces with a nonlinear and reduced:

5.4. Lemma. — *Let $\mathscr{X}$ be a family of $(q;a)$-tic hypersurfaces where $a(t)$ is nonlinear and reduced. The degree $a(q)$ and $a(q) - 1$ equations of the family $\mathscr{X}_1$ of $(q;\mathbf{a}_1)$-tic schemes in $\rho \colon \mathbf{P}\mathscr{T} \to \mathscr{X}$ constructed in 5.3 lie in a subbundle of the form $\mathscr{S}^\vee \otimes \mathscr{O}_\rho(a(q) - 1) \subseteq \mathscr{E}_1$.*

Proof. Since $\mathscr{X}$ is a family of $(q;a)$-tic hypersurfaces, up to twisting by line bundles from S, its structure is given by a tensor $\alpha \colon \mathscr{O}_S \to \mathrm{S}^a(\mathscr{V}^\vee)$ and the evaluation map $\varepsilon \colon \pi^* \mathrm{S}^a(\mathscr{V}^\vee) \to \mathscr{O}_\pi(a(q))$. The construction of 5.3 then shows that the $(q;\mathbf{a}_1)$-tic polynomials defining $\mathscr{X}_1$ in $\rho \colon \mathbf{P}\mathscr{T} \to \mathscr{X}$ take values in the bundle

$$\mathscr{E}_1 = \ker\big(\mathrm{S}^a(\mathscr{S}^\vee) \to \rho^* \mathscr{O}_\pi(a(q))\big)$$

where $\mathscr{S}^\vee$ denotes the tautological rank 2 subbundle on $\mathbf{P}\mathscr{T}$, and the morphism is the canonical quotient that arises upon applying S^a to the short exact sequence of $\mathscr{O}_{\mathbf{P}\mathscr{T}}$-modules

$$0 \to \mathscr{O}_\rho(1) \to \mathscr{S}^\vee \to \rho^* \mathscr{O}_\pi(1) \to 0$$

as in 5.2. The line subbundle $\mathscr{O}_\rho(1)$ gives the fibre coordinates of $\rho \colon \mathbf{P}\mathscr{T} \to \mathscr{X}$, so the degree $a(q)$ and $a(q) - 1$ equations of $\mathscr{X}_1$ lie in the deepest rank 2 subbundle of $\mathscr{E}_1$ with respect to the filtration obtained by applying S^a to the short exact sequence. Since

$$\mathrm{S}^a(\mathscr{S}^\vee) = \mathrm{Sym}^{a_0}(\mathscr{S}^\vee) \otimes \Big(\bigotimes\nolimits_{j \geq 1} \mathrm{Sym}^{a_j}(\mathscr{S}^\vee)^{[j]}\Big),$$

this is the deepest rank 2 subbundle of $\mathrm{Sym}^{a_0}(\mathscr{S}^\vee)$ twisted by the deepest line subbundle of each $\mathrm{Sym}^{a_j}(\mathscr{S}^\vee)^{[j]}$ for $j \geq 1$. General facts about rank 2 bundles shows that this is

$$\big(\mathscr{S}^\vee \otimes \mathscr{O}_\rho(a_0 - 1)\big) \otimes \Big(\bigotimes\nolimits_{j \geq 1} \mathscr{O}_\rho(a_j q^j)\Big) \cong \mathscr{S}^\vee \otimes \mathscr{O}_\rho(a(q) - 1). \qquad ∎$$

5.5. Penultimate tangents. — Let $\mathscr{X}$ be a family of $(q;\mathbf{a})$-tic schemes in $\pi \colon \mathbf{P}\mathscr{V} \to S$, and let $\mathscr{X}_1$ be its scheme of pointed lines equipped with its structure as a family of $(q;\mathbf{a}_1)$-tic schemes in $\rho \colon \mathbf{P}\mathscr{T} \to \mathscr{X}$ as constructed in 5.3. Assume that $\mathbf{a}_{\mathrm{nlr}} \neq \varnothing$ and choose a subbundle $\mathrm{S}^a(\mathscr{V}^\vee) \otimes \mathscr{M} \subseteq \mathscr{A}_{\mathrm{nlr}}$ as in 4.9. To perform the construction 5.1 in families, view the projective bundle $\mu \colon \mathbf{P}\mathscr{M} \to S$ as the linear system parameterizing $(q;a)$-tic hypersurfaces containing $\mathscr{X}$ corresponding to the equations of $\mathrm{S}^a(\mathscr{V}^\vee) \otimes \mathscr{M}$. On the product $\mathbf{P}\mathscr{T} \times_S \mathbf{P}\mathscr{M}$, the chosen subbundle of $\mathscr{A}$ together with the tautological line subbundle $\mathscr{O}_\mu(-1)$, and the computation of 5.4 distinguish subbundles

$$\mathrm{pr}_1^* \mathscr{E}_1 \supseteq \ker\big(\mathrm{S}^a(\mathscr{S}^\vee) \to \rho^* \mathscr{O}_\pi(a(q))\big) \boxtimes \mathscr{O}_\mu(-1) \supseteq \big(\mathscr{S}^\vee \otimes \mathscr{O}_\rho(a(q) - 1)\big) \boxtimes \mathscr{O}_\mu(-1)$$

giving, respectively, the equations of the $(q;a)$-tic hypersurfaces parameterized by $\mathbf{P}\mathscr{M}$, and the corresponding degree $a(q)$ and $a(q) - 1$ equations of $\mathscr{X}_1 \times_S \mathbf{P}\mathscr{M}$ over $\mathbf{P}\mathscr{M}$. The composition

$$\overline{\sigma}_1 \colon \mathscr{O} \to \overline{\mathscr{E}}_1 := \mathrm{pr}_1^* \mathscr{E}_1 / (\mathscr{S}^\vee \otimes \mathscr{O}_\rho(a(q) - 1)) \boxtimes \mathscr{O}_\mu(-1)$$

of $\mathrm{pr}_1^*\sigma_1$ with the quotient map $\mathrm{pr}_1^*\mathcal{E}_1 \to \overline{\mathcal{E}}_1$ defines the desired component of penultimate tangents.

To keep notation consistent with what follows, assume additionally that there is given an r-planing $\mathcal{P} \subseteq \mathcal{X}$ as in 4.12. Let

$$S' := \mathcal{P} \times_S \mathbf{P}\mathcal{M} \quad\text{and}\quad \mathbf{P}\mathcal{V}' := \mathbf{P}\mathcal{T} \times_{\mathcal{X}} S' = \mathbf{P}\mathcal{T}|_{\mathcal{P}} \times_S \mathbf{P}\mathcal{M}$$

where the second projection $\pi' \colon \mathbf{P}\mathcal{V}' \to S'$ exhibits the fibre product as the projective bundle on $\mathcal{V}' := \nu^*(\mathcal{T}|_{\mathcal{P}})$ where $\nu \colon S' \to \mathcal{P}$ is the structure map. View $\mathbf{P}\mathcal{V}'$ as a closed subscheme of $\mathbf{P}\mathcal{T} \times_S \mathbf{P}\mathcal{M}$ and let $\sigma' \colon \mathcal{O}_{\mathbf{P}\mathcal{V}'} \to \mathcal{E}'$ be the restriction of the section $\overline{\sigma}_1$ above. The *scheme of penultimate tangents* associated with $\mathcal{X}$ and the subbundle $\mathrm{S}^a(\mathcal{V}^\vee) \otimes \mathcal{M}$ over $\mathcal{P}$ is the vanishing locus

$$\mathcal{X}' := \mathbf{PenTa}(\mathcal{X}; \mathrm{S}^a(\mathcal{V}^\vee) \otimes \mathcal{M})|_{\mathcal{P}} := \mathrm{V}(\sigma' \colon \mathcal{O}_{\mathbf{P}\mathcal{V}'} \to \mathcal{E}') \subseteq \mathbf{P}\mathcal{V}'$$

of the section σ' in $\mathbf{P}\mathcal{V}'$. All of this data fits into a commutative diagram of schemes

$$\begin{tikzcd}
\mathcal{X}_1|_{\mathcal{P}} \times_{\mathcal{P}} S' \subseteq \mathcal{X}' \arrow[d] & \subseteq & \mathbf{P}\mathcal{V}' \arrow[r, "\pi'"'] \arrow[d, "\mathrm{pr}_1"'] & S' \arrow[d, "\nu"] & & \\
\mathcal{X}_1|_{\mathcal{P}} \arrow[rr, hook] & & \mathbf{P}\mathcal{T}|_{\mathcal{P}} \arrow[r, "\rho"] & \mathcal{P} \subseteq \mathcal{X} \subseteq \mathbf{P}\mathcal{V} \arrow[r, "\pi"] & S\,.
\end{tikzcd}$$

The defining equations σ' of $\mathcal{X}'$ are essentially a subset of the equations σ_1 defining $\mathcal{X}_1$, whence the containment relation in the top left. This relationship between $\mathcal{X}'$ and $\mathcal{X}_1$ further means that the $(q;\mathbf{a}_1)$-tic structure on $\mathcal{X}_1$ induces a $(q;\mathbf{a}')$-tic structure on $\mathcal{X}'$:

5.6. Proposition. — *In the above setting, the scheme $\mathcal{X}'$ of penultimate tangents admits the structure of a family of $(q;\mathbf{a}')$-tic schemes in $\pi' \colon \mathbf{P}\mathcal{V}' \to S'$, where $\mathbf{a}' := \mathbf{a}_1 \setminus (a, a-1)$.* ■

Proof. In describing the $(q;\mathbf{a}')$-tic structure of $\mathcal{X}'$, we may replace S by $\mathbf{P}\mathcal{M}$ to simplify the setting of 5.5 so as to assume that we have chosen a subbundle $\mathrm{S}^a(\mathcal{V}^\vee) \subseteq \mathcal{A}$ corresponding to a family of $(q;a)$-tic hypersurfaces containing $\mathcal{X}$. Thus $S' = \mathcal{P} \subseteq \mathcal{X}$ and $\mathbf{P}\mathcal{V}' = \mathbf{P}\mathcal{T}|_{\mathcal{P}}$. The preimage under $\varepsilon_1 \colon \rho^*\mathcal{A}_1 \to \mathcal{E}_1$ of the rank 2 subbundle from 5.4 corresponding to the degree $a(q)$ and $a(q)-1$ equations of the chosen family of $(q;a)$-tic equations over $\mathcal{X}$ is the pullback of a subbundle $\mathcal{B} \subseteq \mathcal{A}_1$ which fits in an extension

$$0 \to \mathrm{S}^a(\mathcal{T}^\vee) \to \mathcal{B} \to \mathrm{S}^{a-1}(\mathcal{T}^\vee) \otimes \mathcal{O}_\pi(1) \to 0$$

arising from the functor S^a applied to the relative dual Euler sequence in 5.2. Writing $\theta \colon \mathcal{A}_1 \to \overline{\mathcal{A}}_1$ for the corresponding quotient, there is thus an induced map $\bar{\varepsilon}_1 \colon \overline{\mathcal{A}}_1 \to \overline{\mathcal{E}}_1$. The final arguments of 5.3 may then be adapted to show that the maps

$$\boldsymbol{\alpha}' := (\theta \circ \boldsymbol{\alpha}_1)|_{S'} \colon \mathcal{O}_{S'} \to \mathcal{A}' := \overline{\mathcal{A}}_1|_{S'} \quad\text{and}\quad \varepsilon' := \bar{\varepsilon}_1|_{\mathbf{P}\mathcal{V}'} \colon \pi'^*\mathcal{A}' \to \mathcal{E}'$$

define the $(q;\mathbf{a}')$-tic structure of $\mathcal{X}'$ in $\pi' \colon \mathbf{P}\mathcal{V}' \to S'$. ■

5.7. Induced planings. — Let $\mathcal{X}$ be a family of $(q;\mathbf{a})$-tic schemes in a projective bundle $\pi \colon \mathbf{P}\mathcal{V} \to S$. An r-planing $\mathcal{P} \subseteq \mathcal{X}$ induces a canonical $(r-1)$-planing of the scheme $\mathcal{X}_1|_{\mathcal{P}}$ of pointed lines over $\mathcal{P}$ from 5.2: The twisted tangent bundle of $\mathcal{P}$ provides a rank r subbundle

$$\mathcal{T}_{\mathcal{P}} \otimes \mathcal{O}_\pi(-1) \subseteq \mathcal{T}_{\mathbf{P}\mathcal{V}} \otimes \mathcal{O}_\pi(-1)|_{\mathcal{P}} = \mathcal{T}|_{\mathcal{P}}$$

whose associated projective bundle $\mathcal{P}_1$ is contained in $\mathcal{X}_1|_{\mathcal{P}}$; geometrically, $\mathcal{P}_1$ parameterizes pointed lines (x,ℓ) where $\ell \subseteq \mathcal{P}$. Any scheme $\mathcal{X}'$ of penultimate tangents over $\mathcal{P}$ as constructed in 5.5 also inherits an $(r-1)$-planing: simply set $\mathcal{P}' := \mathcal{P}_1 \times_{\mathcal{P}} S'$ and observe that $\mathcal{X}_1|_{\mathcal{P}} \times_{\mathcal{P}} S' \subseteq \mathcal{X}'$.

5.8. Genericity of the induced families. — The remainder of this section is concerned with genericity properties of the families $\mathcal{P}_1 \subseteq \mathcal{X}_1|_{\mathcal{P}}$ of pointed lines over $\mathcal{P}$, and $\mathcal{P}' \subseteq \mathcal{X}'$ of penultimate tangents associated with a generic family, in the sense of 4.14, of r-planed $(q;\mathbf{a})$-tic schemes $\mathcal{P} \subseteq \mathcal{X}$ over an integral base S. First, to see that the families $\mathcal{P}_1 \subseteq \mathcal{X}_1|_{\mathcal{P}}$ and $\mathcal{P}' \subseteq \mathcal{X}'$ are themselves generic, consider the tautological situation over the incidence correspondence of r-planes and $(q;\mathbf{a})$-tic schemes:

5.9. Lemma. — *Let $\mathcal{P} \subseteq \mathcal{X}$ be the tautological family of r-planed $(q;\mathbf{a})$-tic schemes over the incidence correspondence $S := \mathbf{Inc}_{n,r,\mathbf{a}}$. The classifying map*

$$[\mathcal{P}_1 \subseteq \mathcal{X}_1|_{\mathcal{P}}] \colon \mathcal{P} \dashrightarrow \mathbf{Inc}_{n-1,r-1,\mathbf{a}_1}$$

for the associated family of pointed lines is dominant.

Proof. Consider the fibre of $\mathcal{P} \subseteq \mathcal{X} \subseteq \mathbf{P}^n_S$ over a fixed closed point $x \in \mathbf{P}^n$:

$$\mathcal{P}_x = \{([U],[\boldsymbol{\alpha}]) \in \mathbf{Inc}_{n,r,\mathbf{a}} : x \in \mathbf{P}U \subseteq X_{\boldsymbol{\alpha}}\}.$$

There is a choice of classifying morphism whose domain of definition intersects $\mathcal{P}_x$, and such that its restriction $\Psi \colon \mathcal{P}_x \dashrightarrow \mathbf{Inc}_{n-1,r-1,\mathbf{a}_1}$ acts as $([U],[\boldsymbol{\alpha}]) \mapsto ([U/L],[\boldsymbol{\alpha}_x])$, where $x = \mathbf{P}L$ and $\boldsymbol{\alpha}_x$ is the $(q;\mathbf{a}_1)$-tic tensor constructed in 3.14 defining $X_{\boldsymbol{\alpha},1,x}$ in $\mathbf{P}^{n-1} = \mathbf{P}(V/L)$. Since $\mathbf{F}_{r-1}(X_{\boldsymbol{\alpha}_x}) \cong \mathbf{F}_r(X_{\boldsymbol{\alpha}},x)$ as observed in 3.13, Ψ is dominant if and only if the map

$$\{[\boldsymbol{\alpha}] \in (q;\mathbf{a})\textbf{-tics}_{\mathbf{P}^n} : x \in X_{\boldsymbol{\alpha}}\} \dashrightarrow (q;\mathbf{a}_1)\textbf{-tics}_{\mathbf{P}^{n-1}} : [\boldsymbol{\alpha}] \mapsto [\boldsymbol{\alpha}_x]$$

is dominant. This is a product of rational maps determined by the linear maps

$$\{\alpha \in \mathrm{S}^a(V^\vee) : \alpha|_L = 0\} \to \bigoplus\nolimits_{0 \prec b \preceq a} \mathrm{S}^b((V/L)^\vee) \colon \alpha \mapsto (\alpha_b)_{0 \prec b \preceq a}$$

where α_b is the b-homogeneous component of α upon expansion at x. It is straightforward to see from the computations of 3.14 and 3.15 that, upon choosing coordinates so that $x = (0 : \cdots : 0 : 1)$, the α_b are uniquely determined from α by the relation

$$\alpha(x_0,\ldots,x_n) = \sum\nolimits_{0 \prec b \preceq a} \alpha_b(x_0,\ldots,x_{n-1}) \cdot x_n^{a(q)-b(q)}.$$

Combined with 1.5, it follows that the map $\alpha \mapsto (\alpha_b)_{0 \prec b \preceq a}$ is an isomorphism, and so the corresponding map $[\boldsymbol{\alpha}] \mapsto [\boldsymbol{\alpha}_x]$ on multi-projective spaces is dominant. ■

For an analogous statement for penultimate tangents, observe that the $(q;\mathbf{a})$-tic tensor $\boldsymbol{\alpha} \colon \mathcal{O}_S \to \mathcal{A}$ defining the tautological family $\mathcal{X} \subseteq \mathbf{P}^n_S$ takes values in a split bundle of the form

$$\mathcal{A} = \bigoplus\nolimits_{a \in \mathbf{a}} \mathrm{S}^a(\mathcal{V}^\vee)$$

where $\mathcal{V}$ itself is a trivial $\mathcal{O}_S$-module of rank $n+1$. Choose a summand $\mathrm{S}^a(\mathcal{V}^\vee) \subseteq \mathcal{A}$ with nonlinear and reduced profile a, apply the constructions of 5.5 and 5.7, and let $\mathcal{P}' \subseteq \mathcal{X}'$ be the resulting family of $(r-1)$-planed $(q;\mathbf{a}')$-tic schemes, with $\mathbf{a}' = \mathbf{a}_1 \setminus (a, a-1)$, over $S' := \mathcal{P}$.

5.10. Lemma. — *Let $\mathcal{P} \subseteq \mathcal{X}$ be the tautological family of r-planed $(q;\mathbf{a})$-tic schemes over the incidence correspondence $S := \mathbf{Inc}_{n,r,\mathbf{a}}$. For any choice of subbundle $\mathrm{S}^a(\mathcal{V}^\vee) \subseteq \mathcal{A}$ above, the classifying map*

$$[\mathcal{P}' \subseteq \mathcal{X}'] \colon S' \dashrightarrow \mathbf{Inc}_{n-1,r-1,\mathbf{a}'}$$

for the associated family of penultimate tangents is dominant.

Proof. The classifying map factors as the dominant classifying map $S' \dashrightarrow \mathbf{Inc}_{n-1,r-1,\mathbf{a}_1}$ from 5.9, followed by the morphism $\pi \colon \mathbf{Inc}_{n-1,r-1,\mathbf{a}_1} \to \mathbf{Inc}_{n-1,r-1,\mathbf{a}'}$ which projects out the profile a and $a-1$

components corresponding to the tensor of the chosen $(q;a)$-tic subbundle $\mathrm{S}^a(\mathscr{V}^\vee) \subseteq \mathscr{A}$. Writing $\mathbf{P}^{n-1} = \mathbf{P}\overline{V}$, the fibres of π are isomorphic to bi-projective spaces of the form

$$\pi^{-1}([\overline{U}],[\boldsymbol{\alpha}']) \cong \mathbf{P}(\ker(\mathrm{S}^a(\overline{V}^\vee) \to \mathrm{S}^a(\overline{U}^\vee)) \times \mathbf{P}(\ker(\mathrm{S}^{a-1}(\overline{V}^\vee) \to \mathrm{S}^{a-1}(\overline{U}^\vee))$$

parameterizing the missing components of $\boldsymbol{\alpha}'$. Thus π is surjective, and the result follows. ■

Consider an arbitrary family $\mathscr{P} \subseteq \mathscr{X}$ of r-planed $(q;\mathbf{a})$-tic schemes over an integral base S. If the family $\mathscr{P} \subseteq \mathscr{X}$ is generic in the sense of 4.14, then so too are the families $\mathscr{P}_1 \subseteq \mathscr{X}_1|_{\mathscr{P}}$ of pointed lines over $\mathscr{P}$ with its structure from 5.3, and—whenever $\mathbf{a}_{\mathrm{nlr}} \neq \varnothing$—any family $\mathscr{P}' \subseteq \mathscr{X}'$ of penultimate tangents as constructed in 5.5 and 5.6:

5.11. Proposition. — *Let $\mathscr{P} \subseteq \mathscr{X}$ be a generic family of r-planed $(q;\mathbf{a})$-tic schemes.*

(i) *$\mathscr{P}_1 \subseteq \mathscr{X}_1|_{\mathscr{P}}$ is generic as a family of $(r-1)$-planed $(q;\mathbf{a}_1)$-tic schemes.*
(ii) *If $\mathbf{a}_{\mathrm{nlr}} \neq \varnothing$, then any $\mathscr{P}' \subseteq \mathscr{X}'$ of is a generic family of $(r-1)$-planed $(q;\mathbf{a}')$-tic schemes.*

Proof. The family $\mathscr{P} \subseteq \mathscr{X}$ is locally on S the pull back via a classifying map of the tautological family over $\mathbf{Inc}_{n,r,\mathbf{a}}$. The invariant construction of $\mathscr{P}_1 \subseteq \mathscr{X}_1|_{\mathscr{P}}$ from 5.3 and 5.7 means that it, too, is pulled back from the corresponding construction over the tautological family, and so (i) follows from 5.9.

Assume now that $\mathbf{a}_{\mathrm{nlr}} \neq \varnothing$, choose a subbundle $\mathrm{S}^a(\mathscr{V}^\vee) \otimes \mathscr{M} \subseteq \mathscr{A}$ for some nonlinear and reduced profile a, and let $\mathscr{P}' \subseteq \mathscr{X}'$ be the corresponding family of penultimate tangents over $S' = \mathscr{P} \times_S \mathbf{P}\mathscr{M}$ as constructed in 5.5. The base change of the original family $\mathscr{P} \subseteq \mathscr{X}$ to $\mathbf{P}\mathscr{M}$ will remain generic, so as in the proof of 5.6, replace S by $\mathbf{P}\mathscr{M}$ to assume that $\mathscr{M} \cong \mathscr{O}_S$ and $S' = \mathscr{P}$. Then, once again, locally over S, this family $\mathscr{P}' \subseteq \mathscr{X}'$ is pulled back from the corresponding construction over the incidence correspondence, and so (ii) follows from 5.10. ■

Combined with numerical assumptions on the ambient projective bundle dimension n and the integer r, genericity of the family $\mathscr{P}' \subseteq \mathscr{X}'$ yields geometric genericity properties of the families $\mathscr{X}_1|_{\mathscr{P}} \to \mathscr{P}$ and $\mathscr{X}' \to S$. The most useful for what follows is that whenever n is sufficiently large depending on r and $\mathbf{a}$, the general fibre of either family has its expected dimension:

5.12. Proposition. — *Let $\mathscr{P} \subseteq \mathscr{X}$ be a generic family of r-planed $(q;\mathbf{a})$-tic schemes in a $\mathbf{P}^n$-bundle over an integral base S. If*

$$n \geq \max\Big\{2r-1+\#\mathbf{a}_1, r + \frac{1}{r}\sum\nolimits_{a\in\mathbf{a}}\prod\nolimits_{j\geq 0}\binom{a_j+r}{r} - \frac{1}{r}\#\mathbf{a}\Big\},$$

then the general fibre of $\mathscr{X}_1|_{\mathscr{P}} \to \mathscr{P}$ is a $(q;\mathbf{a}_1)$-tic complete intersection in $\mathbf{P}^{n-1}$. Similarly, the general fibre of any family $\mathscr{X}' \to S'$ of penultimate tangents is a $(q;\mathbf{a}')$-tic complete intersection in $\mathbf{P}^{n-1}$.

Proof. The hypothesis on n implies that $n-1-\#\mathbf{a}_1 \geq 0$, so the tautological family over $(q;\mathbf{a}_1)\text{-}\mathbf{tics}_{\mathbf{P}^{n-1}}$ is generically a complete intersection. Since the classifying map $\mathscr{P} \dashrightarrow \mathbf{Inc}_{n-1,r-1,\mathbf{a}_1}$ is dominant by 5.9, the result would follow if the projection $\mathbf{Inc}_{n-1,r-1,\mathbf{a}_1} \to (q;\mathbf{a}_1)\text{-}\mathbf{tics}_{\mathbf{P}^{n-1}}$ were surjective. By 3.2(ii), this is the case whenever $\delta_-(n-1,\mathbf{a}_1,r-1) \geq 0$, and this inequality is equivalent to the hypothesis on n, as seen in 3.16. The same argument applies for penultimate tangents, using the facts that $S' \dashrightarrow \mathbf{Inc}_{n-1,r-1,\mathbf{a}'}$ is dominant by 5.11(ii) and that $\delta_-(n-1,\mathbf{a}',r-1) \geq \delta_-(n-1,\mathbf{a}_1,r-1)$. ■

6. Residual point map

As mentioned at the beginning of §5, extracting the residual point of intersection provides a rational map from a scheme of penultimate lines back to the original projective scheme. The aim of this section is to construct this rational map in the relative setting, for the family $\mathscr{X}'$ of penultimate lines constructed in 5.5, and to study when the resulting map $\mathrm{res}\colon \mathscr{X}' \dashrightarrow \mathscr{X}$ is dominant.

To begin, let $\mathscr{P} \subseteq \mathscr{X}$ be a family of r-planed $(q;\mathbf{a})$-tic schemes in a projective bundle $\pi\colon \mathbf{P}\mathscr{V} \to S$. Assume that $\mathscr{A}_{\mathrm{nlr}} \neq 0$, choose a subbundle $\mathrm{S}^a(\mathscr{V}^\vee) \otimes \mathscr{M} \subseteq \mathscr{A}_{\mathrm{nlr}}$ as in 4.9, and let $\mathscr{X}' \to S'$ be the associated family of penultimate tangents as in 5.5. To give an explicit description of the residual point map, represent geometric points of $\mathscr{X}'$ as quintuples (s,x,ℓ,H,Y) where

- s is a geometric point of S;
- $x \in \ell$ is a pointed line in $\mathbf{P}\mathscr{V}_s$ with $x \in \mathscr{P}_s$;
- H is a $(q;a)$-tic hypersurface parameterized by the linear system $\mathbf{P}\mathscr{M}_s$ as in 5.4; and
- Y is the vanishing locus of the remaining equations so that $\mathscr{X}_s = Y \cap H$ is a presentation of $\mathscr{X}_s$ as a $(q;\mathbf{a})$-tic scheme.

This data is subject to the conditions that $\ell \subseteq Y$ and

$$\mathrm{mult}_x(\ell \cap H) \geq a(q) - 1.$$

Note H is not uniquely determined in this representation. However, any other H' representing the same point of $\mathscr{X}'$ has equation differing from that of H by a $(q;a)$-tic polynomial in the ideal of Y. In particular, since $\ell \subseteq Y$, the multiplicity condition is well-posed. The residual point map is now described, and constructed, as follows:

6.1. Proposition. — *In the above setting, there exists a morphism over S*

$$\mathrm{res}\colon \mathscr{X}' \setminus \mathrm{pr}_1^{-1}(\mathscr{X}_1|_{\mathscr{P}}) \longrightarrow \mathscr{X}$$

acting on geometric points as $\mathrm{res}(s,x,\ell,H,Y) = \ell \cap H - (a(q)-1)x \in \mathscr{X}_s$.

Proof. To construct this map globally, continue with the notation in 5.5, and observe that the section $\mathrm{pr}_1^*\sigma_1\colon \mathscr{O}_{\mathbf{P}\mathscr{V}'} \to \mathrm{pr}_1^*\mathscr{E}_1$ defining $\mathscr{X}_1$ in $\mathbf{P}\mathscr{T}$ pulled back to $\mathscr{X}'$ factors through a section

$$\tau\colon \mathscr{O}_{\mathscr{X}'} \to \big(\mathscr{S}^\vee \otimes \mathscr{O}_\rho(a(q)-1)\big) \boxtimes \mathscr{O}_\mu(-1).$$

Its value on a point (s,x,ℓ,H,Y) may be identified as the degree $a(q)$ polynomial defining $\ell \cap H$. Twisting down by $\mathscr{O}_\rho(1-a(q)) \boxtimes \mathscr{O}_\mu(1)$ factors out the $(a(q)-1)$-fold zero at $x \in \ell \cap H$, at which point τ may be viewed as a family of linear forms on lines in $\mathscr{V}$. Composing τ with the wedge product isomorphism $\mathscr{S}^\vee \cong \mathscr{S} \otimes \mathscr{O}_\rho(1) \otimes \rho^*\mathscr{O}_\pi(1)$, which sends a linear form to its zero locus, yields a section

$$\tau'\colon \big(\mathscr{O}_\rho(-a(q)) \otimes \rho^*\mathscr{O}_\pi(1)\big) \boxtimes \mathscr{O}_\mu(1) \longrightarrow \mathrm{pr}_1^*\mathscr{S}$$

whose value at (s,x,ℓ,H,Y) is the zero locus of the linear form given by τ; in other words, this is the residual point of intersection between ℓ and H. Including $\mathscr{S}$ into the pullback of $\mathscr{V}$ then provides the map to $\mathscr{X} \subseteq \mathbf{P}\mathscr{V}$. It is defined at points where τ' does not vanish which, from the description, are points where $\ell \subseteq H$: this is the preimage of $\mathscr{X}_1|_{\mathscr{P}}$ in $\mathscr{X}'$. ■

When the families $\mathscr{X}_1|_{\mathscr{P}} \to \mathscr{P}$ and $\mathscr{X}' \to S'$ of pointed lines and penultimate tangents associated with an r-planed filtered family $\mathscr{P} \subseteq \mathscr{X}$ of $(q;\mathbf{a})$-tic complete intersections are themselves generically complete intersections, 6.1 provides a rational map $\mathrm{res}\colon \mathscr{X}' \dashrightarrow \mathscr{X}$ which takes a penultimate tangent to its residual point of intersection. The next goal is to show that res is dominant whenever r is sufficiently large depending only on $\mathbf{a}$. In the following statement, a property is said to hold *fibrewise* in a family over S if the property holds upon restriction to each closed point of S.

6.2. Proposition. — *Let $\mathscr{P} \subseteq \mathscr{X}$ be a family of r-planed $(q;\mathbf{a})$-tic complete intersections over a scheme S. If the family $\mathscr{X}_1$ of pointed lines is of expected relative dimension over $\mathscr{P}$ fibrewise over S and*

$$r \geq r_0(\mathbf{a}) := \Big(\sum\nolimits_{a\in\mathbf{a}} \prod\nolimits_{j\geq 0}(a_j+1)\Big) - 2\cdot\#\mathbf{a} - 1,$$

then, for any associated family $\mathscr{X}'$ of penultimate tangents, the residual point map $\mathrm{res}\colon \mathscr{X}' \dashrightarrow \mathscr{X}$ *exists and is dominant fibrewise over S.*

Begin with a few reductions: If $\mathbf{a}_{\mathrm{nlr}} = \varnothing$, then the statement is empty and there is nothing to prove. So assume otherwise and fix any family $\mathscr{X}'$ of penultimate tangents as constructed in 5.5. Using the fact that the first projection $\mathscr{X} \times_S \mathbf{P}\mathscr{M} \to \mathscr{X}$ is surjective, S may be replaced by the linear system $\mathbf{P}\mathscr{M}$ of hypersurfaces relative to which $\mathscr{X}'$ is constructed so as to assume $\mathscr{M} = \mathscr{O}_S$ and $\mathscr{X}'$ is a family over $S' = \mathscr{P}$. Since, fibrewise over S, $\mathscr{X}'$ is obtained from $\mathscr{X}_1$ by omitting two relatively ample divisors over their common base $\mathscr{P}$, the hypothesis that $\mathscr{X}_1$ is of expected dimension implies that $\mathscr{X}'$ is also of expected dimension. This guarantees, via 6.1, that the residual point map $\mathrm{res}\colon \mathscr{X}' \dashrightarrow \mathscr{X}$ exists, and even that its indeterminacy locus does not contain any fibre over S. Given this, since the statement is fibrewise over S, it suffices to consider the absolute case over $\operatorname{Spec}\mathbf{k}$.

For the remainder of the proof, then, let $P \subseteq X$ be an r-planed $(q;\mathbf{a})$-tic complete intersection in $\mathbf{P}^n$ over $\mathbf{k}$. Write $\mathbf{a} = (a_1, \ldots, a_c)$ and fix a presentation of the form

$$X = H_1 \cap \cdots \cap H_c \;\text{ where } H_i \text{ is a } (q;a_i)\text{-tic hypersurface.}$$

Perhaps after reordering, assume that $a_1 \in \mathbf{Prfl}$ is nonlinear and reduced, and let X' be the scheme of penultimate tangents with respect to H_1.

For each point $y \in X \setminus P$, consider the closed subscheme of P given by

$$Z_y := \{z \in P : \mathrm{mult}_z(\ell_{y,z} \cap H_1) \geq a_1(q) - 1 \text{ and } \ell_{y,z} \subset H_i \text{ for } 2 \leq i \leq c\}$$

where $\ell_{y,z}$ is the line between y and z. The task is to show that the open subscheme $Z_y^\circ \subseteq Z_y$ parameterizing lines intersecting H_1 at z with multiplicity exactly $a_1(q) - 1$ is nonempty for general y. Toward this, observe that:

6.3. Lemma. — *If $Z_y^\circ = \varnothing$, then $\ell_{y,z} \subseteq X$ for every $z \in Z_y$.*

Proof. Emptiness of Z_y° means $\ell_{y,z}$ intersects H_1 with multiplicity at least $a_1(q)$ at z. Since $\ell_{y,z}$ also intersects H_1 at y, $\ell_{y,z}$ must be contained in H_1, whence also X. ∎

Equations for Z_y are simple to describe, and yield the following dimension estimate:

6.4. Lemma. — $\dim Z_y \geq r - r_0(\mathbf{a})$ *for all $y \in X \setminus P$.*

Proof. Write $P_y := \langle y, P\rangle$ for the $(r+1)$-plane spanned by y and the r-plane P, and view linear projection in P_y centred at y as a rational map $P_y \dashrightarrow P$ to identify P with the space of lines in P_y through y. This is resolved into a morphism $a\colon \widetilde{P}_y \to P$ on the blowup of P_y at y. As is standard, a exhibits $\widetilde{P}_y$ as the projective bundle on

$$\mathscr{E} \cong \mathscr{O}_P \oplus \mathscr{O}_P(-1) \subseteq \mathscr{O}_P \otimes \mathrm{H}^0(P_y, \mathscr{O}_{P_y}(1))^\vee$$

where $\mathscr{O}_P$ corresponds to the point $y \in P_y$ and $\mathscr{O}_P(-1)$ is the tautological line subbundle in the subspace corresponding to $P \subset P_y$.

Each of the linear sections $H_{i,y} := H_i \cap P_y$ is a $(q;a_i)$-tic hypersurface in $P_y = \mathbf{P}^{r+1}$ by 2.3. As in 4.4, the total transforms $b^{-1}(H_{i,y})$ are then families of $(q;a_i)$-tic hypersurfaces over P whose equations in $\widetilde{P}_y$ correspond to a section

$$\sigma_i\colon \mathscr{O}_P \to \mathrm{S}^{a_i}(\mathscr{E}^\vee) \cong \bigotimes\nolimits_{j=0}^{m_i} \mathrm{Sym}^{a_{i,j}}(\mathscr{O}_P \oplus \mathscr{O}_P(q^j)).$$

Each line bundle summand corresponds to a coefficient of the equation of $H_{i,y}$ restricted to the line $\ell_{y,z} = \mathbf{P}\mathscr{E}_z$ as a function of $z \in P$; thus $Z_{i,y} := \mathrm{V}(\sigma_i)$ parameterizes points $z \in P$ for which

$\ell_{y,z} \subset H_{i,y}$. Observe that some components of σ_i vanish for *a priori* reasons: Write ξ and η for local fibre coordinates of $\mathbf{P}\mathscr{E}$ so that $\xi = 0$ and $\eta = 0$ define the points z and y on $\ell_{y,z} = \mathbf{P}\mathscr{E}_z$. Since $\ell_{y,z}$ intersects $H_{i,y}$ at both y and z, the coefficients of $\xi^{a_i(q)}$ and $\eta^{a_i(q)}$ vanish, and so

$$\operatorname{codim}(Z_{i,y} \subseteq P) \leq \operatorname{rank} \mathrm{S}^{a_i}(\mathscr{E}^\vee) - 2 = \prod\nolimits_{j=0}^{m_i}(a_{i,j}+1) - 2.$$

The condition on $H_{1,y}$ requires only that $\ell_{y,z}$ intersect it at z with multiplicity $a_1(q)-1$, meaning that the scheme of interest is, rather than $Z_{1,y}$, the potentially larger locus

$$Z'_{1,y} := \{z \in P : \operatorname{mult}_z(\ell_{y,z} \cap H_{1,y}) \geq a_1(q) - 1\}.$$

This is cut out by the vanishing of all components of σ_1 other than that corresponding to the coefficient of $\xi^{a_1(q)-1}\eta$. Combined with the above, this shows that the codimension of $Z'_{1,y}$ in P is at most $\operatorname{rank} \mathrm{S}^{a_1}(\mathscr{E}^\vee) - 3$. Since $Z_y = Z'_{1,y} \cap Z_{2,y} \cap \cdots \cap Z_{c,y}$, the codimension estimates give

$$\begin{aligned}
\dim Z_y &= \dim P - \operatorname{codim}(Z_y \subseteq P) \\
&\geq \dim P - \operatorname{codim}(Z'_{1,y} \subseteq P) - \sum\nolimits_{i=2}^{c} \operatorname{codim}(Z_{i,y} \subseteq P) \\
&\geq r + 2c + 1 - \sum\nolimits_{i=1}^{c} \prod\nolimits_{j=0}^{m_i}(a_{i,j}+1) = r - r_0(\mathbf{a}). \qquad \blacksquare
\end{aligned}$$

Proof of 6.2. Comparing the numerical hypothesis with 6.4 shows that Z_y is nonempty for every $y \in X \setminus P$. Suppose, however, that $Z_y^\circ = \varnothing$ for general $y \in X \setminus P$. Derive a contradiction by estimating the dimension of $X_1|_P$ in two ways: On the one hand, it has its expected dimension by assumption; viewing X_1 as the universal line over the Fano scheme $\mathbf{F}_1(X)$ and using 3.1, this is

$$\dim X_1|_P = (\dim \mathbf{F}_1(X) + 1) - \dim X + \dim P = n + c + r - 1 - \sum\nolimits_{i=1}^{c} \prod\nolimits_{j=0}^{m_i}(a_{i,j}+1).$$

On the other hand, emptiness of Z_y° together with 6.3 gives a morphism

$$\{(y,z) \in (X \setminus P) \times P : z \in Z_y\} \to X_1|_P : (y,z) \mapsto (z, [\ell_{y,z}]).$$

Fibres of this map are contained in the points of the lines $\ell_{y,z}$ and so have dimension at most 1. Combined with the dimension estimate 6.4, this gives

$$\dim X_1|_P \geq \dim X + \dim Z_y - 1 \geq n + c + r - \sum\nolimits_{i=1}^{c} \prod\nolimits_{j=0}^{m_i}(a_{i,j}+1).$$

Comparing the two quantities yields a contradiction. Therefore $Z_y^\circ \neq \varnothing$ for general $y \in X$, and this means that the residual point map $\mathrm{res}\colon X' \dashrightarrow X$ is dominant. $\blacksquare$

7. Unirationality

The goal of this section is to establish Theorem B, showing that general $(q;\mathbf{a})$-tic complete intersection is unirational once its dimension is sufficiently large, depending only on the multi-profile $\mathbf{a}$. The proof proceeds by inductively simplifying the equations of the tautological family of $(q;\mathbf{a})$-tic complete intersections via the constructions of §§4–6. Induction takes place over the set Π of all multi-profiles equipped with a somewhat complicated partial ordering; the ordering is designed to keep track of the multi-profiles that appear after passing to one of the following three constructions: Frobenius descent as in 4.11; removing linear equations as in 4.10; and passing to the family of penultimate tangents as in 5.6.

7.1. Ordering multi-profiles. — Let Π be the set of all multi-profiles, and consider the relation $\preceq^\Pi$ defined as follows: The cover relations $\mathbf{a}' \prec\!\!\cdot^\Pi \mathbf{a}$ for this relation come in three flavours, depending on which parts of the canonical type decomposition from 4.8 are present:

(i) If $\mathbf{a} = \mathbf{a}_{\mathrm{pow}}$, then set $\mathbf{a}' := \mathbf{a}/t := (a/t : a \in \mathbf{a})$.

(ii) If $\mathbf{a} = \mathbf{a}_{\mathrm{lin}} \sqcup \mathbf{a}_{\mathrm{pow}}$ and $\mathbf{a}_{\mathrm{lin}} \neq \varnothing$, then set $\mathbf{a}' := \mathbf{a} \setminus \mathbf{a}_{\mathrm{lin}}$.

(iii) If $\mathbf{a}_{\mathrm{nlr}} \neq \varnothing$, then let $a_0 \in \mathbf{a}_{\mathrm{nlr}}$ be any element with maximal coefficient sum $a_0(1)$, and set

$$\mathbf{a}' := (b \in \mathbf{Prfl} : 0 \prec b \preceq a \text{ for } a \in \mathbf{a}) \setminus (a_0, a_0 - 1).$$

In general, two multi-profiles satisfy $\mathbf{a}' \preceq^\Pi \mathbf{a}$ if and only if $\mathbf{a}' = \mathbf{a}$ or else they are connected by a finite sequence of the above cover relations:

$$\mathbf{a}' = \mathbf{a}_n \prec\!\!\cdot^\Pi \mathbf{a}_{n-1} \prec\!\!\cdot^\Pi \cdots \prec\!\!\cdot^\Pi \mathbf{a}_1 \prec\!\!\cdot^\Pi \mathbf{a}_0 = \mathbf{a}.$$

7.2. Examples. — The following are a few examples illustrating properties of the poset $(\Pi, \preceq^\Pi)$:

(i) If $a(t) = d$ is a constant, then the interval $[\varnothing, d]^\Pi$ between $\varnothing$ and d is totally ordered. Explicitly, the Hasse diagrams for $3 \leq d \leq 5$ take the form:

$$(3) - (1) - \varnothing, \quad (4) - (2,1) - (1) - \varnothing, \quad (5) - (3,2,1) - (2,1,1,1) - (1,1,1) - \varnothing.$$

To depict the Hasse diagram for $d = 6$, write k^m for the profile k appearing m times:

$$(6) - (4,3,2,1) - (3,2^3,1^4) - (2^3,1^8) - (2^2,1^{10}) - (2,1^{11}) - (1^{11}) - \varnothing.$$

(ii) The Hasse diagram for the interval $[\varnothing, 1+t]^\Pi$ is: $(1+t) - (1) - \varnothing$.

(iii) The cover relation 7.1(i) appears in the Hasse diagram for $[\varnothing, 1+2t]^\Pi$:

$$(1+2t) - (1+t,t,1) - (t,1,1) - (t) - (1) - \varnothing.$$

(iv) Consider the multi-profile $\mathbf{a} := ((q+1)t+1, t^2+(q+1))$ of $\sqsubseteq$-incomparable profiles of numerical degree q^2+q+1 from 1.3(iii). Specializing to $q = 2$, the two multi-profiles covered by $\mathbf{a}$ via the relation 7.1(iii) are obtained from

$$\mathbf{a}_1 = (3t+1, 3t, 2t+1, 2t, t+1, t, 1) \cup (t^2+3, t^2+2, t^2+1, t^2, 3, 2, 1)$$

by omitting either $(3t+1, 3t)$ or else (t^2+3, t^2+2).

(v) By considering a slight variant of (iv), it is possible to see that lengths of paths from $\mathbf{a}$ to $\varnothing$ need not be the same. For instance, this can be seen with the pair $\mathbf{a} = ((q+2)t+1, t^2+(2q+1))$ of $\sqsubseteq$-incomparable profiles of numerical degree q^2+2q+1.

(vi) Maximal coefficient sums in a multi-profile does not necessarily drop along a cover relation of the form 7.1(iii). This is because the maximum may be achieved by a member of $\mathbf{a}_{\mathrm{pow}}$. For example, consider the Hasse diagram for $\mathbf{a} = (2, 3t)$:

$$(2, 3t) - (3t) - (3) - (1) - \varnothing.$$

7.3. An invariant. — Basic properties of $(\Pi, \preceq^\Pi)$ require some effort to establish since, for example, multi-profiles may grow bigger along the cover relations defined in 7.1 and maximal coefficient sums of profiles does not necessarily decrease along $\preceq^\Pi$ as in 7.2(vi). To address these difficulties, consider the three integer-valued functions

$$\delta(\mathbf{a}) := \max\{\deg_t a(t) : a \in \mathbf{a}\}, \ \ \sigma(\mathbf{a}) := \max\{a(1) : a \in \mathbf{a}_{\mathrm{nlr}}\}, \ \ \mu(\mathbf{a}) := \#\{a \in \mathbf{a}_{\mathrm{nlr}} : a(1) = \mu_{\mathrm{nlr}}(\mathbf{a})\},$$

where the maximum of an empty set is 0. Define $\phi : \Pi \to \mathbf{Z}^4_{\geq 0}$ by $\phi(\mathbf{a}) := (\delta(\mathbf{a}), \sigma(\mathbf{a}), \mu(\mathbf{a}), \#\mathbf{a}_{\mathrm{nlr}})$. A case analysis shows that

$$\text{if } \mathbf{a}' \prec\!\!\cdot^\Pi \mathbf{a} \text{ is a cover relation in } \Pi, \text{ then } \phi(\mathbf{a}') <^{\mathrm{lex}} \phi(\mathbf{a}),$$

where $<^{\mathrm{lex}}$ is the lexicographical ordering on $\mathbf{Z}^4_{\geq 0}$. Thus this invariant provides a relation preserving function $\phi : (\Pi, \preceq^\Pi) \to (\mathbf{Z}^4_{\geq 0}, \leq^{\mathrm{lex}})$. This makes it possible to establish the basic properties of $\preceq^\Pi$:

7.4. Lemma. — $\preceq^\Pi$ *defines a partial ordering on* Π *with a unique bottom element* $\varnothing \in \Pi$.

Proof. To see that $\preceq^\Pi$ is a partial ordering, it remains to establish antisymmetry. Assume $\mathbf{a} \preceq^\Pi \mathbf{b}$ and $\mathbf{b} \preceq^\Pi \mathbf{a}$. Since $(\mathbf{Z}^4_{\geq 0}, \leq^{\mathrm{lex}})$ is itself a poset, 7.3 implies $\phi(\mathbf{a}) = \phi(\mathbf{b})$. But then the sequence of cover relations witnessing $\mathbf{a} \preceq^\Pi \mathbf{b}$ must be empty, since would ϕ strictly decrease along each step, so $\mathbf{a} = \mathbf{b}$.

Since nothing can precede $\varnothing$ in $\preceq^\Pi$, the remaining statement is that $\varnothing$ can be reached from any multi-profile $\mathbf{a}$ through a finite sequence of cover relations. Since the invariant $\phi(\mathbf{a})$ is a sequence of nonnegative integers and lexicographically strictly decreases along each cover relation, it must eventually reach $(0,0,0,0)$ after finitely many steps. But $\phi^{-1}(0,0,0,0) = \{\varnothing\}$, so $\varnothing \preceq^\Pi \mathbf{a}$. ■

7.5. Lemma. — *For any* $\mathbf{a} \in \Pi$, *the interval* $[\varnothing, \mathbf{a}]^\Pi := \{\mathbf{a}' \in \Pi : \varnothing \preceq^\Pi \mathbf{a}' \preceq^\Pi \mathbf{a}\}$ *is finite.*

Proof. Since $\preceq^\Pi$ is locally finite by construction, it suffices to see that the length of any path from $\mathbf{a}$ down to $\varnothing$ is bounded. Proceed by induction on $\phi(\mathbf{a}) \in \mathbf{Z}^4_{\geq 0}$ ordered lexicographically. The base case is $\phi(\varnothing) = (0,0,0,0)$, in which there is nothing to prove. Given a nonzero $(\delta, \sigma, \mu, \lambda) \in \mathbf{Z}^4_{\geq 0}$, inductively assume that for every multi-profile $\mathbf{a}'$ with $\phi(\mathbf{a}') <^{\mathrm{lex}} (\delta, \sigma, \mu, \lambda)$, there exists an integer $L = L(\mathbf{a}')$ such that any sequence of cover relations between $\mathbf{a}'$ and $\varnothing$ has length at most L. Consider a multi-profile $\mathbf{a} \in \phi^{-1}(\delta, \sigma, \mu, \lambda)$ and consider three cases:

Suppose that $\sigma = 0$, so that $\mathbf{a}_{\mathrm{nlr}} = \varnothing$. In the case $\delta = 0$, then $\mathbf{a} = \mathbf{a}_{\mathrm{nlr}}$ and a single cover relation of type 7.1(ii) brings $\mathbf{a}$ to $\varnothing$. If $\delta > 0$, then any path from $\mathbf{a}$ to $\varnothing$ begins with a step of the relation 7.1(i). This produces a new multi-profile $\mathbf{a}'$ with $\delta(\mathbf{a}') = \delta - 1 < \delta$ and so induction applies. Therefore, in this case, any path from $\mathbf{a}$ to $\varnothing$ has length at most $L(\mathbf{a}) := L(\mathbf{a}') + 1$.

If $\sigma \neq 0$, then any sequence of cover relations from $\mathbf{a}$ down to $\varnothing$ begins with the relation 7.1(iii). Each of the μ choices for applying this relation leads to a multi-profile $\mathbf{a}'$ with a lexicographically smaller $\phi(\mathbf{a}')$. Taking L' to be the maximum over these $\mathbf{a}'$ of the bounds $L(\mathbf{a}')$ provided by induction, any path from $\mathbf{a}$ to $\varnothing$ is bounded by $L(\mathbf{a}) := L' + 1$ cover relations. This concludes the induction. ■

7.6. Numbers. — Inductively define numerical functions on the poset $(\Pi, \preceq^\Pi)$ as follows: Let $r \geq 0$ be an integer and $\mathbf{a}$ be a multi-profile. Assuming that $\mathbf{a}_{\mathrm{nlr}} \neq \varnothing$, define

$$\begin{aligned}
r(\mathbf{a}) &:= \max\{r_0(\mathbf{a})\} \cup \{r(\mathbf{a}') + 1 : \mathbf{a}' \prec^\Pi \mathbf{a}\}, \\
n_1(\mathbf{a}, r) &:= \max\{2r - 1 + \#\mathbf{a}_1\} \cup \{n_1(\mathbf{a}', r-1) + 1 : \mathbf{a}' \prec^\Pi \mathbf{a}\}, \text{ and} \\
n_2(\mathbf{a}, r) &:= \max\Big\{\Big\lceil r + \frac{1}{r} \sum\nolimits_{a \in \mathbf{a}} \prod\nolimits_{j \geq 0} \binom{a_j + r}{r} - \frac{1}{r}\#\mathbf{a}\Big\rceil\Big\} \cup \{n_2(\mathbf{a}', r-1) + 1 : \mathbf{a}' \prec^\Pi \mathbf{a}\},
\end{aligned}$$

where $r_0(\mathbf{a})$ is defined in 6.2, $\mathbf{a}'$ in the second set ranges over elements of Π covered by $\mathbf{a}$ as in 7.1, and $\lceil \cdot \rceil$ denotes the ceiling function. One may verify that $n_1(\mathbf{a}, r) \leq n_2(\mathbf{a}, r)$ unless $\mathbf{a} = (2t^k) \cup \mathbf{a}'$ where $\mathbf{a}'$ consist of profiles of the form t^m. In the case that $\mathbf{a}_{\mathrm{nlr}} = \varnothing$, define

$$\begin{aligned}
r(\mathbf{a}) &:= \max\{r_0(\mathbf{a})\} \cup \{r(\mathbf{a}') : \mathbf{a}' \prec^\Pi \mathbf{a}\}, \\
n_1(\mathbf{a}, r) &:= \max\{2r - 1 + \#\mathbf{a}_1\} \cup \{n_1(\mathbf{a}', r) : \mathbf{a}' \prec^\Pi \mathbf{a}\}, \text{ and} \\
n_2(\mathbf{a}, r) &:= \max\Big\{\Big\lceil r + \frac{1}{r} \sum\nolimits_{a \in \mathbf{a}} \prod\nolimits_{j \geq 0} \binom{a_j + r}{r} - \frac{1}{r}\#\mathbf{a}\Big\rceil\Big\} \cup \{n_2(\mathbf{a}', r) : \mathbf{a}' \prec^\Pi \mathbf{a}\},
\end{aligned}$$

where the notation is as before. Here, there is precisely one multi-profile $\mathbf{a}'$ covered by $\mathbf{a}$, and a direct computation shows that $r_0(\mathbf{a}) = r_0(\mathbf{a}')$, and hence $r(\mathbf{a}) = r(\mathbf{a}')$, and also $n_i(\mathbf{a}', r) = n_i(\mathbf{a}, r) - \#\mathbf{a}_{\mathrm{lin}}$

for $i = 1, 2$. Finally, for all $\mathbf{a} \in \Pi$ and $r \geq 0$, define

$$n_0(\mathbf{a}, r) := \max\{n_1(\mathbf{a}, r), n_2(\mathbf{a}, r)\}$$

With these, it is possible to formulate the precise unirationality result:

7.7. Proposition. — *Let $\mathscr{P} \subseteq \mathscr{X}$ be a generic family of r-planed $(q;\mathbf{a})$-tic complete intersections in a $\mathbf{P}^n$-bundle over an integral base scheme S with $r \geq r(\mathbf{a})$. If $n \geq n_0(\mathbf{a}, r)$, then the general fibre of $\mathscr{X}$ over S is unirational.*

Proof. Proceed by induction on $\mathbf{a}$ along the poset $(\Pi, \preceq^\Pi)$. The base case is when $\mathbf{a} = \varnothing$, in which case $\mathscr{X} = \mathbf{P}\mathscr{V}$ and each fibre over S is even rational. Let $\mathbf{a} \neq \varnothing$ and inductively assume that the conclusion holds for all multi-profiles preceding it in Π. The task is to construct a new generic r'-planed family $\mathscr{P}' \subseteq \mathscr{X}'$ of $(q;\mathbf{a}')$-tic complete intersections in a $\mathbf{P}^{n'}$-bundle over an integral base S' with $\mathbf{a}' \prec^\Pi \mathbf{a}$, $r' \geq r(\mathbf{a}')$, $n' \geq n_0(\mathbf{a}', r')$, and all fitting into a commutative diagram of the form

$$\begin{tikzcd} \mathscr{X}' \arrow[r, dashed] \arrow[d] & \mathscr{X} \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

where the horizontal maps are dominant. Induction would then give the result since $\varnothing$ is the unique bottom element by 7.4 and the interval $[\varnothing, \mathbf{a}]^\Pi$ is finite by 7.5. Decompose the multi-profile as $\mathbf{a} = \mathbf{a}_{\mathrm{lin}} \sqcup \mathbf{a}_{\mathrm{pow}} \sqcup \mathbf{a}_{\mathrm{nlr}}$ into types as in 4.8, and proceed depending on which types are present:

If $\mathbf{a}_{\mathrm{nlr}} \neq \varnothing$, apply 4.9 to choose a nonzero subbundle $\mathrm{S}^{a_0}(\mathscr{V}^\vee) \otimes \mathscr{M} \subseteq \mathscr{A}_{\mathrm{nlr}}$ where $a_0 \in \mathbf{a}_{\mathrm{nlr}}$ has maximal coefficient sum $a_0(1)$. Applying the penultimate line constructions of 5.5, 5.6, and 5.7 provides a family of $(r-1)$-planed $(q;\mathbf{a}')$-tic schemes in a $\mathbf{P}^{n-1}$-bundle over $\mathscr{P} \times_S \mathbf{P}\mathscr{M}$, where

$$\mathbf{a}' = (a' \in \mathbf{Prfl} : 0 \prec a' \preceq a \text{ for } a \in \mathbf{a}) \setminus (a_0, a_0 - 1).$$

This family is generic by 5.11. The choice of n together with 5.12 provide a nonempty open subscheme $S' \subseteq \mathscr{P} \times_S \mathbf{P}\mathscr{M}$ over which the family $\mathscr{P}' \subseteq \mathscr{X}'$ is a $(q;\mathbf{a}')$-tic complete intersection. The residual point map $\mathrm{res}\colon \mathscr{X}' \dashrightarrow \mathscr{X}$ from 6.1 exists and is dominant by 6.2 and the choice of r, providing the sought after rational map over the dominant map $S' \to S$. The choice of numbers in 7.6 implies $r' = r - 1 \geq r(\mathbf{a}')$ and $n' = n - 1 \geq n_0(\mathbf{a}', r')$, completing the induction in this case.

If $\mathbf{a}_{\mathrm{nlr}} = \varnothing$ and $\mathbf{a}_{\mathrm{lin}} \neq \varnothing$, then apply 4.10 to view $\mathscr{P} \subseteq \mathscr{X}$ as a family of r-planed $(q;\mathbf{a}')$-tic complete intersections in a projective subbundle $\mathbf{P}\mathscr{V}' \subseteq \mathbf{P}\mathscr{V}$ where $\mathbf{a}' := \mathbf{a} \setminus \mathbf{a}_{\mathrm{lin}}$. Thus $\mathscr{X}' = \mathscr{X}$ and $S' = S$, but $\mathscr{X}'$ is equipped with the structure of a $(q;\mathbf{a}')$-tic complete intersection in a different projective bundle. By 4.15, the family $\mathscr{P} \subseteq \mathscr{X}$ is also generic with this structure. Now

$$r' = r \geq r(\mathbf{a}) = r(\mathbf{a}') \;\text{ and }\; n' = n - \#\mathbf{a}_{\mathrm{lin}} \geq n_0(\mathbf{a}, r) - \#\mathbf{a}_{\mathrm{lin}} = n_0(\mathbf{a}', r'),$$

completing the inductive step here.

Finally, if $\mathbf{a} = \mathbf{a}_{\mathrm{pow}}$, apply 4.11 to obtain a family $\mathscr{X}'$ of $(q;\mathbf{a}')$-tic complete intersections in a $\mathbf{P}^n$-bundle over $S' = S$ with $\mathbf{a}' := \mathbf{a}/t$ and fitting into a diagram

$$\begin{tikzcd} \mathscr{X}' \arrow[r] \arrow[d] & \mathscr{X} \arrow[d] \\ S \arrow[r, "\mathrm{Fr}"] & S \end{tikzcd}$$

where the horizontal map up top is a universal homeomorphism. Pulling back the family $\mathscr{P}$ of r-planes equips $\mathscr{X}'$ with a family of r-planes. It is straightforward to check that genericity of the family $\mathscr{P} \subseteq \mathscr{X}$ propagates to genericity of $\mathscr{P}' \subseteq \mathscr{X}'$. Since $r' := r$ and $n' := n$, but also $r(\mathbf{a}) = r(\mathbf{a}/t)$ and $n_0(\mathbf{a}, r) = n_0(\mathbf{a}/t, r)$, the inductive step is settled in this case too. ∎

Proof of Theorem B. Given a multi-profile $\mathbf{a}$, set

$$r := r(\mathbf{a}) \text{ and } n := n_0(\mathbf{a}) := n_0(\mathbf{a}, r)$$

where $r(\mathbf{a})$ and $n_0(\mathbf{a}, r)$ are as in 7.6. Consider the incidence correspondence

$$\mathbf{Inc}_{n,r,\mathbf{a}} := \{([U],[\boldsymbol{\alpha}]) \in \mathbf{G}(r+1,n+1) \times (q;\mathbf{a})\text{-}\mathbf{tics}_{\mathbf{P}^n} : \mathbf{P}U \subseteq X_{\boldsymbol{\alpha}}\}$$

between r-planes and $(q;\mathbf{a})$-tic schemes in $\mathbf{P}^n$. This is a projective bundle over the Grassmannian via the first projection, and so it is integral. Let $S \subseteq \mathbf{Inc}_{n,r,\mathbf{a}}$ be the open subscheme parameterizing r-planed $(q;\mathbf{a})$-tic complete intersections $P \subseteq X$. Restricting the tautological family to S therefore provides a family $\mathscr{P} \subseteq \mathscr{X}$ of r-planed $(q;\mathbf{a})$-tic complete intersections in $\mathbf{P}^n_S$ satisfying the hypotheses of 7.7. Therefore each fibre of $\mathscr{X} \to S$ is unirational. Since the choice of n and r imply, via 3.16, that the projection $S \to (q;\mathbf{a})\text{-}\mathbf{tics}_{\mathbf{P}^n}$ is dominant and the result follows. ∎

7.8. Example. — The bound $n_0(\mathbf{a})$ may be computed for small multi-profiles $\mathbf{a}$. Some examples:

$$n_0(t+1) = 4, \quad n_0(2, t+1) = 9, \quad n_0(t+1, t+1) = 13, \quad n_0(t^2+t+1) = 48.$$

The bounds obtained for constant profiles $a(t) = d$, corresponding to degree d hypersurfaces as in 2.2(i), are surprisingly small:

$$n_0(3) = 4, \quad n_0(4) = 9, \quad n_0(5) = 22, \quad n_0(6) = 160, \quad n_0(7) = 20376,$$
$$n_0(8) = 11914188890, \quad n_0(9) = 8616199237736295920955120,$$

and $n_0(10) = 192884152577980851363553858004926940342106493833715693762179 \approx 10^{59}$. In comparison, the best bounds $n_0'(d)$ for the classical unirationality construction are from [Ram90], and they give

$$n_0'(3) = 3, \quad n_0'(4) = 20, \quad n_0'(5) = 8855, \quad n_0'(6) = 454205040715033146,$$
$$n_0'(7) \approx 10^{103}, \quad n_0'(8) \approx 10^{717}, \quad n_0'(9) \approx 10^{5738}, \quad n_0'(10) \approx 10^{51641}.$$

Asymptotically, $n_0(d)$ grows quite a bit slower than $n_0'(d)$; details shall appear elsewhere.

References

[BC66] Raj C. Bose and Indra M. Chakravarti, *Hermitian varieties in a finite projective space* PG(N, q^2), Canadian J. Math. **18** (1966), 1161–1182.

[Bea90] Arnaud Beauville, *Sur les hypersurfaces dont les sections hyperplanes sont à module constant. (On hypersurfaces the hyperplane sections of which are of constant module)*, The Grothendieck Festschrift, Collect. Artic. in Honor of the 60th Birthday of A. Grothendieck. Vol. I, Prog. Math. 86, 121-133 (1990)., 1990.

[Ben12] Olivier Benoist, *Degrés d'homogénéité de l'ensemble des intersections complètes singulières*, Ann. Inst. Fourier (Grenoble) **62** (2012), no. 3, 1189–1214.

[BLRT23] Roya Beheshti, Brian Lehmann, Eric Riedl, and Sho Tanimoto, *Rational curves on del Pezzo surfaces in positive characteristic*, Trans. Am. Math. Soc., Ser. B **10** (2023), 407–451.

[BR21] Roya Beheshti and Eric Riedl, *Linear subspaces of hypersurfaces*, Duke Math. J. **170** (2021), no. 10, 2263–2288.

[CG72] C. Herbert Clemens and Phillip A. Griffiths, *The intermediate Jacobian of the cubic threefold*, Ann. Math. (2) **95** (1972), 281–356.

[Che22] Raymond Cheng, *Geometry of q-bic Hypersurfaces*, ProQuest LLC, Ann Arbor, MI, 2022, Thesis (Ph.D.)–Columbia University.

[Che24] ______, *q-bic threefolds and thier surface of lines*, preprint at arXiv:2402.09884.

[Che25a] ______, *q-bic forms*, J. Algebra **675** (2025), 196–236.

[Che25b] ______, *q-bic hypersurfaces and their Fano schemes*, Pure Appl. Math. Q. **21** (2025), no. 4, 1721–1773.

[DL76] Pierre Deligne and George Lusztig, *Representations of reductive groups over finite fields*, Ann. Math. (2) **103** (1976), 103–161.

[DM98] Olivier Debarre and Laurent Manivel, *Sur la variété des espaces linéaires contenus dans une intersection complète*, Math. Ann. **312** (1998), no. 3, 549–574.

[Ein88] Lawrence Ein, *Subvarieties of generic complete intersections*, Invent. Math. **94** (1988), no. 1, 163–169.

[Hef85] Abramo Hefez, *Duality for projective varieties*, Ph.D. thesis, Massachusetts Institute of Technology, 1985.

[HMP98] Joe Harris, Barry Mazur, and Rahul Pandharipande, *Hypersurfaces of low degree*, Duke Math. J. **95** (1998), no. 1, 125–160.

[HRS04] Joe Harris, Mike Roth, and Jason Starr, *Rational curves on hypersurfaces of low degree*, J. Reine Angew. Math. **571** (2004), 73–106.

[KKP+21] Zhibek Kadyrsizova, Jennifer Kenkel, Janet Page, Jyoti Singh, Karen E. Smith, Adela Vraciu, and Emily E. Witt, *Cubic surfaces of characteristic two*, Trans. Am. Math. Soc. **374** (2021), no. 9, 6251–6267.

[KKP+22] ———, *Lower bounds on the F-pure threshold and extremal singularities*, Trans. Amer. Math. Soc. Ser. B **9** (2022), 977–1005.

[Kol96] János Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 32, Springer-Verlag, Berlin, 1996.

[Kol01] ———, *Which are the simplest algebraic varieties?*, Bull. Am. Math. Soc., New Ser. **38** (2001), no. 4, 409–433.

[KP91] Steven Kleiman and Ragni Piene, *On the inseparability of the Gauss map*, Enumerative algebraic geometry (Copenhagen, 1989), Contemp. Math., vol. 123, Amer. Math. Soc., Providence, RI, 1991, pp. 107–129.

[Mor42] Ugo Morin, *Sull'unirazionalità dell'ipersuperficie algebrica di qualunque ordine e dimensione sufficientemente alta*, Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, Ed. Cremonese, Rome, 1942, pp. 298–302.

[Mur72] Jacob P. Murre, *Algebraic equivalence modulo rational equivalence on a cubic threefold*, Compos. Math. **25** (1972), 161–206.

[MW25] Gebhard Martin and Réka Wagener, *Classification of non-F-split del Pezzo surfaces of degree* 1, Arch. Math. (2025).

[Pac04] Gianluca Pacienza, *Subvarieties of general type on a general projective hypersurface*, Trans. Am. Math. Soc. **356** (2004), no. 7, 2649–2661.

[Pre49] Arno Predonzan, *Sull'unirazionalità della varietà intersezione completa di più forme*, Rend. Sem. Mat. Univ. Padova **18** (1949), 163–176.

[PS92] Kapil H. Paranjape and V. Srinivas, *Unirationality of the general complete intersection of small multidegree*, Flips and abundance for algebraic threefolds, Astérisque, vol. 211, Société Mathématique de France, Paris, 1992, pp. 241—-248.

[Ram90] Lorenzo Ramero, *Effective estimates for unirationality*, Manuscripta Math. **68** (1990), no. 4, 435–445.

[Rot55] Leonard Roth, *Algebraic threefolds, with special regard to problems of rationality*, Ergebnisse der Mathematik und ihrer Grenzgebiete, (N.F.), vol. Heft 6, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955.

[RY20] Eric Riedl and David Yang, *Rational curves on general type hypersurfaces*, J. Differ. Geom. **116** (2020), no. 2, 393–403.

[Sai17] Natsuo Saito, *Characterization of non-F-split del Pezzo surfaces of degree 2*, Tokyo J. Math. **40** (2017), no. 1, 247–253.

[Shi74] Tetsuji Shioda, *An example of unirational surfaces in characteristic p*, Math. Ann. **211** (1974), 233–236.

[Shi92] Ichiro Shimada, *Unirationality of certain complete intersections in positive characteristics*, Tôhoku Math. J. (2) **44** (1992), no. 3, 379–393.

[Shi95] ———, *A generalization of Morin-Predonzan's theorem on the unirationality of complete intersections*, J. Algebr. Geom. **4** (1995), no. 4, 597–638.

[Shi01] ———, *Lattices of algebraic cycles on Fermat varieties in positive characteristics*, Proc. London Math. Soc. (3) **82** (2001), no. 1, 131–172.

[Shi12] ———, *On Frobenius incidence varieties of linear subspaces over finite fields*, Finite Fields Appl. **18** (2012), no. 2, 337–361.

[Stacks] The Stacks Project Authors, *Stacks Project*, https://stacks.math.columbia.edu.

[Voi96] Claire Voisin, *On a conjecture of Clemens on rational curves on hypersurfaces*, J. Differ. Geom. **44** (1996), no. 1, 200–213.

[Wal56] Andrew H. Wallace, *Tangency and duality over arbitrary fields*, Proc. London Math. Soc. (3) **6** (1956), 321–342.

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