UNBOUNDED NEGATIVITY ON RATIONAL SURFACES IN POSITIVE CHARACTERISTIC

RAYMOND CHENG AND REMY VAN DOBBEN DE BRUYN

ABSTRACT. We give explicit blowups of the projective plane in positive characteristic that contain arbitrarily negative smooth rational curves, showing that the Bounded Negativity Conjecture fails even for rational surfaces in positive characteristic.

INTRODUCTION

A smooth projective surface X over an algebraically closed field is said to have *Bounded Negativity* if there exists a positive integer $b(X)$ such that $C^2 \geq -b(X)$ for any reduced curve $C \subset X$. A folklore conjecture, going back to Enriques and discussed in [Har10, Conjecture I.2.1] and [BHK+13, Conjecture 1.1], is the

Bounded Negativity Conjecture. — *Any smooth projective surface in characteristic* 0 *has Bounded Negativity.*

The assumption on the characteristic cannot be dropped: if C is a curve over $\bar{\mathbf{F}}_p$, then the graph $\Gamma_{F^e} \subseteq C \times C$ of the p^e-th power Frobenius endomorphism has self-intersection $p^e(2-2g)$, which becomes arbitrarily negative as $e \to \infty$ when $g \geq 2$. Nonetheless, it is conceivable that certain geometric assumptions on the surface may still guarantee Bounded Negativity in positive characteristic. For instance, [BBC+12, discussion preceding Example 3.3.3] and [Har18, Conjecture 2.1.2] ask whether smooth rational surfaces over a field of positive characteristic have Bounded Negativity. We give a negative answer to this question:

Main Theorem. — *Let k be an algebraically closed field of characteristic $p > 0$, let m be a positive integer invertible in k, and let R_m be the blowup of $\mathbf{P}^2$ along*

$$Z_m := \left\{ [x_0 : x_1 : x_2] \,\middle|\, x_0^m = x_1^m = x_2^m \right\}.$$

Let $C_1 = V(x_0 + x_1 + x_2) \subseteq \mathbf{P}^2$, and for $d \geq 1$ invertible in k, write $C_d \subseteq \mathbf{P}^2$ for the image of

$$\begin{aligned} \phi_d \colon C_1 &\to \mathbf{P}^2 \\ [x_0 : x_1 : x_2] &\mapsto [x_0^d : x_1^d : x_2^d]. \end{aligned}$$

If $dm = p^e - 1$ for some positive integer e, then the strict transform $\widetilde{C}_d \subseteq R_m$ of C_d is a smooth rational curve with $\widetilde{C}_d^2 = d(3-m) - 1$. Thus, if $m > 3$, the rational surface R_m does not have Bounded Negativity over k.

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Since $\mathbf{P}^2$ has Bounded Negativity, this shows that [BDRH+15, Problem 1.2] has a negative answer in positive characteristic:

Corollary. — *Bounded Negativity is not a birational property of smooth projective surfaces in positive characteristic.* ∎

In fact, since every smooth projective surface X admits a finite morphism $X \to \mathbf{P}^2$, pulling back the blowup $R_m \to \mathbf{P}^2$ gives a blowup $\widetilde{X} \to X$ with a finite morphism $\widetilde{X} \to R_m$. Pulling back the curves $\widetilde{C}_d$ to $\widetilde{X}$ shows:

Corollary. — *If X is a smooth projective surface over an algebraically closed field k of positive characteristic, then there exists a blowup $\widetilde{X} \to X$ such that $\widetilde{X}$ does not have Bounded Negativity.* ∎

In §1, we give a direct proof of the **Main Theorem**. In §2, we realise the plane curves C_d as norms of line configuration, thereby deriving equations for them. In §3, we view R_m as an isotrivial family of diagonal curves over C_1 and relate the curves $\widetilde{C}_d$ on R_m to graphs of Frobenius morphisms on Fermat curves. Finally, we close in §4 with some questions and remarks towards characteristic zero.

Sections 2 and 3 each give alternative methods for computing the self-intersections of $\widetilde{C}_d$. Given the simplicity of the formulas for $\widetilde{C}_d$ and the many connections to other well-studied examples, it is surprising that these curves have not been found before.

Notation

Throughout the paper, k will be an algebraically closed field of arbitrary characteristic, and m and d will denote positive integers invertible in k. We will use the notation of the **Main Theorem** throughout.

1. Proof of Main Theorem

In this section, fix m and write $R := R_m$ for the blowup of $\mathbf{P}^2$ along $Z := Z_m$. The generators $s_0 = x_1^m - x_2^m$, $s_1 = x_2^m - x_0^m$, and $s_2 = x_0^m - x_1^m$ of the ideal of Z give a closed immersion $R \hookrightarrow \mathbf{P}^2 \times \mathbf{P}^2$. Since $s_0 + s_1 + s_2 = 0$, one of the s_i can be eliminated at the expense of breaking the symmetry in the computations below.

1.1. Lemma. — *The embedding $R \hookrightarrow \mathbf{P}^2 \times \mathbf{P}^2$ realises R as the complete intersection*

$$\left\{ \big([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]\big) \in \mathbf{P}^2 \times \mathbf{P}^2 \;\middle|\; \begin{array}{c} y_0 + y_1 + y_2 = 0 \\ x_0^m y_0 + x_1^m y_1 + x_2^m y_2 = 0 \end{array} \right\}$$

of degrees $(0,1)$ and $(m,1)$ in $\mathbf{P}^2 \times \mathbf{P}^2$. In particular, $K_R = \mathcal{O}_R(m-3,-1)$.

Proof. The generators s_0, s_1, s_2 of the ideal of Z identify R as

$$\left\{ \big([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]\big) \in \mathbf{P}^2 \times \mathbf{P}^2 \;\middle|\; \begin{array}{c} y_0(x_2^m - x_0^m) = y_1(x_1^m - x_2^m) \\ y_1(x_0^m - x_1^m) = y_2(x_2^m - x_0^m) \\ y_2(x_1^m - x_2^m) = y_0(x_0^m - x_1^m) \end{array} \right\}.$$

The relation $s_0+s_1+s_2=0$ shows that R is contained in the locus $y_0+y_1+y_2=0$. The equation $y_0(x_2^m-x_0^m)=y_1(x_1^m-x_2^m)$ can be rewritten as $(y_0+y_1)x_2^m=x_0^m y_0+x_1^m y_1$, which is equivalent to $x_0^m y_0+x_1^m y_1+x_2^m y_2=0$ since $y_0+y_1+y_2=0$. The same holds for the other two equations by symmetry. The final statement then follows from the adjunction formula since $K_{\mathbf{P}^2\times\mathbf{P}^2}=\mathcal{O}(-3,-3)$. ■

Alternatively, one can observe that the complete intersection of **Lemma 1.1** maps birationally onto its first factor, where the fibres are points when $[x_0^m:x_1^m:x_2^m]\neq[1:1:1]$ and lines otherwise.

1.2. Lemma. — *If* $\operatorname{char} k=p>0$ *and* $dm=p^e-1$ *for some positive integer* e*, then the map* $\widetilde{\phi}_d\colon C_1\to\mathbf{P}^2\times\mathbf{P}^2$ *given by*

$$[x_0:x_1:x_2]\mapsto\big([x_0^d:x_1^d:x_2^d],[x_0:x_1:x_2]\big)$$

lands in R*. In particular, it is the unique map lifting* $\phi_d\colon C_1\to\mathbf{P}^2$.

Proof. Since $x_0+x_1+x_2=0$, the image of $\widetilde{\phi}_d$ is contained in the locus $y_0+y_1+y_2=0$. Since $dm=p^e-1$, the expression $x_0^m y_0+x_1^m y_1+x_2^m y_2$ pulls back to $x_0^{p^e}+x_1^{p^e}+x_2^{p^e}$, which vanishes because the p^e-th power Frobenius is an endomorphism. Thus $\widetilde{\phi}_d$ is a lift of ϕ_d to R, and it is the unique lift since the first projection $\mathrm{pr}_1\colon R\to\mathbf{P}^2$ is birational. ■

1.3. Corollary. — *The map* $\widetilde{\phi}_d\colon C_1\to R$ *is a closed immersion, whose image* $\widetilde{C}_d$ *is a smooth rational curve in* R *with* $\widetilde{C}_d^2=d(3-m)-1$.

Proof. The first two statements follow from the coordinate expression in **Lemma 1.2**, since $\widetilde{\phi}_d$ embeds C_1 linearly into the second factor. The same expression shows that $\widetilde{\phi}_d^*\mathcal{O}_R(a,b)=\mathcal{O}_{C_1}(da+b)$, so $K_R\cdot\widetilde{C}_d=d(m-3)-1$ by **Lemma 1.1**. Then the adjunction formula shows that

$$\widetilde{C}_d^2=-2-K_R\cdot\widetilde{C}_d=d(3-m)-1. \qquad ■$$

This completes the proof of the **Main Theorem**. ■

A consequence of **Corollary 1.3** is that the singularities of C_d are contained in Z. However, the individual multiplicities are not so easy to determine. For example, in **Lemma 3.8** we will compute the multiplicity of C_d at $[1:1:1]$ in terms of point counts on Fermat curves.

2. Relation with Line Configurations

In this section, we observe that the d-th power maps $\pi_d\colon\mathbf{P}^2\to\mathbf{P}^2$ are finite Galois morphisms such that $\pi_d^*C_d$ is the union of the Galois translates of C_1. Thus the C_d are norms of line configurations, from which we derive in **Corollary 2.3** a formal product formula for the equation of the plane curves C_d. In the second half of this section, we observe that in characteristic $p>0$ and for q a power of p, the curve C_{q-1} comes from a subconfiguration of the set of all $\mathbf{F}_q$-rational lines. This allows us to show in **Corollary 2.7** that an equation of C_{q-1} in this case is the complete homogeneous polynomial of degree $q-1$.

2.1. Power Maps. For any integer $a\geq 1$ invertible in k, write π_a for the a-th power map $\mathbf{P}^2\to\mathbf{P}^2$. Since $\pi_a^*Z_m=Z_{am}$, the map π_a lifts to a finite morphism $\widetilde{\pi}_a\colon R_{am}\to R_m$ given by

$$\big([x_0:x_1:x_2],[y_0:y_1,y_2]\big)\mapsto\big([x_0^a:x_1^a:x_2^a],[y_0:y_1,y_2]\big).$$

Since a is invertible in k, both π_a and $\tilde{\pi}_a$ are finite Galois with group $G = \boldsymbol{\mu}_a^3/\boldsymbol{\mu}_a$, where $(\zeta_0, \zeta_1, \zeta_2) \in G$ acts on $\mathbf{P}^2$ via

$$[x_0 : x_1 : x_2] \mapsto [\zeta_0 x_0 : \zeta_1 x_1 : \zeta_2 x_2].$$

This gives a tower of extensions

$$\begin{array}{ccccccc} R_4 & & R_6 & & R_9 & & \iddots \\ & \diagdown & | & \diagdown & | & \diagup & \\ & & R_2 & & R_3 & & \iddots \\ & & & \diagdown & | & \diagup & \\ & & & & R_1 & & \end{array}$$

indexed by the poset of positive integers invertible in k under the divisibility relation.

2.2. Lemma. — *If $a, d \geq 1$ are invertible in k, then*

(i) *The map $\phi_d : C_1 \to \mathbf{P}^2$ is unramified and birational onto its image;*
(ii) *The inverse image $\pi_a^* C_{ad}$ is totally split into the G-translates of C_d.*

Proof. The Jacobian $(d \cdot x_0^{d-1}, d \cdot x_1^{d-1}, d \cdot x_2^{d-1})$ of ϕ_d only vanishes when $x_0 = x_1 = x_2 = 0$, showing that ϕ_d is unramified. Then the map $C_1 \to C_d^\nu$ to the normalisation of C_d is unramified, hence an isomorphism since it is an étale map of smooth projective rational curves, proving (i).

Since $\pi_a \circ \phi_d = \phi_{ad}$, part (i) shows that π_a maps C_d birationally onto its image. This shows that the decomposition group of C_d is trivial, so no two G-translates ζC_d of C_d coincide and C_{ad} is totally split under π_a. ■

2.3. Corollary. — *If d is invertible in k, then the homogeneous ideal of $C_d \subseteq \mathbf{P}^2$ is generated by*

$$f_d := N_{\pi_{d,*}\mathcal{O}_{\mathbf{P}^2}/\mathcal{O}_{\mathbf{P}^2}}\big(x_0^{1/d} + x_1^{1/d} + x_2^{1/d}\big) = \prod_{\zeta,\zeta' \in \boldsymbol{\mu}_d} \big(x_0^{1/d} + \zeta x_1^{1/d} + \zeta' x_2^{1/d}\big).$$

Proof. By **Lemma 2.2** (ii), the inverse image $\pi_d^{-1}(C_d)$ is the union of lines $\bigcup_{\zeta \in \boldsymbol{\mu}_d^3/\boldsymbol{\mu}_d} \zeta C_1$. The result follows since C_1 is cut out by $x_0 + x_1 + x_2 = 0$. ■

2.4. In general, the f_d are complicated symmetric polynomials. However, in **Corollary 2.7** we will show that the coefficients of f_{q-1} are congruent to 1 modulo p if q is a power of a prime p. For example, for $q = p = 3$, we get

$$\begin{aligned} N\big(x_0^{\frac12} + x_1^{\frac12} + x_2^{\frac12}\big) &= \big(x_0^{\frac12} + x_1^{\frac12} + x_2^{\frac12}\big)\big(x_0^{\frac12} + x_1^{\frac12} - x_2^{\frac12}\big)\big(x_0^{\frac12} - x_1^{\frac12} + x_2^{\frac12}\big)\big(x_0^{\frac12} - x_1^{\frac12} - x_2^{\frac12}\big) \\ &= x_0^2 + x_1^2 + x_2^2 - 2x_0x_1 - 2x_1x_2 - 2x_2x_0. \\ &\equiv x_0^2 + x_1^2 + x_2^2 + x_0x_1 + x_1x_2 + x_2x_0 \pmod 3. \end{aligned}$$

In the remainder of this section, assume $\operatorname{char} k = p > 0$ and let q be a power of p.

2.5. Finite Field Line Configurations. The configuration of $\mathbf{F}_q$-rational lines in $\mathbf{P}^2$ is the union of the lines $L_a = \{a_0x_0 + a_1x_1 + a_2x_2 = 0\}$ indexed by $a = [a_0 : a_1 : a_2] \in \check{\mathbf{P}}^2(\mathbf{F}_q)$. Their union is the divisor in $\mathbf{P}^2$ cut out by the polynomial

$$\det\begin{pmatrix} x_0 & x_1 & x_2 \\ x_0^q & x_1^q & x_2^q \\ x_0^{q^2} & x_1^{q^2} & x_2^{q^2} \end{pmatrix} = x_0^q x_1^{q^2} x_2 - x_0^{q^2} x_1^q x_2 + x_0 x_1^q x_2^{q^2} - x_0 x_1^{q^2} x_2^q + x_0^{q^2} x_1 x_2^q - x_0^q x_1 x_2^{q^2},$$

since the three columns become linearly dependent when x_0, x_1, and x_2 satisfy a linear relation over $\mathbf{F}_q$, and the degree equals $q^2+q+1 = |\check{\mathbf{P}}^2(\mathbf{F}_q)|$. Now **Lemma 2.2**(ii) shows that $\pi_{q-1}^* C_{q-1}$ consists of the q^2-2q+1 lines L_a with all coordinates of $a = [a_0 : a_1 : a_2]$ nonzero. We can thus derive an equation for C_{q-1} by extracting factors cutting out the lines L_a in which a has a vanishing coordinate. A neat description of the result comes from the following polynomial identity, also observed in [RVVZ01, p. 90]:

2.6. Lemma. — *For any nonnegative integer n, define the polynomials*

$$g_n := \sum_{n_0+n_1+n_2=n} x_0^{n_0} x_1^{n_1} x_2^{n_2} \quad \text{and} \quad h_n := x_0 x_1^n - x_0^n x_1 + x_1 x_2^n - x_1^n x_2 + x_2 x_0^n - x_2^n x_0$$

in $\mathbf{Z}[x_0, x_1, x_2]$. *Then* $h_2 = (x_2-x_1)(x_0-x_2)(x_1-x_0)$ *and* $h_n = h_2 g_{n-2}$ *for* $n \geq 3$.

Proof. Let $G(t) := \sum_{n\geq 0} g_n t^n$ and $H(t) := \sum_{n\geq 0} h_n t^n$ be the generating functions of the g_n and h_n, respectively. A standard computation gives

$$G(t) = \frac{1}{(1-x_0 t)(1-x_1 t)(1-x_2 t)}.$$

On the other hand, writing $h_n = (x_2-x_1)x_0^n + (x_0-x_2)x_1^n + (x_1-x_0)x_2^n$ gives

$$H(t) = \frac{x_2-x_1}{1-x_0 t} + \frac{x_0-x_2}{1-x_1 t} + \frac{x_1-x_0}{1-x_2 t} = \frac{(x_2-x_1)x_2x_1 + (x_0-x_2)x_0x_2 + (x_1-x_0)x_0x_1}{(1-x_0 t)(1-x_1 t)(1-x_2 t)} t^2.$$

The result follows by recognising the numerator as h_2. ■

2.7. Corollary. — *Suppose* $\operatorname{char} k = p > 0$ *and q is a power of p. Then* g_{q-1} *generates the homogeneous ideal of* $C_{q-1} \subseteq \mathbf{P}^2$. *In particular,* $f_{q-1} \equiv g_{q-1} \pmod p$.

Proof. Since $C_1 = L_{[1:1:1]}$ is among the $\mathbf{F}_q$-rational lines of **2.5** and is not a coordinate axis, the points $[x_0 : x_1 : x_2]$ of $C_1 = V(x_0+x_1+x_2)$ satisfy the equation there divided by $x_0x_1x_2$:

$$x_0^{q-1}x_1^{q^2-1} - x_0^{q^2-1}x_1^{q-1} + x_1^{q-1}x_2^{q^2-1} - x_1^{q^2-1}x_2^{q-1} + x_2^{q-1}x_0^{q^2-1} - x_2^{q^2-1}x_0^{q-1} = 0.$$

Since $C_1 \to C_{q-1}$ is $[x_0 : x_1 : x_2] \mapsto [x_0^{q-1} : x_1^{q-1} : x_2^{q-1}]$, any $[x_0 : x_1 : x_2] \in C_{q-1}$ satisfies

$$x_0 x_1^{q+1} - x_0^{q+1} x_1 + x_1 x_2^{q+1} - x_1^{q+1} x_2 + x_2 x_0^{q+1} - x_2^{q+1} x_0 = 0.$$

So h_{q+1} vanishes on C_{q-1}, which by **Lemma 2.6** equals $(x_2-x_1)(x_0-x_2)(x_1-x_0)g_{q-1}$. The result follows since C_{q-1} is not contained in any of the lines $\{x_2 = x_1\}$, $\{x_0 = x_2\}$, or $\{x_1 = x_0\}$, and $\deg g_{q-1} = q-1 = \deg C_{q-1}$. ■

2.8. Negative Curves via Equations. If $m > 3$ and q is a power of p congruent to 1 modulo m, then the curves $\widetilde{C}_d \subseteq R_m$ with $dm = q^e - 1$ of the **Main Theorem** can therefore be obtained by starting with the very explicit equations

$$C_{q^e-1} = V\left(\sum_{n_0+n_1+n_2=q^e-1} x_0^{n_0} x_1^{n_1} x_2^{n_2}\right) \subseteq \mathbf{P}^2,$$

blowing up at $[1:1:1]$, pulling back along $\widetilde{\pi}_m \colon R_m \to R_1$, and taking one of the m^2 isomorphic components $\zeta\widetilde{C}_d$ for $\zeta \in \boldsymbol{\mu}_m^3/\boldsymbol{\mu}_m$. From this point of view, the self-intersection may be computed as

$$\widetilde{C}_d^2 = \widetilde{C}_d \cdot \widetilde{\pi}_m^*(\widetilde{C}_{q^e-1}) - \sum_{\zeta\neq 1} \widetilde{C}_d \cdot (\zeta\widetilde{C}_d) = (2dm-1) - 3(m-1)d = d(3-m) - 1,$$

since the intersection number between $\widetilde{C}_d$ and a Galois translate by $\zeta = (\zeta_0, \zeta_1, \zeta_2) \in G \setminus \{1\}$ is

$$\widetilde{C}_d \cdot (\zeta \widetilde{C}_d) = \begin{cases} d, & \zeta_0 = \zeta_1 \text{ or } \zeta_1 = \zeta_2 \text{ or } \zeta_2 = \zeta_0, \\ 0, & \text{otherwise.} \end{cases}$$

Indeed, $\zeta \widetilde{C}_d$ is the image of the morphism $\zeta \circ \widetilde{\phi}_d$ given by

$$[x_0 : x_1 : x_2] \mapsto \big([\zeta_0 x_0^d : \zeta_1 x_1^d : \zeta_2 x_2^d], [x_0 : x_1 : x_2]\big).$$

Thus, $\widetilde{C}_d$ and $\zeta \widetilde{C}_d$ only intersect when $\zeta \widetilde{\phi}_d([x_0 : x_1 : x_2]) = \widetilde{\phi}_d([x_0 : x_1 : x_2])$. At most one of the x_i can vanish since $x_0 + x_1 + x_2 = 0$, so there are no intersections when $\zeta_i \neq \zeta_j$ for $i \neq j$, and a single intersection with multiplicity d at $V(x_k)$ when $\zeta_i = \zeta_j$ and $\{i, j, k\} = \{0, 1, 2\}$.

3. Relation with Fermat varieties and Frobenius morphisms

By **Lemma 1.1**, the second projection $\mathrm{pr}_2 \colon R_m \to V(y_0 + y_1 + y_2)$ realises R_m as the family of diagonal degree m curves over $C_1 \cong \mathbf{P}^1$ given by

$$x_0^m y_0 + x_1^m y_1 + x_2^m y_2 = 0.$$

If $\operatorname{char} k = p > 0$ and m is invertible in k, then the curves $\widetilde{C}_d \subseteq R_m$ for $dm = p^e - 1$ are given by sections $\widetilde{\phi}_d \colon C_1 \to R_m$ of pr_2. In this section, we pull back the family $R_m \to C_1$ and the sections $\widetilde{\phi}_d$ along finite covers of C_1. Pulling back along covers by Fermat curves allows us to relate the $\widetilde{C}_d$ in **Corollary 3.4** with graphs of Frobenius on products of Fermat curves. Pulling back along the Frobenius morphism of C_1 allows us to realise the $\widetilde{C}_d$ in **Corollary 3.6** as pullbacks of a constant section $\widetilde{C}_0$ under powers of a horizontal Frobenius morphism of R_m over C_1.

3.1. Intermediate Surfaces. For positive integers m and n invertible in k and $r \in \mathbf{N}$, denote by $R_{m,n,r}$ the normal surface

$$R_{m,n,r} = \left\{ \big([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]\big) \in \mathbf{P}^2 \times \mathbf{P}^2 \;\middle|\; \begin{array}{c} y_0^n + y_1^n + y_2^n = 0 \\ x_0^m y_0^r + x_1^m y_1^r + x_2^m y_2^r = 0 \end{array} \right\}.$$

It is smooth if and only if $m = 1$ or $r \in \{0, 1\}$; in all other cases, the singular locus $V(x_0 y_0, x_1 y_1, x_2 y_2)$ consists of the $3n$ points

$$\Big\{\big([1:0:0],[0:s:t]\big), \big([0:1:0],[s:0:t]\big), \big([0:0:1],[s:t:0]\big) \;\Big|\; s^n + t^n = 0\Big\}.$$

Note that $R_{m,1,1}$ is none other than the surface R_m of **Lemma 1.1**. If X_n denotes the Fermat curve $V(y_0^n + y_1^n + y_2^n) \subseteq \mathbf{P}^2$ of degree n, then $R_{m,n,0}$ coincides with $X_m \times X_n$. The surfaces $R_{m,n,r}$ for $r > 0$ come with a projection

$$\mathrm{pr}_2 \colon R_{m,n,r} \to X_n$$

that is smooth away from the $3n$ fibres above $V(y_0 y_1 y_2) \subseteq X_n$, and whose singular fibres consist of m lines meeting at a point.

3.2. Generalized Power Maps. For positive integers a and b invertible in k, define the finite morphism

$$\pi_{a,b} \colon R_{am,bn,br} \to R_{m,n,r}$$

$$\big([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]\big) \mapsto \big([x_0^a : x_1^a : x_2^a], [y_0^b : y_1^b : y_2^b]\big).$$

For $b=1$ and $n=r=1$, it coincides with the morphism $\widetilde{\pi}_a$ from 2.1. When $a=1$, these fit into pullback squares

$$\begin{array}{ccc} R_{m,bn,br} & \xrightarrow{\pi_{1,b}} & R_{m,n,r} \\ {\scriptstyle \mathrm{pr}_2}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_2} \\ X_{bn} & \longrightarrow & X_n\,. \end{array}$$

If $F^e\colon X_n \to X_n$ is the p^e-th power Frobenius morphism of X_n, there are pullback squares

$$\begin{array}{ccc} R_{m,n,p^e r} & \longrightarrow & R_{m,n,r} \\ {\scriptstyle \mathrm{pr}_2}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_2} \\ X_n & \xrightarrow{F^e} & X_n\,, \end{array}$$

so $R_{m,n,p^e r}$ is the Frobenius twist $R^{(e)}_{m,n,r}$ of $R_{m,n,r}$ over X_n. We denote the top map by $\pi^{(e)}$.

3.3. Lemma. — *Let m and n be positive integers invertible in k, let $a, r \in \mathbf{Z}$, and assume that r and $r+am$ are nonnegative. Then the map*

$$\psi_a\colon \mathbf{P}^2 \times \mathbf{P}^2 \overset{\sim}{\dashrightarrow} \mathbf{P}^2 \times \mathbf{P}^2$$
$$\big([x_0:x_1:x_2],[y_0:y_1:y_2]\big) \longmapsto \big([x_0y_0^a : x_1y_1^a : x_2y_2^a],[y_0:y_1:y_2]\big)$$

maps $R_{m,n,r+am}$ birationally onto $R_{m,n,r}$.

Proof. Note that ψ_a is a birational map with rational inverse ψ_{-a}. The result follows since ψ_a takes $R_{m,n,r+am}$ into $R_{m,n,r}$ and ψ_{-a} does the opposite, and neither surface is contained in the locus where ψ_a or ψ_{-a} is undefined. ∎

This allows us to relate $R_{m,m,m}$ and $X_m \times X_m$:

3.4. Corollary. — *The surfaces $X_m \times X_m \cong R_{m,m,0}$ and $R_{m,m,m}$ are birational via*

$$\psi\colon X_m \times X_m \overset{\sim}{\dashrightarrow} R_{m,m,m}$$
$$\big([x_0:x_1:x_2],[y_0:y_1:y_2]\big) \longmapsto \big(\big[\tfrac{x_0}{y_0} : \tfrac{x_1}{y_1} : \tfrac{x_2}{y_2}\big],[y_0:y_1:y_2]\big).$$

The composition $\rho\colon X_m \times X_m \dashrightarrow R_m$ of ψ with $\pi_{1,m}$ is given by

$$\big([x_0:x_1:x_2],[y_0:y_1:y_2]\big) \longmapsto \big(\big[\tfrac{x_0}{y_0} : \tfrac{x_1}{y_1} : \tfrac{x_2}{y_2}\big],[y_0^m:y_1^m:y_2^m]\big).$$

If char $k = p > 0$, *m is invertible in k, and $dm = p^e - 1$ for some positive integer e, then the strict transform of $\pi_{1,m}^*\widetilde{C}_d$ under ψ is the transpose $\Gamma_{F^e}^{\top}$ of the graph of the p^e-power Frobenius.*

Proof. The first statement follows by applying Lemma 3.3 to $m=n=r$ and $a=-1$, and the second is immediate from the definitions. For the final statement, recall that $\Gamma_{F^e}^{\top}$ is given by the section $s\colon X_m \to X_m \times X_m$ of pr_2 given by

$$[y_0:y_1:y_2] \mapsto \big([y_0^{p^e} : y_1^{p^e} : y_2^{p^e}],[y_0:y_1:y_2]\big).$$

By the first pullback square of 3.2, the curve $\pi_{1,m}^*\widetilde{C}_d$ is the image of the section $X_m \to R_{m,m,m}$ given by

$$[y_0:y_1:y_2] \mapsto \big([y_0^{dm} : y_1^{dm} : y_2^{dm}],[y_0:y_1:y_2]\big),$$

which agrees with $\psi \circ s$. ∎

3.5. The curves $\Gamma_{F^e} \subseteq X_m \times X_m$ are the standard example of curves with unbounded negative self-intersection: the condition $m > 3$ of the **Main Theorem** is exactly the condition $g(X_m) > 1$ that makes $\Gamma_{F^e}^2 = p^e(2-2g)$ negative. In fact, since $\Gamma_{F^e}^\top$ passes through $3m$ of the $3m^2$ points of indeterminacy of ψ, resolving the map shows that $m^2\,\widetilde{C}_d^2 = \Gamma_{F^e}^2 - 3m$.

On the other hand, $R_m \to C_1$ is an isotrivial family of diagonal degree m curves that becomes rationally trivialised over the m-th power cover $X_m \to C_1$. Thus, we can also look directly at the pullback $\pi^{(e)}\colon R_m^{(e)} \to R_m$ of the Frobenius $F^e\colon C_1 \to C_1$. Note that $R_m^{(e)} = R_{m,1,p^e}$ by **3.2**, so we get:

3.6. Corollary. — *If $p^e = dm+1$, then $R_m^{(e)}$ is birational to R_m via*

$$\psi\colon R_m \overset{\sim}{\dashrightarrow} R_m^{(e)}$$
$$\big([x_0 : x_1 : x_2],[y_0 : y_1 : y_2]\big) \longmapsto \Big(\big[\tfrac{x_0}{y_0^d} : \tfrac{x_1}{y_1^d} : \tfrac{x_2}{y_2^d}\big],[y_0 : y_1 : y_2]\Big).$$

If $\widetilde{\phi}_0\colon C_1 \to R_m$ denotes the constant section $[y_0 : y_1 : y_2] \mapsto \big([1:1:1],[y_0:y_1:y_2]\big)$ and $\widetilde{C}_0 \subseteq R_m$ denotes its image, then $\widetilde{C}_d$ is the strict transform of $\pi^{(e),*}\widetilde{C}_0$ under ψ.

Proof. The first statement follows from **Lemma 3.3** applied to $n=1$, $r=p^e$, and $a=-d$. For the second, by the second pullback square of **3.2**, the curve $\pi^{(e),*}\widetilde{C}_0$ is the image of the constant section $C_1 \to R_m^{(e)}$ given by

$$[y_0 : y_1 : y_2] \mapsto \big([1:1:1],[y_0:y_1:y_2]\big),$$

which agrees with $\psi \circ \widetilde{\phi}_d$. ■

3.7. Instead of the transpose $\Gamma_{F^e}^\top$ of the graph of $F^e\colon X_m \to X_m$, one can also look at the negative curves $\Gamma_{F^e} \subseteq X_m \times X_m$, which are given by pulling back the diagonal along the relative Frobenius of $\mathrm{pr}_2\colon X_m \times X_m \to X_m$. Their images under the rational map ρ of **Corollary 3.4** are given by the parametrised rational curves

$$C_1 \to R_m$$
$$[y_0 : y_1 : y_2] \mapsto \Big(\big[y_1^d y_2^d : y_0^d y_2^d : y_0^d y_1^d\big],\big[y_0^{p^e} : y_1^{p^e} : y_2^{p^e}\big]\Big),$$

where $dm = p^e - 1$ as usual. These are obtained from the curve $\widetilde{C}_0$ of **Corollary 3.6** by pulling back the strict transform of $\widetilde{C}_0$ under $\psi^{-1}\colon R_m^{(e)} \dashrightarrow R_m$ along the relative Frobenius $F^e_{R_m/C_1}$. Note that the images of these curves in $\mathbf{P}^2$ differ from the curves C_d by the Cremona transformation

$$[x_0 : x_1 : x_2] \mapsto [x_0^{-1} : x_1^{-1} : x_2^{-1}].$$

Finally, we relate the multiplicity of C_d at $[1:1:1]$ to point counts on the Fermat curve X_m if $dm = p^e - 1$ for some positive integer e.

3.8. Lemma. — *If $dm = p^e - 1$, then a point $x \in C_1$ maps to $[1:1:1]$ in C_d if and only if there exists $y \in X_m(\mathbf{F}_{p^e})$ with nonzero coordinates mapping to x under the m-th power map $X_m \to C_1$. In particular,*

$$\mathrm{mult}_{[1:1:1]}\,C_d = \frac{|X_m(\mathbf{F}_{p^e})| - 3m}{m^2}.$$

Proof. The first statement follows since $X_m \to C_1$ is surjective and a point $y = [y_0 : y_1 : y_2]$ on X_m with nonzero coordinates maps to $[1:1:1]$ under the (p^e-1)-st power map $X_m \to C_d$ if and only if $y \in X_m(\mathbf{F}_{p^e})$.

For the second statement, note that $\mathrm{mult}_{[1:1:1]} C_d$ equals the number of preimages of $[1:1:1]$ in C_1, since $\widetilde{\phi}_d \colon C_1 \to \widetilde{C}_d$ is an isomorphism by **Corollary 1.3**. The result now follows since $X_m \to C_1$ is finite étale of degree m^2 away from the coordinate axes, so each point $x \in C_1 \setminus V(x_0x_1x_2)$ has exactly m^2 preimages in X_m. ■

For example, if $p^\nu \equiv -1 \pmod{m}$ for some positive integer ν, then

$$|X_m(\mathbf{F}_{p^e})| = 1 - \frac{(m-1)(m-2)}{2} p^{e/2} + p^e$$

whenever $p^e \equiv 1 \pmod{m}$ [SK79, Lem. 3.3].

4. Remarks towards characteristic 0

4.1. Although the Bounded Negativity Conjecture is currently still open in characteristic 0, the Weak Bounded Negativity Conjecture is known [Hao19]: for any smooth projective complex surface X and any $g \in \mathbf{N}$, there exists a constant $b(X,g)$ such that $C^2 \geq -b(X,g)$ for every reduced curve $C = \sum_i C_i$ whose components C_i have geometric genus at most g.

Our examples in the **Main Theorem** certainly violate this, and, as we now verify, arise from the failure of the logarithmic Bogomolov–Miyaoka–Yau inequality for the pair $(R_m, \widetilde{C}_d)$ when d is large with respect to m. In the next three paragraphs, assume $\operatorname{char} k = p > 0$ and $dm = p^e - 1$. To ease notation, write $(R, \widetilde{C})$ for $(R_m, \widetilde{C}_d)$. We will use logarithmic sheaves of differentials; see for example [EV92, §2].

4.2. Lemma. — *The Chern numbers of the pair $(R, \widetilde{C})$ are*

$$c_1^2(R,\widetilde{C}) := c_1^2\big(\Omega_R^1(\log \widetilde{C})\big) = d(m-3) - m^2 + 6,$$
$$c_2(R,\widetilde{C}) := c_2\big(\Omega_R^1(\log \widetilde{C})\big) = m^2 + 1.$$

In particular, the Chern slopes $c_1^2(R,\widetilde{C})/c_2(R,\widetilde{C})$ are unbounded for fixed m and growing d.

Proof. The logarithmic sheaf of differentials fit into a short exact sequence

$$0 \to \Omega_R^1 \to \Omega_R^1(\log \widetilde{C}) \to \mathcal{O}_{\widetilde{C}} \to 0,$$

so $c_1^2(R,\widetilde{C}) = (K_R + \widetilde{C})^2$ and $c_2(R,\widetilde{C}) = c_2(\Omega_R^1) + \widetilde{C}(K_R + \widetilde{C})$. Since R is the blowup of $\mathbf{P}^2$ in m^2 points, we get $K_R^2 = 9 - m^2$ and $c_2(\Omega_R^1) = 3 + m^2$, so the result follows from the computations of the intersection numbers in **Corollary 1.3**. ■

4.3. Lemma. — *If $m > 3$ and d is such that*

$$\chi\big(2(K_R + \widetilde{C})\big) = d(m-3) - m^2 + 5 > 0$$

then $\mathrm{H}^0(R, 2(K_R + \widetilde{C})) \neq 0$. In particular, $K_R + \widetilde{C}$ is pseudoeffective.

Proof. The Euler characteristic statement follows from Riemann–Roch, so it remains to show that $\mathrm{H}^0\big(R, 2(K_R+\widetilde{C})\big) \neq 0$ once $\chi\big(2(K_R+\widetilde{C})\big) > 0$. But $\mathrm{H}^2\big(R, 2(K_R+\widetilde{C})\big) = \mathrm{H}^0(R, -K_R - 2\widetilde{C})^\vee$, and the latter vanishes since $\widetilde{C}$ is effective and $-K_R = \mathcal{O}_R(3-m, 1)$ by **Lemma 1.1**. ■

For d large with respect to m, this shows that $(R, \widetilde{C})$ falls into the final case considered in [Hao19, §1.2, Case 2], and that the failure of Weak Bounded Negativity stems from the failure of the logarithmic Bogomolov–Miyaoka–Yau inequality:

4.4. Corollary. — *If $m > 3$ and $d > \frac{5m^2-2}{m-3}$, then $K_R + \widetilde{C}$ is pseudoeffective and*

$$c_1^2(R, \widetilde{C})/c_2(R, \widetilde{C}) > 4.$$

Moreover, the pair $(R, \widetilde{C})$ does not lift to the second Witt vectors $W_2(k)$.

Proof. The first part follows from **Lemma 4.2** and **Lemma 4.3**. The final statement follows from [Lan16, Proposition 4.3], since $(R, \widetilde{C})$ violates the logarithmic Bogomolov–Miyaoka–Yau inequality. ■

4.5. On the other hand, the surface R_m itself does lift to characteristic 0. This gives new examples of surfaces $X \to \operatorname{Spec} \mathbf{Z}$ such that almost all special fibres $X_{\bar{\mathbf{F}}_p}$ (namely those with $p \nmid m$) violate bounded negativity. The same property holds for the square $C \times C$ of a curve $C \to \operatorname{Spec} \mathbf{Z}$ of genus ≥ 2, which is the classical counterexample to bounded negativity in positive characteristic. However, the rational surface $X = R_m$ has the additional property that the specialisation maps $\operatorname{NS}(X_{\bar{\mathbf{Q}}}) \to \operatorname{NS}(X_{\bar{\mathbf{F}}_p})$ are isomorphisms for every prime $p \nmid m$.

4.6. Question. — *Is it possible to determine the effective cone of R_m for some $m \geq 4$? How does it depend on the characteristic of k?*

For example, the curves in $\mathbf{P}^2$ cut out by the polynomials g_{m-1} of **Lemma 2.6** are smooth of genus $\frac{(m-2)(m-3)}{2}$ in characteristic 0 [RVVZ01, Thm. 1], and the equation $g_{m-1}h_2 = h_{m+1}$ shows that $V(g_{m-1}) \cup V(x_0 - x_1) \cup V(x_1 - x_2) \cup V(x_2 - x_0)$ contains

$$Z'_m := V\begin{pmatrix} x_0x_1^{m+1} - x_0^{m+1}x_1, \\ x_1x_2^{m+1} - x_1^{m+1}x_2, \\ x_2x_0^{m+1} - x_2^{m+1}x_0 \end{pmatrix} = Z_m \cup \left\{[s:t:0], [s:0:t], [0:s:t] \,\middle|\, s^m = t^m\right\}.$$

Since $V(g_{m-1})$ has self-intersection $(m-1)^2$ and passes through the $m^2 - 3m + 2$ points of Z_m whose coordinates are pairwise distinct, its strict transform on R_m has self-intersection $m - 1$. On the further blowup R'_m of $\mathbf{P}^2$ in Z'_m, the strict transform has self-intersection $-2m + 2$, but unlike the situation described in **2.8**, there does not appear to be an obvious way to produce infinitely many negative curves on a *single* rational surface this way.

When $m = p^e$ for some prime p, the specialisation to characteristic p collapses Z_m onto the point $[1:1:1]$, and the smooth curve $V(g_{m-1})$ becomes a rational curve that is highly singular at $[1:1:1]$. Even though these curves are not negative yet (see **2.8**), taking different values of e does give infinitely many curves on the same rational surface.

4.7. As far as we are aware, all known counterexamples to bounded negativity on a smooth projective surface X over an algebraically closed field k of characteristic $p > 0$ consist of a family C_i of curves on X for which there exist constants a, b such that $C_i^2 = ap^i + b$ for all $i \in \mathbf{N}$.

4.8. Question. — *If X is a surface over an algebraically closed field k of characteristic $p > 0$, is there a finite set $\{(a_i, b_i) \in \mathbf{Q}^2 \mid i \in I\}$ such that all integral curves $C \subseteq X$ with $C^2 < 0$ satisfy*

$$C^2 = a_ip^e + b_i$$

for some positive integer e and some $i \in I$? If not, is there some other way in which the self-intersections of negative curves on X are "not too scattered"?

We can also consider the following uniform version:

4.9. Question. — *If $X \to S$ is a smooth projective surface over a finitely generated integral base scheme S, does there exist a finite set $\{(a_i, b_i) \in \mathbf{Q}^2 \mid i \in I\}$ such that every geometrically integral curve $C \subseteq X_s$ of negative self-intersection in a fibre X_s with $\operatorname{char} \kappa(s) > 0$ satisfies*

$$C^2 = a_i p^e + b_i$$

for some positive integer e and some $i \in I$, where $p = \operatorname{char} \kappa(s)$?

For example, for the surfaces $R_m \to \operatorname{Spec} \mathbf{Z}[1/m]$ and the curves $\widetilde{C}_d$ of the **Main Theorem**, we may take $a = \frac{3-m}{m}$ and $b = \frac{-3}{m}$, which do not depend on the characteristic of $\kappa(s)$.

4.10. Despite the failure of bounded negativity in positive characteristic, a positive answer to **Question 4.9** still implies bounded negativity in characteristic 0 via reduction modulo primes. Indeed, the minimum $b_{\min} = \min\{b_i \mid i \in I\}$ is a lower bound for the self-intersection C^2 of a geometrically integral curve C in the generic fibre, since the specialisations C_s of C satisfy $C_s^2 = C^2$ for all $s \in S$ and remain geometrically integral for s in a dense open set $U \subseteq S$, and

$$\bigcap_{p \in P} \left\{ a_i p^e + b_i \,\middle|\, i \in I, e \in \mathbf{Z}_{>0} \right\} \subseteq [b_{\min}, \infty)$$

for any infinite set of primes P. Thus, **Question 4.9** is a natural analogue of the Bounded Negativity Conjecture in positive characteristic.

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DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY, 2990 BROADWAY, NEW YORK, NY, 10027, UNITED STATES OF AMERICA

Email address: rcheng@math.columbia.edu

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, FINE HALL, WASHINGTON ROAD, PRINCETON, NJ 08544, UNITED STATES OF AMERICA

INSTITUTE FOR ADVANCED STUDY, 1 EINSTEIN DRIVE, PRINCETON, NJ 08540, UNITED STATES OF AMERICA

Email address: rdobben@math.princeton.edu