FROBENIUS–TSCHIRNHAUSEN AMPLENESS

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ABSTRACT. We study smooth projective varieties whose Frobenius-trace kernel, also known as the Tschirnhausen bundle associated with the Frobenius morphism, is ample. We show that any non-constant morphism out of such a variety must be finite onto its image, providing strong evidence that these varieties must be Fano varieties of Picard rank 1. Using infinitesimal representation theory and by constructing special Frobenius splittings, we show that generalized Grassmannian of classical type and type G_2 have ample Frobenius-trace kernel, with the exception of certain low characteristic examples which are explained by the existence of exotic isogenies of the associated algebraic groups.

INTRODUCTION

Positivity of natural vector bundles often imposes strong restrictions on the global geometry of a smooth projective variety X, the prototypical statement being Mori's theorem from [Mor79] on ampleness of the tangent bundle: X must be projective space $\mathbf{P}^n$. Relaxing ampleness to nefness, Campana and Peternell conjectured in [CP91] that X, at least over the complex numbers and up to passing to an étale cover, must fibre into rational homogeneous spaces over its Albanese variety. Another series of statements of a similar flavour pertain to X which are *Tschirnhausen ample*, by which we mean that for any finite surjective morphism $f : Y \to X$ from another smooth projective variety, the associated Tschirnhausen bundle $\mathscr{E}_f := (f_*\mathscr{O}_Y/\mathscr{O}_X)^\vee$ is ample. Lazarsfeld observed in [Laz80] that $\mathbf{P}^n$ has this property. Peternell and Sommese later showed in [PS00] that any such X must be simply connected and has no contractions; in particular, complex Fano examples must have Picard rank 1. Kim and Manivel found in [Kim96a, Kim96b, Man97, KM99] that, over the complex numbers, quadrics of dimension at most 6, Grassmannians, and Lagrangian Grassmannians are Tschirnhausen ample; they conjectured the same for any complex rational homogeneous space of Picard rank 1.

Call a smooth projective variety X over a field of positive characteristic *Frobenius–Tschirnhausen ample* if the Tschirnhausen bundle $\mathscr{E}_X := (F_*\mathscr{O}_X/\mathscr{O}_X)^\vee$ associated with its absolute Frobenius morphism $F : X \to X$ is ample. As explained below, this property is best understood as a common specialization of the conditions described above. Few general restrictions and examples are known: In their initial work [CRP25], Carvajal-Rojas and Patakfalvi showed that such X must be a Fano variety which neither has a generically smooth fibration nor a divisorial contraction onto another smooth variety; moreover, they found that $\mathbf{P}^n$ and quadrics of dimension at least 3—with the exception of the quadric threefold in characteristic $p = 2$!—are Frobenius–Tschirnhausen ample. Carvajal-Rojas and the second named author studied this property for toric varieties in [CRÖ25], showing that the only example amongst toric varieties is $\mathbf{P}^n$. Finally, Mallory found a geometric obstruction in [Mal25] which implies, for instance, that Fano varieties containing projective spaces of low anti-canonical degree are not Frobenius–Tschirnhausen ample.

The aim of this work is to provide new restrictions on and examples of Frobenius–Tschirnhausen ample varieties. Our first result completes the initial results of Carvajal-Rojas and Patakfalvi and shows that such varieties have no nontrivial contractions:

Theorem A. — *Let X be a smooth projective variety which is Frobenius–Tschirnhausen ample. Then any non-constant morphism $f : X \to Y$ is finite onto its image.*

This follows from the more technical result 2.2 that any connected closed subscheme $Z \subseteq X$ satisfies Hironaka–Matsumara's G1 property: the formal functions on the completion of X along Z are just the constants. Consequently, every semiample divisor on a Frobenius–Tschirnhausen ample variety X is ample, and X has no nontrivial extremal contractions. Since X is also a Fano variety, this suggests that X ought to have Picard rank 1; we summarize this with the following:

Corollary. — *If the contraction theorem holds for X, then X has Picard rank* 1.

Existence of contractions as in [KM98, Theorem 3.7] is wide open in positive characteristic, and is known only up to dimension 3 by the work of Kollár in [Kol91]. Contractions are also known to exist for special classes of varieties such as rational homogeneous spaces, or potentially more generally, those with nef tangent bundle by [KW23, Theorem 1.5], and toric varieties; applying the corollary to the latter shows that the only Frobenius–Tschirnhausen ample toric variety is $\mathbf{P}^n$, giving another proof of the smooth case of [CRÖ25, Theorem A, (a)$\Leftrightarrow$(b)].

Which smooth Fano varieties of Picard rank 1 are Frobenius–Tschirnhausen ample? Only $\mathbf{P}^n$ has this property in dimensions 1 and 2. Mallory shows in [Mal25, Theorem 1.2] that only smooth quadrics away from characteristic $p = 2$ and $\mathbf{P}^3$ have this property in dimension 3; in higher dimensions, he shows that smooth complete intersections in $\mathbf{P}^n$ of total degree $d = n$ or $d = n-1$ are *not* Frobenius–Tschirnhausen ample. The same is expected for any $2 < d \leq n$, and the following provides some evidence toward this:

Theorem B. — *Let $p < d \leq n$. There exists smooth Fano hypersurfaces of degree d in $\mathbf{P}^n$ over a field of characteristic p which are not Frobenius–Tschirnhausen ample.*

This follows from 2.6 and 2.7, where an explicit family of hypersurfaces is shown to be not Frobenius–Tschirnhausen ample. The defining equations of these hypersurfaces are taken to be of a special form: In particular, they have a nontrivial *profile* with respect to p in the sense of [Che25a]; the simplest examples are the *q-bic* or *extremal* hypersurfaces of [Che25b, KKP+22], such as the Fermat hypersurface of degree $q+1$ with $q = p^e$ for some $e \geq 1$.

Besides complete intersections, the most accessible smooth Fano varieties of Picard rank 1 are projective rational homogeneous spaces $X = \mathbf{G}/\mathbf{P}$ where $\mathbf{P}$ is a reduced maximal parabolic subgroup of a reductive algebraic group $\mathbf{G}$. A first observation—one which already implicitly appeared in Mehta and Ramanathan's seminal work on Frobenius splittings in [MR85, Remark (ii) on p.40]—is that $\mathscr{E}_X$ is always globally generated, even for X not necessarily of Picard rank 1: see 3.3. Representation-theoretic methods relate ampleness of $\mathscr{E}_X$ to the existence of special Frobenius splittings on X and, away from certain low characteristic exceptions, such splittings may be explicitly constructed for Grassmannian of classical type and of type G_2; note that examples such as the quadric threefold in characteristic $p = 2$ show that conditions on the characteristic are necessary. Our technique is notably different from prior works in that we need not explicitly decompose $F_*\mathscr{O}_X$. The result is:

Theorem C. — *A projective rational homogeneous space X of Picard rank* 1 *of classical type or type* G_2 *is Frobenius–Tschirnhausen ample if and only if X is not the*

(i) *symplectic Grassmannian $\mathbf{SGr}(k, 2m)$ with $2 \leq k \leq m$ in characteristic $p = 2$; or the*
(ii) *orthogonal Grassmannian $\mathbf{OGr}(m-1, 2m+1)$ in characteristic $p = 2$; or the*
(iii) G_2*-Grassmannian in characteristics $p = 2$ or $p = 3$.*

Ampleness follows from the stronger result 4.1, which shows that the twist $\mathscr{E}_X \otimes \mathscr{O}_X(-1)$ by the negative generator of $\operatorname{Pic} X$ is globally generated. In particular, quadrics of dimension ≥ 4 in characteristic $p = 2$ are Frobenius–Tschirnhausen ample, correcting a slight inaccuracy in [CRP25, Corollary 4.8]. Non-ampleness for the exceptions is established in §5 and, in most cases, is explained by the existence of exotic isogenies between the relevant algebraic groups. Optimistically, we would guess that the generalized Grassmannian in the remaining exceptional types are also Frobenius–Tschirnhausen ample, at least away from characteristic $p = 2$; more conservatively, we conjecture that this is the case as soon as the prime is good, say $p > 5$.

The quadric threefold Q in characteristic $p = 2$ appears twice in this list of exceptions, and non-ampleness of $\mathscr{E}_Q$ is explained by the existence of the finite purely inseparable morphism

$$\psi\colon \mathbf{P}^3 \to Q = \{(x_0 : \cdots : x_4) \in \mathbf{P}^4 : x_0x_1 + x_2x_3 + x_4^2 = 0\}$$
$$(x_0 : x_1 : x_2 : x_3) \mapsto (x_0^2 : x_1^2 : x_2^2 : x_3^2 : x_0x_1 + x_2x_3)$$

albeit for different reasons: Viewing $Q = \mathbf{SGr}(2,4)$ as in (i), the point is that the Tschirnhausen bundle $\mathscr{E}_\psi$ is a quotient of $\mathscr{E}_Q$ and has rank 3 and only degree 2 on lines in Q. Viewing $Q = \mathbf{OGr}(1,5)$ as in (ii), non-ampleness is due to the fact that the restriction of ψ to the quadric surface $S \subset \mathbf{P}^3$ defined by $x_0x_1 + x_2x_3 = 0$ is the Frobenius morphism onto the hyperplane section $S \cong Q \cap \mathrm{V}(x_4)$, so that $\mathscr{E}_\psi|_S \cong \mathscr{E}_S$, and the latter is not ample; in more geometric language, the leaves of the foliation associated with ψ are quadric surfaces, and this obstructs ampleness of $\mathscr{E}_\psi$, whence $\mathscr{E}_Q$.

To close this introduction, we indicate as to how the Frobenius–Tschirnhausen bundle $\mathscr{E}_X$ may be related to the tangent bundle of X. First, Kunz's theorem from [Kun69] says that the Frobenius–Tschirnhausen bundle detects nonsingularity: $\mathscr{E}_X$ is locally free if and only if X is regular. Next, positive characteristic foliation theory as in [Eke87] identifies the Frobenius morphism $F\colon X \to X$ as the quotient map for the infinitesimal action of the tangent bundle $\mathscr{T}_X$ on X itself; as such, $\mathscr{E}_X$ is dual to the sheaf of coinvariants for the action of $\mathscr{T}_X$ on $\mathscr{O}_X$. Finally, and in relation to this, there exists a canonical filtration of $F^*\mathscr{E}_X$ whose graded pieces are certain (truncated) divided powers of $\mathscr{T}_X$: see [Sun08, §3]. This final point, especially, indicates that positivity of $\mathscr{T}_X$ contributes to that of $\mathscr{E}_X$; the spirit of this work is a conviction that the converse should also be true to some extent.

Notation. — Throughout, $\mathbf{k}$ denotes an algebraically closed field of characteristic $p > 0$, over which all schemes appearing are taken over. A *variety* refers to an integral scheme which is separated and of finite type over $\mathbf{k}$. For a scheme X and a finite locally free $\mathscr{O}_X$-module $\mathscr{E}$, we follow geometric conventions and write $\mathbf{P}\mathscr{E} := \mathrm{Proj}\,\mathrm{Sym}\,\mathscr{E}^\vee$ for the projective bundle of lines in $\mathscr{E}$; the pushforward along $\pi\colon \mathbf{P}\mathscr{E} \to X$ of the relatively ample tautological line bundle is therefore $\pi_*\mathscr{O}_{\mathbf{P}\mathscr{E}}(1) = \mathscr{E}^\vee$. As such, $\mathscr{E}$ is *ample* if the tautological line bundle of $\mathbf{P}\mathscr{E}^\vee$ is ample.

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1. Frobenius–Tschirnhausen ampleness

The central object of study in this work is the Tschirnhausen bundle $\mathscr{E}_X$ of the absolute Frobenius morphism $F\colon X \to X$ of a smooth projective variety X. Explicitly, and more generally for each $e \geq 1$, define the *e-th Frobenius cokernel* to be the $\mathscr{O}_X$-module

$$\mathscr{B}_{X,e} := \mathrm{coker}(F^{e,\#}\colon \mathscr{O}_X \to F^e_*\mathscr{O}_X).$$

Since X is smooth, Kunz's theorem [Kun69] implies that $\mathscr{B}_{X,e}$ is locally free of rank $p^{ne} - 1$ where $n := \dim X$. Dually, the *e-th Frobenius–Tschirnhausen bundle* or *Frobenius-trace kernel* is

$$\mathscr{E}_{X,e} := \mathscr{B}_{X,e}^\vee \cong \ker(\tau_e\colon F^e_*(\omega_X^{\otimes 1-p^e}) \to \mathscr{O}_X)$$

where the second identification arises from Grothendieck duality along the finite flat morphism $F^e : X \to X$; here, ω_X is the dualizing line bundle of X, and the canonical map τ_e is referred to as the *p^e-th Frobenius-trace morphism*. In the case $e = 1$, write also $\mathscr{B}_X := \mathscr{B}_{X,1}$ and $\mathscr{E}_X := \mathscr{E}_{X,1}$.

An initial question regarding positivity of the Frobenius–Tschirnhausen bundles is: might there be a hierarchy of positivity conditions depending on the sequence of indices $e \geq 0$ for which $\mathscr{E}_{X,e}$ is positive? Happily, there is just one possibility:

1.1. Lemma. — *For a smooth projective variety X, the following statements are equivalent:*

(i) *$\mathscr{E}_X$ is ample;*
(ii) *$\mathscr{E}_{X,e}$ is ample for some $e \geq 1$; and*
(iii) *$\mathscr{E}_{X,e}$ is ample for all $e \geq 1$.*

The equivalence also holds with nefness in place of ampleness.

Proof. It is easier to work with the dual statements and to show anti-ampleness of the $\mathscr{B}_{X,e}$. Clearly, (iii) $\Rightarrow$ (ii), and (ii) $\Rightarrow$ (i) was observed in [CRP25, Remark 5.2]. So it remains to show that (i) $\Rightarrow$ (iii). Suppose that $\mathscr{B}_X = \mathscr{B}_{X,1}$ is anti-ample. As already observed in *loc. cit.*, for any integer $e > 0$, factoring $F^e = F \circ F^{e-1}$ provides a short exact sequence

$$0 \to \mathscr{B}_X \to \mathscr{B}_{X,e} \to F_*\mathscr{B}_{X,e-1} \to 0.$$

By induction on e, we may suppose that $\mathscr{B}_{X,e-1}$ is anti-ample. Since $F : X \to X$ is finite and surjective, $F_*\mathscr{B}_{X,e-1}$ is also anti-ample by [BLNN24, Theorem 1.1]—although stated in characteristic 0, their proof works in any characteristic. Thus $\mathscr{B}_{X,e}$ is an extension of anti-ample vector bundles and so is anti-ample itself. The proof for nefness is entirely analogous. ■

Positivity of $\mathscr{E}_X$ propagates to positivity of Tschirnhausen bundles of all finite purely inseparable surjective morphisms $\varphi : Y \to X$ from smooth varieties; for brevity, we also say that Y is a *purely inseparable covering* of X. The main point is that any such morphism factors through a power of the absolute Frobenius morphism: there is a morphism $\psi : X \to Y$ and integer $e \geq 0$ such that the maps $\varphi \circ \psi$ and $\psi \circ \varphi$ are the p^e-power Frobenius morphisms of X and Y, respectively; the minimal such e is called the *height* of φ. With this, we have:

1.2. Proposition. — *A smooth projective variety X is Frobenius–Tschirnhausen ample if and only if $\mathscr{E}_\varphi$ is ample for any purely inseparable covering $\varphi : Y \to X$ by a smooth projective variety.*

Proof. Let $\psi : X \to Y$ be a morphism such that $\varphi \circ \psi$ is the p^e-power Frobenius morphism of X, and consider the short exact sequence

$$0 \to \mathscr{O}_Y \to \psi_*\mathscr{O}_X \to \mathscr{B}_\psi \to 0.$$

Pushing this along φ, identifying $\varphi_*\psi_*\mathscr{O}_X = F^e_*\mathscr{O}_X$, and passing to cokernels of the canonical maps $\mathscr{O}_X \to \varphi_*\mathscr{O}_Y$ and $\mathscr{O}_X \to F^e_*\mathscr{O}_X$ provides a short exact sequence of $\mathscr{O}_X$-modules:

$$0 \to \mathscr{B}_\varphi \to \mathscr{B}_{X,e} \to \varphi_*\mathscr{B}_\psi \to 0$$

wherein the result follows from **1.1** upon taking duals. ■

1.3. Numerical restrictions. — Grothendieck–Riemann–Roch shows that

$$c_1(\mathscr{E}_X) = -c_1(F_*\mathscr{O}_X) = \frac{1}{2}p^{n-1}(p-1)c_1(\mathscr{T}_X)$$

so that if $\mathscr{E}_X$ is ample, then X must be Fano; see also [CRP25, Proposition 5.9]. This implies that, in dimension $n = 1$, $X = \mathbf{P}^1$ is the only curve with ample Frobenius-trace kernel. In higher dimensions, by restricting to smooth rational curves, this provides a little more. The next result is phrased for X

of dimension $n \geq 3$ as it provides a slightly stronger statement. When $n = 2$, the argument shows that if X has a curve of anti-canonical degree ≤ 2, then $\mathscr{E}_X$ is not ample. Since an exceptional curve on a del Pezzo surface has degree 1, if $\mathscr{E}_X$ is ample, then X must be minimal. In the case of $\mathbf{P}^1 \times \mathbf{P}^1$, a fibre of one of the projections has degree 2, and so its Frobenius-trace kernel is also not ample. Thus the only surface with ample $\mathscr{E}_X$ is $X = \mathbf{P}^2$, providing a simple proof of [CRP25, Corollary 5.15].

1.4. Lemma. — *Let X be a smooth projective variety of dimension $n \geq 3$ and suppose that X contains a smooth rational curve on which $c_1(\mathscr{T}_X)$ has degree k. If either $p = 2$ and $k \leq 3$ or $p \geq 3$ and $k \leq 2$, then $\mathscr{E}_X$ is not ample.*

Proof. Prove the contrapositive: If $\mathscr{E}_X$ were ample, then the degree k of $c_1(\mathscr{T}_X)$ on an arbitrary smooth rational curve $C \subset X$ satisfies $k \geq 4$ if $p = 2$ and $k \geq 3$ if $p \geq 3$. In this case, $\mathscr{E}_X|_C$ is an ample vector bundle of rank $r := p^n - 1$ and degree $d := \frac{1}{2}p^{n-1}(p-1)k$ by the above calculation. Since vector bundles on $C \cong \mathbf{P}^1$ split into a sum of line bundles, this implies that $d \geq r$, and isolating k leads to

$$k \geq \frac{2(p^n - 1)}{p^{n-1}(p-1)} = 2 + 2\frac{p^{n-1} - 1}{p^{n-1}(p-1)}$$

The second term is positive as soon as $n \geq 2$, so integrality implies $k \geq 3$; if $p = 2$ and $n \geq 3$, the second term is greater than 1, whence $k \geq 4$. ■

Applying this to complete intersections in projective space provides a simple proof to a slight refinement of [Mal25, Theorem 5.1]:

1.5. Corollary. — *Let $X \subset \mathbf{P}^n$ be a smooth complete intersection of dimension ≥ 3 and total degree d. If either $p = 2$ and $d \geq n - 2$ or $p \geq 3$ and $d \geq n - 1$, then $\mathscr{E}_X$ is not ample.*

Proof. This follows directly from 1.4 since $c_1(\mathscr{T}_X) = (n + 1 - d)H$ for $H = c_1(\mathscr{O}_X(1))$ the hyperplane class and any such X has a line, that is, a curve of H-degree 1: see [DM98] or [Kol96, V.4.10]. ■

The examples of homogeneous spaces as considered in §4 show that these simple numerical considerations cannot be improved, and that 1.4 is in a sense optimal. Finer considerations are therefore required to obtain further obstructions to ampleness of $\mathscr{E}_X$.

2. Subschemes

If $\mathscr{E}_X$ were ample, so would its restriction to any closed subscheme $Z \subseteq X$. This method is made effective by Mallory's observation from [Mal25] that, dually, the kernel of the restriction map $F_*\mathscr{O}_X|_Z \to F_*\mathscr{O}_Z$ looks like a conormal bundle, related to the Frobenius neighbourhoood of Z in X. In this section, we refine some of Mallory's techniques to show in 2.2 that subschemes of Frobenius–Tschirnhausen ample varieties support only constant formal functions; consequently, as explained in 2.4, all non-constant morphisms out of such varieties are finite onto their image.

2.1. Restriction to subschemes. — Mallory observed in [Mal25, Lemma 3.2] that the Frobenius cokernel of a scheme X and that of a closed subscheme Z defined by the ideal sheaf $\mathscr{I}$ are related via a short exact sequence of $\mathscr{O}_Z$-modules of the form

$$0 \to F^e_*(\mathscr{I}/\mathscr{I}^{[p^e]}) \to \mathscr{B}_{X,e}|_Z \to \mathscr{B}_{Z,e} \to 0$$

where $\mathscr{I}^{[p^e]}$ is the ideal sheaf generated by p^e-powers of sections of $\mathscr{I}$. This sequence arises from the Frobenius direct image of the sequence of $\mathscr{O}_X$-modules

$$0 \to \mathscr{I}/\mathscr{I}^{[p^e]} \to \mathscr{O}_X/\mathscr{I}^{[p^e]} \to \mathscr{O}_Z \to 0$$

indicating that $\mathscr{B}_{X,e}|_Z$ carries information about certain infinitesimal neighbourhoods of Z in X.

In the case that X is Frobenius–Tschirnhausen ample, this provides strong restrictions on the internal geometry of X, one such being best phrased in terms of the formal completion $\hat{X}$ along a closed subscheme $Z \subset X$. Recall from [HM68, Definition 2.9] that Z is said to be *G1* in X if the canonical map $\mathrm{H}^0(X, \mathscr{O}_X) \to \mathrm{H}^0(\hat{X}, \mathscr{O}_{\hat{X}})$ is an isomorphism. With this, we have:

2.2. Theorem. — *Any connected closed subscheme of positive dimension in a smooth projective Frobenius–Tschirnhausen ample variety X is G1 in X.*

Proof. Let $Z \subset X$ be a closed connected subscheme and let $\hat{X}$ be the completion of X along Z. Writing $\mathscr{I}$ for its ideal sheaf, note that the two sequences of ideal sheaves $\{\mathscr{I}^k\}_{k\geq 0}$ and $\{\mathscr{I}^{[p^e]}\}_{e\geq 0}$ are mutually cofinal, so

$$\mathrm{H}^0(\hat{X}, \mathscr{O}_{\hat{X}}) = \lim_k \mathrm{H}^0(X, \mathscr{O}_X/\mathscr{I}^k) = \lim_e \mathrm{H}^0(X, \mathscr{O}_X/\mathscr{I}^{[p^e]}).$$

Thus it suffices to show that $\mathrm{H}^0(X, \mathscr{O}_X/\mathscr{I}^{[p^e]}) \cong \mathbf{k}$ for all $e \geq 0$. Restricting the e-th Frobenius cokernel $\mathscr{B}_{X,e}$ to Z and using 2.1 provides a short exact sequence

$$0 \to F_*^e(\mathscr{I}/\mathscr{I}^{[p^e]}) \to \mathscr{B}_{X,e}|_Z \to \mathscr{B}_{Z,e} \to 0.$$

Ampleness of the Frobenius-trace kernel implies via 1.1 that $\mathscr{B}_{X,e}|_Z$ is anti-ample, whence has no sections by 2.3 below. Since Frobenius and any closed immersion is affine,

$$\mathrm{H}^0(X, \mathscr{I}/\mathscr{I}^{[p^e]}) = \mathrm{H}^0(Z, F_*^e(\mathscr{I}/\mathscr{I}^{[p^e]})) \subseteq \mathrm{H}^0(Z, \mathscr{B}_{X,e}|_Z) = 0.$$

So for each $e \geq 0$, cohomology of the sequence

$$0 \to \mathscr{I}/\mathscr{I}^{[p^e]} \to \mathscr{O}_X/\mathscr{I}^{[p^e]} \to \mathscr{O}_Z \to 0$$

together with connectedness and projectivity of Z gives $\mathrm{H}^0(X, \mathscr{O}_X/\mathscr{I}^{[p^e]}) \cong \mathrm{H}^0(Z, \mathscr{O}_Z) \cong \mathbf{k}$. ■

In the proof, we used the fact that the dual of an ample vector bundle on a positive-dimensional projective scheme has no sections. For lack of suitable reference, we include a proof:

2.3. Lemma. — *Let X be a projective scheme over* $\mathbf{k}$ *and $\mathscr{E}$ a finite locally free $\mathscr{O}_X$-module. If* $\dim X > 0$ *and $\mathscr{E}$ is ample, then $\Gamma(X, \mathscr{E}^\vee) = 0$.*

Proof. Suppose $\sigma : \mathscr{E} \to \mathscr{O}_X$ is the morphism dual to a nonzero section of $\mathscr{E}^\vee$. Since X is of positive dimension, we may choose a finite morphism $\nu : C \to X$ from a smooth projective curve such that $\nu^*\sigma \neq 0$: for instance, take C to be the normalization of a component of a general complete intersection curve in X. The image of $\nu^*\sigma : \nu^*\mathscr{E} \to \mathscr{O}_C$ must then be a line subbundle of $\mathscr{O}_C$, whence of degree ≤ 0. This implies $\nu^*\mathscr{E}$ is not ample, and so neither is $\mathscr{E}$ since ν is finite. ■

A well-known consequence of 2.2 is that any nontrivial morphism out of X must be finite:

2.4. Corollary. — *If X is a smooth projective variety which is Frobenius–Tschirnhausen ample, then any non-constant $f : X \to Y$ is finite onto its image.*

Proof. Let $f : X \to Y$ be a non-constant morphism where the fibre $X_y := f^{-1}(y)$ over some point $y \in Y$ is positive-dimensional; by replacing f by its Stein factorization, we may assume that X_y is connected. Writing $\hat{X}$ for the completion of X along X_y, the theorem on formal functions, as in [Stacks, 02OD] for instance, implies

$$\mathrm{H}^0(\hat{X}, \mathscr{O}_{\hat{X}}) \cong (f_*\mathscr{O}_X)^\wedge_y.$$

Since f is non-constant, Y has positive dimension, and so the completion on the right contains a subring of the form $\mathbf{k}[[t]]$; in particular, it has infinite dimension as a $\mathbf{k}$-vector space. Thus X_y is not G1 in X, so the Frobenius-trace kernel of X cannot be ample by 2.2. ■

Together with Mori's theory of extremal contractions, this strongly suggests that Frobenius–Tschirnhausen ample varieties have Picard rank 1. Since contractions exist for smooth Fano threefolds by [Kol91], this gives a direct proof of [CRP25, Theorem 5.19]: a Frobenius–Tschirnhausen ample threefold is a Fano variety of Picard rank 1. Numerical considerations as in 1.4 together with the fact that Fano threefolds of Picard rank 1 of index 1 or 2 contain anti-canonical conics implies that the only Frobenius–Tschirnhausen ample threefolds are $\mathbf{P}^3$ and, when $p \neq 2$, a smooth quadric threefold: see [Mal25, Theorem 4.1] for details.

2.5. Filtration. — A more computable obstruction to Frobenius–Tschirnhausen ampleness can be extracted from 2.1 by considering the $\mathscr{I}$-adic filtration of the $\mathscr{O}_X$-module $\mathscr{I}/\mathscr{I}^{[p]}$:

$$\mathscr{I}^k \cdot \mathscr{I}/\mathscr{I}^{[p]} = \mathscr{I}^{k+1}/\mathscr{I}^{k+1} \cap \mathscr{I}^{[p]} \text{ for each } k \geq 0.$$

Since $\mathscr{I}^k \subseteq \mathscr{I}^{[p]}$ for $k \gg 0$, this provides a separated exhaustive filtration of $\mathscr{I}/\mathscr{I}^{[p]}$. If Z is a local complete intersection subscheme of X of codimension c, [Mal25, Proposition 3.4] provides a neat description of the graded pieces of this filtration in terms of the conormal bundle $\mathscr{C}_{Z/X} := \mathscr{I}/\mathscr{I}^2$:

$$\begin{aligned}\mathrm{gr}^k(\mathscr{I}/\mathscr{I}^{[p]}) &:= (\mathscr{I}^k \cdot \mathscr{I}/\mathscr{I}^{[p]})/(\mathscr{I}^{k+1} \cdot \mathscr{I}/\mathscr{I}^{[p]}) \\ &\cong \mathrm{coker}(\mathrm{Sym}^{k+1-p}\,\mathscr{C}_{Z/X} \otimes F^*\mathscr{C}_{Z/X} \to \mathrm{Sym}^{k+1}\,\mathscr{C}_{Z/X}) =: T^{k+1}(\mathscr{C}_{Z/X})\end{aligned}$$

where T^{k+1} is sometimes called the $(k+1)$-*st truncated symmetric power functor*; here, the displayed map is multiplication and negative symmetric powers are taken to zero. Notably, the last nonzero piece of the $\mathscr{I}$-adic filtration provides a submodule of $\mathscr{I}/\mathscr{I}^{[p]}$ of the form

$$\det(\mathscr{C}_{Z/X})^{\otimes p-1} \cong T^{c(p-1)}(\mathscr{C}_{Z/X}) \cong \mathrm{gr}^{c(p-1)-1}(\mathscr{I}/\mathscr{I}^{[p]}) = \mathscr{I}^{c(p-1)}/\mathscr{I}^{c(p-1)} \cap \mathscr{I}^{[p]} \subseteq \mathscr{I}/\mathscr{I}^{[p]}.$$

Combined with the exact sequence from 2.1 and 2.3, this gives Mallory's criterion [Mal25, Theorem 1.1]: if there exists a local complete intersection subscheme $Z \subset X$ such that $\det(\mathscr{C}_{Z/X})^{\otimes p-1}$ has a section, then X is not Frobenius–Tschirnhausen ample. This may be used to show that, for example, cubic fourfolds containing planes are not Frobenius–Tschirnhausen ample.

Other steps of this filtration may be effectively used to obstruct Frobenius–Tschirnhausen ampleness when the filtration splits. The following identifies a special class of Fano hypersurfaces in projective space for which this method may be applied:

2.6. Proposition. — *A smooth Fano hypersurface $X \subset \mathbf{P}^n$ of degree $d > p$ projectively equivalent to*

$$(x_0^{d-1}x_n + x_1^{d-1}x_{n-1} + x_2^pF_2 + \cdots + x_n^pF_n = 0) \subset \mathbf{P}^n$$

for some $F_2, \ldots, F_n \in \mathrm{H}^0(\mathbf{P}^n, \mathscr{O}_{\mathbf{P}^n}(d-p))$ is not Frobenius–Tschirnhausen ample.

Proof. Change coordinates and assume that X is defined by the displayed equation. Then it contains the line ℓ whose ideal sheaf $\mathscr{I}$ is generated by the coordinates $x_2, \ldots, x_n$. We show X is not Frobenius–Tschirnhausen ample by showing that $\mathrm{H}^0(X, F_*(\mathscr{I}/\mathscr{I}^{[p]})) \neq 0$, at which point the exact sequence of 2.1 together with 2.3 yields the conclusion. The observation is that, due to the form of the equation, the $\mathscr{I}$-adic filtration on $\mathscr{I}/\mathscr{I}^{[p]}$ from 2.5 splits upon viewing it as an $\mathscr{O}_\ell$-module through F_*, so there is an isomorphism of $\mathscr{O}_\ell$-modules

$$F_*(\mathscr{I}/\mathscr{I}^{[p]}) \cong \bigoplus\nolimits_{k=1}^{(n-2)(p-1)} F_*T^k(\mathscr{C}_{\ell/X}).$$

Indeed, trivialize this bundle on the two standard affine open subschemes of ℓ given by the nonvanishing locus of x_0 and x_1; for example, on the chart where $x_0 = 1$, the equation of X implies

$$x_n = -x_1^{d-1}x_{n-1} - x_2^p F_2 - \cdots - x_n^p F_n \equiv -x_1^{d-1}x_{n-1} + \mathscr{I}^{[p]}$$

so that an $\mathscr{O}_\ell$-module basis for the restriction of $F_*(\mathscr{I}/\mathscr{I}^{[p]})$ thereon is given by the monomials

$$x_1^a x_2^{b_2} \cdots x_{n-2}^{b_{n-2}} x_{n-1}^{b_{n-1}} \text{ with } 0 \leq a, b_2, \ldots, b_{n-2}, b_{n-1} \leq p-1$$

and $b_2 + \cdots + b_{n-2} + b_{n-1} > 0$. Passing to the chart where $x_1 = 1$ transforms the displayed basis element to a similar one where x_1 and x_{n-1} are replaced by x_0 and x_n; more precisely, if m and $0 \leq a' \leq p-1$ are the integers determined by the relation

$$mp + a' = -(a + b_2 + \cdots + b_{n-2}) + (d-2)b_{n-1},$$

then $x_1^a x_2^{b_2} \cdots x_{n-2}^{b_{n-2}} x_{n-1}^{b_{n-1}} \mapsto x_0^m \cdot x_0^{a'} x_2^{b_2} \cdots x_{n-2}^{b_{n-2}} x_n^{b_{n-1}}$; note that the $\mathscr{O}_\ell$-module structure is through p-th powers, so that $x_0 \cdot s = x_0^p s$ for a local section s of $F_*(\mathscr{I}/\mathscr{I}^{[p]})$. The transition function in the given basis is block diagonal and preserves the exponents $b_2, \ldots, b_{n-2}, b_{n-1}$, and this implies the claimed splitting. Finally, by considering the basis elements where $b_2 + \cdots + b_{n-2} + b_{n-1} = 1$ or else by a direct computation, the conormal bundle may be determined to be

$$\mathscr{C}_{\ell/X} = \mathscr{I}/\mathscr{I}^2 \cong \mathscr{O}_\ell(-1)^{\oplus n-3} \oplus \mathscr{O}_\ell(d-2).$$

This has sections, whence the result follows. ∎

2.7. — To complete the proof of Theorem B, it remains to see that there are smooth hypersurfaces whose equation takes the form in 2.6. Suppose first that $p \nmid d$ and that $n = 2m-1$ is odd. Consider the hypersurface $X \subset \mathbf{P}^{2m-1}$ of degree d defined by

$$f := \begin{cases} \sum_{k=0}^{m-1} (x_k^{d-1}x_{n-k} + x_k x_{n-k}^{d-1}) & \text{if } d \equiv 1 \pmod{p}, \text{ and} \\ \sum_{k=0}^{m-1} (x_k^{d-1}x_{n-k} + x_{n-k}^d) & \text{otherwise.} \end{cases}$$

By the Jacobian criterion, a singular point of X satisfies, for each $k = 0, \ldots, m-1$:

$$0 = \frac{\partial f}{\partial x_k} = \begin{cases} x_{n-k}^{d-1} \\ (d-1)x_k^{d-2}x_{n-k} \end{cases} \text{ and } 0 = \frac{\partial f}{\partial x_{n-k}} = \begin{cases} x_k^{d-1} & \text{if } d \equiv 1 \pmod{p}, \\ x_k^{d-1} + dx_{n-k}^{d-1} & \text{otherwise.} \end{cases}$$

In either case, the only solution to all these equations is that given by $x_k = 0$ for all $k = 0, \ldots, 2m-1$, so X is smooth. When $p \nmid d$ and $n = 2m$ is even, let f be defined by the same formula as above and simply take X to be the hypersurface defined by $f + x_m^d$. Next, suppose that $p \mid d$, and consider

$$f := x_0^{d-1}x_n + x_n^{d-1}x_{n-2} + x_{n-2}^{d-1}x_{n-3} + \cdots + x_2^{d-1}x_1 + x_1^{d-1}x_{n-1} + x_{n-1}^{d-1}x_0.$$

Up to reindexing the coordinates, this equation generalizes that defining the Klein cubic hypersurface; it is shown in [AGR23, Lemma 4] that when $p \nmid n+1$, the hypersurface $X \subset \mathbf{P}^n$ cut out by f is smooth. Finally, when $p \mid n+1$, take X to be defined by $f + x_n^d$; then a minor variation on the argument of *loc. cit.* shows that X is smooth. ∎

3. Homogeneous spaces

Frobenius-trace kernels of projective rational homogeneous spaces tend to be positive. To describe this, fix notation and write throughout this and the following sections $\mathbf{T} \subseteq \mathbf{P} \subseteq \mathbf{G}$ for a maximal torus and reduced parabolic subgroup contained in a connected reductive algebraic group, and let $X := \mathbf{G}/\mathbf{P}$ be the associated projective rational homogeneous space. The first observation, recorded in 3.3, is that Frobenius splitness of X combined with the infinitesimal representation theory of the opposite unipotent radical $\mathbf{U}$ of $\mathbf{P}$ implies that the homogeneous vector bundle $F_*(\omega_X^{\otimes 1-p})$, and hence also the Frobenius-trace kernel $\mathscr{E}_X$, is globally generated.

3.1. Global generation. — To explain, observe that $F_*(\omega_X^{\otimes 1-p})$ and $\mathscr{E}_X$, being constructed from canonical operations on X, are homogeneous vector bundles, so they are globally generated if and only if global sections generate them at the point $o \in X$ corresponding to the identity of $\mathbf{G}$. Since X is Frobenius split by [MR85, Theorem 2], the trace sequence

$$0 \to \mathscr{E}_X \to F_*(\omega_X^{\otimes 1-p}) \to \mathscr{O}_X \to 0$$

shows that $\mathscr{E}_X$ is generated at o if and only if $F_*(\omega_X^{\otimes 1-p})$ is so. Since $F \colon X \to X$ is an affine morphism, the latter statement is equivalent to surjectivity of the restriction map

$$\mathrm{res}\colon \mathrm{H}^0(X, \omega_X^{\otimes 1-p}) \to \mathrm{H}^0(X, \omega_X^{\otimes 1-p} \otimes \mathscr{O}_X/\mathfrak{m}^{[p]})$$

where $\mathfrak{m} \subset \mathscr{O}_X$ is the ideal sheaf of $o \in X$; in the terminology of [MS14], this means that $\mathscr{E}_X$ is globally generated if and only if $\omega_X^{\otimes 1-p}$ *separates p-Frobenius jets* at o.

To analyze this restriction map, decompose the parabolic subgroup $\mathbf{P} = \mathbf{U}^+ \rtimes \mathbf{L}$ into its unipotent radical and a Levi factor, and let $\mathbf{U} \subset \mathbf{G}$ be the opposite unipotent radical. The canonical map $\mathbf{U} \to \mathbf{G}/\mathbf{P} = X$ is an open immersion, equivariant with respect to multiplication by $\mathbf{U}$ and conjugation by $\mathbf{L}$, and such that the Frobenius kernel $\mathbf{U}_1 := \ker(F : \mathbf{U} \to \mathbf{U})$ is mapped onto the subscheme $F^{-1}(o)$. The restriction map may therefore be identified as a map of $\mathbf{U}_1 \rtimes \mathbf{L}$-representations

$$\mathrm{res}\colon \mathrm{H}^0(X, \omega_X^{\otimes 1-p}) \to \mathbf{k}_{-\lambda} \otimes \mathbf{k}[\mathbf{U}_1]$$

where $\mathbf{k}[\mathbf{U}_1]$ is the coordinate ring of the finite infinitesimal group scheme $\mathbf{U}_1$, and $\mathbf{k}_{-\lambda}$ is the 1-dimensional $\mathbf{L}$-representation whose character $-\lambda$ is the restriction of the $\mathbf{P}$-character corresponding to the line bundle dual to $\omega_X^{\otimes 1-p}$ under the usual identification of $\operatorname{Pic} X$ with the character group of $\mathbf{P}$.

3.2. — To describe the coordinate ring $\mathbf{k}[\mathbf{U}_1]$ explicitly, let $\mathrm{X}^*(\mathbf{T}) \supset \Phi \supset \Phi^+ \supset \Phi^+_{\mathbf{P}}$ be, respectively, the character lattice of the maximal torus, the roots of $\mathbf{G}$, positive roots compatible with $\mathbf{P}$, and the roots appearing in $\mathbf{U}^+$. Choose an isomorphism of varieties

$$\mathbf{U} \cong \prod\nolimits_{\alpha \in \Phi^+_{\mathbf{P}}} \mathbf{U}_{-\alpha}$$

given by an ordered product of root subgroups: see [Jan03, II.1.2 and II.1.10]. Since $\mathbf{U}_{-\alpha} \cong \mathbf{G}_a$ for each α, this identifies $\mathbf{U}$ as an affine space with coordinates x_α for each $\alpha \in \Phi^+_{\mathbf{P}}$. In this way, the coordinate ring of $\mathbf{U}_1 \cong F^{-1}(o)$ may be identified as the Artinian ring

$$\mathbf{k}[\mathbf{U}_1] \cong \mathrm{H}^0(X, \mathscr{O}_X/\mathfrak{m}^{[p]}) \cong \mathbf{k}[x_\alpha : \alpha \in \Phi^+_{\mathbf{P}}]/(x_\alpha^p : \alpha \in \Phi^+_{\mathbf{P}}).$$

Under any such isomorphism, the monomials span weight spaces for $\mathbf{T} \subseteq \mathbf{L}$ where a single coordinate x_α is of weight α. There is a unique monomial of highest weight given by

$$\tau := \Big(\prod\nolimits_{\alpha \in \Phi^+_{\mathbf{P}}} x_\alpha\Big)^{p-1}$$

with weight $\lambda = (p-1)\sum_{\alpha \in \Phi^+_{\mathbf{P}}} \alpha$. Since $\mathbf{U}_1$ is unipotent, its action on $\mathbf{k}[\mathbf{U}_1]$ cannot increase weights, and this implies that the complementary $\mathbf{T}$-subspace $\mathbf{k}[\mathbf{U}_1]_{<\lambda}$ spanned by all monomials other than τ is a $\mathbf{U}_1$-stable hyperplane; in fact, it is the unique $\mathbf{U}_1$-stable hyperplane by [Kem81, Lemma 5.4]. This observation leads to:

3.3. Proposition. — *The Frobenius-trace kernel of a rational homogeneous space is globally generated.*

Proof. By the discussion of 3.1, it suffices to show that $\omega_X^{\otimes 1-p}$ separates p-Frobenius jets at $o \in X$. Upon choosing an isomorphism of $\mathbf{U}$ with a product of root groups as in 3.2, view the restriction map as a map of $\mathbf{U}_1 \rtimes \mathbf{L}$-representations

$$\mathrm{res}\colon \mathrm{H}^0(X, \omega_X^{\otimes 1-p}) \to \mathbf{k}_{-\lambda} \otimes \mathbf{k}[x_\alpha : \alpha \in \Phi_{\mathbf{P}}^+]/(x_\alpha^p : \alpha \in \Phi_{\mathbf{P}}^+).$$

If res were not surjective, the discussion of 3.2 implies that its image must be contained in the hyperplane $\mathbf{k}_{-\lambda} \otimes \mathbf{k}[\mathbf{U}_1]_{<\lambda}$. But τ is in the image of res since X is Frobenius split: see [BK05, Theorem 1.3.8]. Therefore res is surjective, and so $\mathscr{E}_X$ is globally generated. ■

Although $\mathscr{E}_X$ is always globally generated for $X = \mathbf{G}/\mathbf{P}$, it certainly not always ample: If $\mathbf{P}$ were contained in a larger parabolic $\mathbf{P}' \subsetneq \mathbf{G}$, then there is a fibration $X \to \mathbf{G}/\mathbf{P}'$ with fibres $\mathbf{P}'/\mathbf{P}$, and 2.4 implies $\mathscr{E}_X$ is not ample. Ampleness thus forces $\mathbf{P}$ to be a *maximal* parabolic subgroup; geometrically, this means X must be a *generalized Grassmannian*, that is, a rational homogeneous space of of Picard rank 1. Unfortunately, it is still too optimistic to hope that $\mathscr{E}_X$ is always ample in this case, and the following example shows that some hypothesis on the characteristic p is required:

3.4. Example. — A smooth quadric hypersurface $X \subset \mathbf{P}^4$, a homogeneous space for $\mathbf{G} = \mathbf{SO}_5$, has ample Frobenius-trace kernel $\mathscr{E}_X$ if and only if $p > 2$. When $p > 2$, [Ach12, Theorem 2] shows that all the summands appearing in $\mathscr{E}_X$ are ample, whereas for $p = 2$, Theorem 1 of *loc. cit.* gives

$$\mathscr{E}_X \cong \mathscr{O}_X(1)^{\oplus 5} \oplus \mathscr{S}^\vee$$

where the dual spinor bundle $\mathscr{S}^\vee$ is not ample. More directly, this also follows from 1.5. ■

Nevertheless, we conjecture that, up to excluding low characteristic exceptions, the Frobenius-trace kernel $\mathscr{E}_X$ of a generalized Grassmannian is ample. In the remainder of the section, we refine the techniques used in 3.3 to develop a method of showing the stronger property that $\mathscr{E}_X \otimes \mathscr{O}_X(-1)$ is globally generated, where $\mathscr{O}_X(1)$ is the ample generator of $\operatorname{Pic} X$.

3.5. Ampleness. — Let X be a generalized Grassmannian, write $\mathscr{O}_X(1)$ for the ample generator of $\operatorname{Pic} X$, and consider the twisted Frobenius-trace sequence

$$0 \to \mathscr{E}_X \otimes \mathscr{O}_X(-1) \to F_*(\omega_X^{\otimes 1-p} \otimes \mathscr{O}_X(-p)) \to \mathscr{O}_X(-1) \to 0.$$

The considerations of 3.2 and 3.3 imply that, at $o \in X$, the trace map to $\mathscr{O}_X(-1)$ is canonically identified as the projection onto the highest weight space of $\mathbf{k}[\mathbf{U}_1]$ spanned by the element τ. Since $\mathscr{O}_X(-1)$ has no sections, the restriction map from global sections of $\omega_X^{\otimes 1-p} \otimes \mathscr{O}_X(-p)$ to p-Frobenius jets at o factors through the hyperplane

$$\mathrm{res}'\colon \mathrm{H}^0(X, \omega_X^{\otimes 1-p} \otimes \mathscr{O}_X(-p)) \to \mathbf{k}_{-\lambda+p\varpi} \otimes \mathbf{k}[\mathbf{U}_1]_{<\lambda}$$

where ϖ is the restriction to $\mathbf{L}$ of the weight corresponding to $\mathscr{O}_X(1)$. Arguing as in 3.1 shows that $\mathscr{E}_X \otimes \mathscr{O}_X(-1)$ is globally generated if and only if res' is surjective. Observe that such a statement is optimal: the line bundle $\omega_X^{\otimes 1-p} \otimes \mathscr{O}_X(1-p)$ separates p-Frobenius jets because X is Frobenius split compatibly with the Schubert divisor $X \setminus \mathbf{U}$ opposite to $\mathbf{U}$: see [BK05, Theorems 2.2.5 and 1.4.10].

3.6. — To formulate a suitable surjectivity criterion for res', rephrase the observation recalled in 3.2 as follows: $\mathbf{k}[\mathbf{U}_1]$ itself is the only $\mathbf{U}_1$-submodule containing the trace element τ. Applying the Demazure–Gabriel correspondence from [DG70, II, §7 n°4]—see also [Jan03, I.9.6]—which gives an equivalence between the representation theory of the height 1 infinitimesmal group scheme $\mathbf{U}_1$ and its p-Lie algebra $\mathfrak{u} = \operatorname{Lie} \mathbf{U}_1 = \operatorname{Lie} \mathbf{U}$, this equivalently says that every element of $\mathbf{k}[\mathbf{U}_1]$ may be expressed as a linear combination of elements obtained by successively acting by $\mathfrak{u}$ on τ:

$$\mathbf{k}[\mathbf{U}_1] = \operatorname{span}\{\tau\} + \sum\nolimits_{k \geq 1} \operatorname{span}\{X_{i_1} \cdots X_{i_k} \cdot \tau : X_{i_1}, \dots, X_{i_k} \in \mathfrak{u}\}.$$

Since the action of $\mathfrak{u}$ decreases weights, the sum on the right must be the hyperplane $\mathbf{k}[\mathbf{U}_1]_{<\lambda}$. This gives the following generalization of Kempf's criterion: $\mathbf{k}[\mathbf{U}_1]_{<\lambda}$ is itself the only $\mathbf{U}_1$-submodule that contains the image $\mathfrak{u}\cdot\tau$ of the trace element under the action of its Lie algebra. The following makes this more explicit, and also gives a geometric consequence; for simplicity, we suppress the global weight shift by $-\lambda+p\varpi$ in the statement:

3.7. Proposition. — *The twisted Frobenius-trace kernel $\mathcal{E}_X\otimes\mathcal{O}_X(-1)$ is globally generated if and only if the image of* res' *contains for each $\alpha\in\Phi^+_{\mathbf{P}}$ the element*

$$-\partial_\alpha\tau := x_\alpha^{p-2}\Big(\prod\nolimits_{\beta\in\Phi^+_{\mathbf{P}}\setminus\alpha} x_\beta\Big)^{p-1}\in\mathbf{k}[\mathbf{U}_1].$$

In particular, in this case, X is Frobenius split compatibly with the Schubert divisor $X\setminus\mathbf{U}$ opposite to $\mathbf{U}$ and relative to each root hyperplane of $\mathbf{U}$.

Proof. Following the discussion of 3.5 and 3.6, it suffices to show that the action on τ of the element $X_\alpha\in\mathfrak{u}$ dual to the root coordinate x_α is, up to scalars, by partial differentiation with respect to x_α. By construction, X_α acts on $\mathbf{k}[\mathbf{U}_1]$ by derivations of weight $-\alpha$; since $\mathbf{U}_{-\alpha}\cong\mathbf{G}_a$, it acts on x_α by its dual derivation ∂_α. Thus it may be expressed as an operator of the form

$$X_\alpha=\partial_\alpha+\sum\nolimits_{\beta\in\Phi^+_{\mathbf{P}}\setminus\alpha} f_\beta\partial_\beta$$

for some $f_\beta\in\mathbf{k}[\mathbf{U}_1]$ of weight $\beta-\alpha$. Weight considerations imply that any nonzero term appearing in f_β must contain a root coordinate x_γ other than x_β; since $x_\gamma^p=0$ in $\mathbf{k}[\mathbf{U}_1]$, this implies that $f_\beta\partial_\beta\tau=0$ for each β, yielding the first statement. For the second statement, simply note that the linear function x_α defining the α root hyperplane of $\mathbf{U}$ is a section of $\mathcal{O}_X(1)$, so that $x_\alpha\partial_\alpha\tau$ is a section of $\omega_X^{\otimes 1-p}\otimes\mathcal{O}_X(1-p)$ that provides a Frobenius splitting relative to the root hyperplane defined by x_α and compatible with the Schubert divisor $X\setminus\mathbf{U}$, as per the remark closing 3.5. ■

To make this criterion more effective, notice that since $\mathbf{L}$ normalizes $\mathbf{U}_1$, the action of $\mathbf{L}$ on $\mathbf{k}[\mathbf{U}_1]$ intertwines the action of the Lie algebra $\mathfrak{u}$, where $\mathbf{L}$ acts on $\mathfrak{u}$ via the adjoint representation. This gives the following structure:

3.8. Corollary. — *The $\partial_\alpha\tau$ with $\alpha\in\Phi^+_{\mathbf{P}}$ span an* $\mathbf{L}$*-submodule of* $\mathbf{k}[\mathbf{U}_1]$*, isomorphic to*

$$\mathrm{span}\{\partial_\alpha\tau:\alpha\in\Phi^+_{\mathbf{P}}\}\cong\mathfrak{u}\cdot\tau\cong\mathfrak{u}\otimes\mathbf{k}_\lambda. \qquad ■$$

3.9. Levels. — More precise structure on the representations of $\mathbf{L}$ will prove useful. For this, let $\Delta\subset\Phi^+$ be a set of simple roots, and let $\Delta_{\mathbf{L}}\subset\Delta$ be the subset of simple roots in $\mathbf{L}$. Given a weight $\beta\in\mathrm{X}^*(\mathbf{T})$ in the cone spanned by Φ^+, write

$$\beta=\sum\nolimits_{\alpha\in\Delta_{\mathrm{L}}} c_\alpha\alpha+\sum\nolimits_{\alpha\in\Delta\setminus\Delta_{\mathrm{L}}} d_\alpha\alpha\ \text{ for integers } c_\alpha,d_\alpha\geq 0.$$

The integer $\sum_\alpha d_\alpha$ is the *level* of β, and the tuple $(d_\alpha)_{\alpha\in\Delta\setminus\Delta_{\mathrm{L}}}$ is the *shape* of β. Since the Levi action shifts weights by combinations of $\Delta_{\mathbf{L}}$, levels are preserved, so any representation of $\mathbf{L}$ has a direct sum decomposition into levels. In particular, this applies to its adjoint representation on $\mathfrak{u}$ and also its action on the coordinate ring of $\mathbf{U}_1$; in the latter case, this provides a multiplicative grading and the level $\ell\geq 0$ piece may be described in terms of the root coordinates x_α from 3.2 as

$$\mathbf{k}[\mathbf{U}_1]_\ell:=\mathrm{span}\{x_{\alpha_1}\cdots x_{\alpha_k}:\alpha_1,\ldots,\alpha_k\in\Phi^+_{\mathbf{P}}\text{ with level}(\alpha_1+\cdots+\alpha_k)=\ell\}.$$

In particular, the constants span level 0 and the trace element τ spans the top level N. The elements $\partial_\alpha\tau$ of interest from 3.7 are typically spread across several levels near the top of $\mathbf{k}[\mathbf{U}_1]$ depending, by 3.8, on the level decomposition of $\mathfrak{u}$. See [ABS90] and [MT11, §17.1] for more.

4. Isotropic Grassmannian

In this section, we apply the methods developed in §3 to show that the twisted Frobenius-trace kernel is globally generated for most generalized Grassmannian $X = \mathbf{G}/\mathbf{P}$. In particular, this implies that $\mathcal{E}_X$ is ample for such homogeneous spaces, yielding sufficiency in Theorem C. Our method relies on some explicit computations and hence works best on *isotropic Grassmannian*, by which we mean any X that may be described as a subvariety of the usual Grassmannian parameterizing linear spaces isotropic for some additional structure. Most familiar are those of *classical type*:

- (A) the Grassmannian $\mathbf{Gr}(k,n)$ of k-dimensional linear subspaces in an n-dimensional vector space, where $n \geq 2$ and $1 \leq k \leq n-1$;
- (C) the symplectic Grassmannian $\mathbf{SGr}(k,n)$ parameterizing totally isotropic subspaces for a given nondegenerate symplectic form ω, where $n = 2m \geq 4$ and $1 \leq k \leq m$; and
- (B/D) the orthogonal Grassmannian $\mathbf{OGr}(k,n)$ parameterizing totally isotropic subspaces for a given nondegenerate quadratic form q, where $n \geq 5$ and $1 \leq k < \lfloor \frac{n-1}{2} \rfloor$ or $k = \lfloor \frac{n}{2} \rfloor$.

In terms of Dynkin diagrams, these are of types A_{n-1}, C_m, and, in the orthogonal case, B_m if $n = 2m+1$ and D_m if $n = 2m$. We also study the two generalized Grassmannian of type G_2: one being a quadric fivefold, and the other—referred to as the G_2-*Grassmannian*—is a 5-dimensional subvariety of $\mathbf{Gr}(2,7)$ parameterizing subspaces totally isotropic for a generic alternating 3-form. We refer to [Bel26] for an interactive guide to these objects. With this notation, the result is:

4.1. Theorem. — *Let X be an isotropic Grassmannian. Then $\mathcal{E}_X(-1)$ is globally generated except possibly when: $X = \mathbf{SGr}(k,2m)$ with $2 \leq k \leq m$ for $p = 2$; $X = \mathbf{OGr}(m-1, 2m+1)$ for $p = 2$; and X is the G_2-Grassmannian for $p = 2,3$.*

4.2. — The proof of 4.1 occupies the remainder of this section. To explain the strategy, coordinatize the canonical affine open subscheme of X provided by the opposite unipotent radical $\mathbf{U}$ as in 3.1 as the big Schubert cell parameterizing isotropic k-dimensional subspaces in an n-dimensional space given by the row span of the $k \times n$ matrix of indeterminates

$$M := \begin{pmatrix} x_{1,1} & \cdots & x_{1,k} & x_{1,k+1} & \cdots & x_{1,n-k} & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{k,1} & \cdots & x_{k,k} & x_{k,k+1} & \cdots & x_{k,n-k} & 0 & \cdots & 1 \end{pmatrix}$$

where, other than in type A, the $x_{i,j}$ are subject to equations which express the isotropicity condition. With the exception of the *spinor varieties* dealt with in 4.11, the ample generator $\mathcal{O}_X(1) \in \mathrm{Pic}\,X$ is the restriction of the Plücker line bundle, so the task prescribed by 3.7 of constructing sections of $\omega_X^{\otimes 1-p} \otimes \mathcal{O}_X(-p)$ realizing the elements $\partial_{i,j}\tau$ amounts to producing suitable polynomials in the minors of M. In the cases considered, the level decomposition from 3.9 of the Lie algebra $\mathfrak{u}$ splits it into at most 2 indecomposable representations for $\mathbf{L}$, and so we need only explicitly construct a small number of splittings by 3.8.

4.3. Rolling determinants. — In most of the cases considered, the device with which we construct the requisite sections is the following: Let $s > 0$ be the Fano index of X, so that $\omega_X^\vee \cong \mathcal{O}_X(s)$. Form an auxiliary $k \times (s+k-2)$ matrix consisting of columns from M of the form

$$N = \begin{pmatrix} 0 & \cdots & 0 & y_{1,1} & \cdots & y_{1,s-1} \\ 1 & \cdots & 0 & y_{2,1} & \cdots & y_{2,s-1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & y_{k,1} & \cdots & y_{k,s-1} \end{pmatrix}$$

where the initial block consists of a size $k-1$ identity matrix shifted down one row, followed by other columns of M for some $y_{i,j} = x_{i,j_i}$. Construct a section of $\omega_X^{\otimes 1-p} \otimes \mathscr{O}_X(1-p) \cong \mathscr{O}_X((p-1)(s-1))$ by taking the $(p-1)$-power *rolling determinant* of N,

$$(\operatorname{roll}\det N)^{p-1} := \prod\nolimits_{\ell=1}^{s-1} (\det N_{\ell,\dots,\ell+k-1})^{p-1}$$

where $N_{\ell,\dots,\ell+k-1}$ is the $k \times k$-submatrix of N formed by the columns from the ℓ-th to the $(\ell+k-1)$-st. With a suitable choice of N, this, or a minor variation thereof, often provides a Frobenius splitting of X compatible with the Schubert divisor $X \setminus \mathbf{U}$ that is additionally divisible by $(\det N_{1,\dots,k})^{p-1} = y_{1,1}^{p-1}$. Dividing out $y_{1,1}$ once produces a section of $\omega_X^{\otimes 1-p} \otimes \mathscr{O}_X(-p)$ whose image under the restriction map res' to $\mathbf{k}[\mathbf{U}_1]_{<\tau}$ contains $\pm\partial\tau/\partial y_{1,1}$ and maybe other terms, which may be determined by weight considerations. We then use the action of $\mathfrak{u}$ to show that the extraneous terms also lie in the image of res', at which point we may conclude.

The purpose of the rolling determinant is the following:

4.4. Lemma. — *As an element of* $\mathbf{k}[y_{1,1},\dots,y_{k,s-1}]/(y_{1,1}^p,\dots,y_{k,s-1}^p)$,

$$(\operatorname{roll}\det N)^{p-1} = \Big(\prod\nolimits_{j=1}^{s-k-1}\prod\nolimits_{i=1}^{k} y_{i,j}\Big)^{p-1}\Big(\prod\nolimits_{j=s-k}^{s-1}\prod\nolimits_{i=1}^{s-j} y_{i,j}\Big)^{p-1}.$$

This can be established with an elementary inductive argument. Rather than spell out the proof, we illustrate the key point by expanding the first three factors in the rolling determinant. Thanks to the initial shifted identity matrix block, this is the left side of

$$y_{1,1}^{p-1}\begin{vmatrix} y_{1,1} & y_{1,2} \\ y_{2,1} & y_{2,2}\end{vmatrix}^{p-1}\begin{vmatrix} y_{1,1} & y_{1,2} & y_{1,3} \\ y_{2,1} & y_{2,2} & y_{2,3} \\ y_{3,1} & y_{3,2} & y_{3,3}\end{vmatrix}^{p-1} = y_{1,1}^{p-1}\begin{vmatrix} 0 & y_{1,2} \\ y_{2,1} & y_{2,2}\end{vmatrix}^{p-1}\begin{vmatrix} 0 & y_{1,2} & y_{1,3} \\ y_{2,1} & y_{2,2} & y_{2,3} \\ y_{3,1} & y_{3,2} & y_{3,3}\end{vmatrix}^{p-1}$$

Since $y_{i,j}^p = 0$, the initial $y_{1,1}^{p-1}$ implies that subsequent $y_{1,1}$ do not contribute, and so we may set all later occurrences of $y_{1,1}$ to 0, as in the right side above. The second matrix is now triangular, so its determinant is the product of its anti-diagonal terms $y_{2,1}$ and $y_{1,2}$. As before, we may set these variables to 0 in the third determinant, yielding again a triangular matrix, from which it follows that the initial product of three determinants yields the element

$$(y_{1,1}y_{2,1}y_{3,1}y_{1,2}y_{2,2}y_{1,3})^{p-1} \in \mathbf{k}[y_{1,1},\dots,y_{k,s-1}]/(y_{1,1}^p,\dots,y_{k,s-1}^p).$$

Iterating this argument, each successive term may be reduced to a determinant of a triangular matrix, from which the lemma follows easily.

4.5. Grassmannian. — As the first case of 4.1, consider $X = \mathbf{Gr}(k,n)$ with $1 \leq k \leq n-1$. Take $\mathbf{G} = \mathbf{GL}_n$, $\mathbf{T}$ the subgroup of diagonal matrices, $\mathbf{P}$ the parabolic subgroup of matrices with vanishing lower left $(n-k) \times k$ block, and Levi factor $\mathbf{L}$ the block diagonal subgroup $\mathbf{GL}_k \times \mathbf{GL}_{n-k}$. The action of $\mathbf{L}$ on $\mathbf{U}$ may be described in terms of the matrix M from 4.2 as that induced by the simultaneous action the left by $\mathbf{GL}_k$ via row operations and on the right by $\mathbf{GL}_{n-k}$ via column operations on the initial $n-k$ columns. This implies that the adjoint representation on the Lie algebra of $\mathbf{U}$ is irreducible, isomorphic to the tensor product $\mathfrak{u} \cong \mathrm{Std}_{\mathbf{GL}_k} \otimes \mathrm{Std}_{\mathbf{GL}_{n-k}}$ of the standard representations of $\mathbf{GL}_k$ and $\mathbf{GL}_{n-k}$. By 3.8, it suffices to construct the element $\partial_{1,1}\tau$.

Since $\omega_X^{-1} \cong \mathscr{O}_X(n)$ where $\mathscr{O}_X(1)$ is the Plücker, the strategy from 4.2 is to produce a polynomial of degree $(p-1)(n-1)$ in the minors of M which provides a Frobenius splitting divisible by the minor evaluating to $x_{1,1}$. As in 4.3, consider the $k \times (n+k-2)$ matrix formed from the columns of M

$$A := \begin{pmatrix} 0 & \cdots & 0 & 0 & x_{1,1} & \cdots & x_{1,n-k} & 1 & 0 & \cdots & 0 \\ 1 & \cdots & 0 & 0 & x_{2,1} & \cdots & x_{2,n-k} & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & x_{k-1,1} & \cdots & x_{k-1,n-k} & 0 & 0 & \cdots & 1 \\ 0 & \cdots & 0 & 1 & x_{k,1} & \cdots & x_{k,n-k} & 0 & 0 & \cdots & 0 \end{pmatrix}$$

consisting of a size $k-1$ identity matrix shifted down one, followed by the $k \times (n-k)$ block of coordinates of $\mathbf{U}$, followed by another size $k-1$ identity, this time shifted up one. A slight modification of 4.4 accounting for the final identity matrix gives

$$(\operatorname{roll}\det A)^{p-1} := \prod\nolimits_{\ell=1}^{n-1} (\det A_{\ell,\ldots,\ell+k-1})^{p-1} = \Big(\prod\nolimits_{j=1}^{n-k}\prod\nolimits_{i=1}^{k} x_{i,j}\Big)^{p-1} \in \mathbf{k}[\mathbf{U}_1].$$

Since the weight of $x_{1,1}$ is indecomposable amongst the roots of $\mathbf{U}^+$, the weight space in $\mathbf{k}[\mathbf{U}_1]$ containing $\partial_{1,1}\tau$ is 1-dimensional. Dividing the displayed element by $\det A_{1,\ldots,k} = \pm x_{1,1}$ thus produces a section of $\omega_X^{\otimes 1-p} \otimes \mathcal{O}_X(-p)$ restricting to $\pm\partial_{1,1}\tau$, completing the proof of 4.1 in type A. ■

4.6. Example. — Beyond the classical case of $\mathbf{P}^{n-1}$, the case $X = \mathbf{Gr}(2,n)$ may be alternatively seen using work of Raedschelders–Špenko–Van den Bergh, in which they explicitly compute the decomposition of $F_*\mathcal{O}_X$. Namely, writing $\mathcal{S}$ and $\mathcal{Q}$ for the tautological sub- and quotient bundles of ranks 2 and $n-2$, respectively, they show in [RŠVdB19, Theorem 16.5 and Corollary 16.6] that

$$\mathcal{E}_X \cong \Big(\bigoplus\nolimits_{k>0} \mathcal{O}_X(k)^{\oplus a_k}\Big) \oplus \Big(\bigoplus\nolimits_{k>0}\bigoplus\nolimits_{j=1}^{n-3} \wedge^j \mathcal{Q}(k)^{\oplus b_{j,k}}\Big) \oplus \Big(\bigoplus\nolimits_{k>0}\bigoplus\nolimits_{j=1}^{n-3} \mathcal{F}_j(j+k)^{\oplus c_{j,k}}\Big)$$

for some multiplicities a_k, $b_{j,k}$, and $c_{j,k}$. The vector bundles $\mathcal{F}_j$ are given by

$$\begin{cases} \mathcal{F}_j \cong \operatorname{Sym}^j(\mathcal{S}) & \text{if } 0 \leq j \leq p-1, \\ 0 \to \operatorname{Sym}^{2p-2-j}(\mathcal{S}) \to \mathcal{F}_j \to \operatorname{Sym}^j(\mathcal{S}) \to 0 & \text{if } p \leq j \leq 2p-2, \text{ and} \\ \mathcal{F}_j \cong \mathcal{F}_{j_0} \otimes F^*\mathcal{F}_{j_1} \otimes \cdots \otimes F^{e,*}\mathcal{F}_{j_e} & \text{if } j = j_0 + j_1 p + \cdots + j_e p^e \end{cases}$$

where the extension is non-split in the second case, and $p-1 \leq j_i \leq 2p-2$ for $i < e$ and $0 \leq j_e \leq p-1$ in the third case—note that our conventions are dual to those in [RŠVdB19, §4.3.1]. Standard arguments, using the wedge product isomorphism $\mathcal{S} \cong \mathcal{S}^\vee(-1)$ for instance, then show that all terms appearing in $\mathcal{E}_X(-1)$ are globally generated. ■

4.7. Symplectic Grassmannian. — Next, consider $X = \mathbf{SGr}(k,n)$ with $n = 2m \geq 4$ and $2 \leq k \leq m$; we omit the case $k = 1$ since then $\mathbf{SGr}(1,n) \cong \mathbf{P}^{n-1}$ and this has been dealt with in 4.5. Take $\mathbf{G} = \mathbf{Sp}_{2m}$ to be the symplectic group of matrices in $\mathbf{SL}_{2m}$ preserving the symplectic form

$$\omega := \sum\nolimits_{s=1}^{m} \mathrm{d}x_s \wedge \mathrm{d}x_{2m+1-s} = \mathrm{d}x_1 \wedge \mathrm{d}x_{2m} + \mathrm{d}x_2 \wedge \mathrm{d}x_{2m-1} + \cdots + \mathrm{d}x_m \wedge \mathrm{d}x_{m+1}.$$

With this choice, we may take $\mathbf{T}$ to be the subgroup of diagonal matrices, and $\mathbf{P}$ the parabolic subgroup preserving the two-step symplectic flag consisting of the subspace spanned by the first k basis vectors together with its ω-orthogonal space. The variables $x_{i,j}$ in the matrix M from 4.2 on the big Schubert cell of X satisfy the following equations, which expresses isotropicity with respect to ω:

$$x_{j,k+1-i} = x_{i,k+1-j} + \sum\nolimits_{s=k+1}^{m} (x_{i,s}x_{j,2m+1-s} - x_{i,2m+1-s}x_{j,s}) \text{ for each } 1 \leq i < j \leq k.$$

We use these equations to eliminate the variables $x_{j,k+1-i}$ strictly below the anti-diagonal in $M_{1,\ldots,k}$, so that the root variables of $\mathbf{U}$ are those on and above the anti-diagonal and also the variables appearing in the central $k \times (2m-2k)$ block. To first order, $M_{1,\ldots,k}$ is a matrix symmetric along its anti-diagonal; for instance $x_{k,k} = x_{1,1} + \mathfrak{m}^2$ where $\mathfrak{m}$ is the maximal ideal generated by all the variables. Globally, X

is cut out of $\mathbf{Gr}(k,2m)$ by the regular section of the second wedge power $\wedge^2\mathscr{S}^\vee$ of the dual of the tautological rank k subbundle, so

$$\omega_X^{-1} \cong \omega_{\mathbf{Gr}(k,2m)}^{-1} \otimes \det \wedge^2\mathscr{S}|_X \cong \mathscr{O}_X(2m-k+1)$$

where $\mathscr{O}_X(1)$ is the Plücker line bundle.

A Levi factor $\mathbf{L}$ is isomorphic to $\mathbf{GL}_k \times \mathbf{Sp}_{2m-2k}$. Considering its action on $\mathbf{U}$ shows that the level decomposition of the adjoint representation on $\mathfrak{u}$ has two steps with

$$\mathfrak{u} \cong \mathrm{Std}_{\mathbf{GL}_k} \otimes \mathrm{Std}_{\mathbf{Sp}_{2m-2k}} \oplus \mathrm{Div}^2\,\mathrm{Std}_{\mathbf{GL}_k}$$

where level 1 is the tensor product of the standard representations arising from the simultaneous left action of $\mathbf{GL}_k$ by row operations and right action of $\mathbf{Sp}_{2m-2k}$ by column operations on the middle $k \times (2m-2k)$ block; and level 2 is the symmetric quadratic tensors of the standard representation of $\mathbf{GL}_k$ corresponding to its action on the anti-diagonally symmetric $M_{1,\dots,k}$. When $p \geq 3$, both summands are irreducible representations for $\mathbf{L}$, and so by 3.8, it suffices to construct the elements $\partial_{1,k+1}\tau$ and $\partial_{1,1}\tau$ as sections of $\mathscr{O}_X((p-1)(2m-k)-1)$.

For $\partial_{1,k+1}\tau$, 4.4 shows that the $(p-1)$-power rolling determinant of the $k \times (2m-1)$ matrix

$$C := \begin{pmatrix} 0 & \cdots & 0 & x_{1,k+1} & \cdots & x_{1,2m-k} & x_{1,1} & \cdots & x_{1,k} \\ 1 & \cdots & 0 & x_{2,k+1} & \cdots & x_{2,2m-k} & x_{2,1} & \cdots & x_{2,k} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & x_{k,k+1} & \cdots & x_{k,2m-k} & x_{k,1} & \cdots & x_{k,k} \end{pmatrix}$$

provides an expression for τ divisible by $\det C_{1,\dots,k} = \pm x_{1,k+1}$. Arguing via indecomposability of the weight of $x_{1,k+1}$ as in 4.5 shows that the $(p-1)$-power rolling determinant of C divided by this factor produces a suitable expression for $\partial_{1,k+1}\tau$.

For $\partial_{1,1}\tau$, let C' be the $k \times (2m-1)$ matrix obtained from C by moving the $x_{i,1}$ column to the k-th position. Since the variables $x_{j,k+1-i}$ with $1 \leq i < j \leq k$ that lie strictly below the anti-diagonal of $M_{1,\dots,k}$ still do not appear in the product given by applying 4.4 to C', the $(p-1)$-power rolling determinant of C' still provides an expression for τ divisible by $\det C'_{1,\dots,k} = \pm x_{1,1}$. However, $x_{1,1}$ has the same weight as $x_{1,j}x_{k,2m+1-j}$ for each $k < j < 2m-k$, and so $\gamma' := (\mathrm{roll}\det C')^{p-1}/\det C'_{1,\dots,k}$ may contain terms of the form $\partial_{1,j}\partial_{k,2m+1-j}\tau$ in addition to $\partial_{1,1}\tau$. But γ', being a product of determinants, may be seen to be a homogeneous polynomial of total degree $(p-1)\dim X - 1$ in the matrix entries of C. Thus every term in γ' has total degree at least $(p-1)\dim X - 1$ in the root coordinates of $\mathbf{k}[\mathbf{U}_1]$. This forces $\gamma' = \pm\partial_{1,1}\tau$, completing the proof of 4.1 in type C. ■

4.8. Orthogonal Grassmannian. — Let $X = \mathbf{OGr}(k,n)$ with $n \geq 5$ and either $1 \leq k < \lfloor \frac{n-1}{2} \rfloor$ or $k = \lfloor \frac{n}{2} \rfloor$. Take $\mathbf{G} = \mathbf{SO}_n$ to be the subgroup of $\mathbf{SL}_n$ preserving the quadratic form

$$q := \begin{cases} x_1x_{2m+1} + x_2x_{2m} + \cdots + x_mx_{m+2} + x_{m+1}^2 & \text{if } n = 2m+1, \text{ and} \\ x_1x_{2m} + x_2x_{2m-1} + \cdots + x_mx_{m+1} & \text{if } n = 2m. \end{cases}$$

With this choice, we may take $\mathbf{T}$ to be the subgroup of diagonal matrices, and $\mathbf{P}$ the parabolic subgroup preserving the two-step orthogonal flag consisting of the subspace spanned by the first k basis vectors together with its orthogonal space with respect to the polar form associated to q. The variables $x_{i,j}$ of M from 4.2 on the big Schubert cell of X satisfy two sets of equations corresponding to isotropicity with respect to q. The first set expresses the $x_{i,k+1-i}$ for $1 \leq i \leq k$ on the anti-diagonal of $M_{1,\dots,k}$ as the quadratic form applied to the coordinates in the middle $k \times (n-2k)$ block:

$$\begin{cases} x_{i,k+1-i} = -x_{i,k+1}x_{i,n-k} - \cdots - x_{i,m}x_{i,m+2} - x_{i,m+1}^2 & \text{if } n = 2m+1, \text{ and} \\ x_{i,k+1-i} = -x_{i,k+1}x_{i,n-k} - \cdots - x_{i,m}x_{i,m+1} & \text{if } n = 2m. \end{cases}$$

The second set of equations shows that $M_{1,\dots,k}$ is, to first-order, anti-symmetric along the anti-diagonal: that is, for $1 \leq i < j \leq k$,

$$\begin{cases} x_{j,k+1-i} = -x_{i,k+1-j} - x_{i,k+1}x_{j,n-k} - \cdots - x_{i,m}x_{j,m+2} - 2x_{i,m+1}x_{j,m+1} & \text{if } n = 2m+1 \text{, and} \\ x_{j,k+1-i} = -x_{i,k+1-j} - x_{i,k+1}x_{j,n-k} - \cdots - x_{i,m}x_{j,m+1} & \text{if } n = 2m. \end{cases}$$

We use these equations to eliminate the variables on and below the anti-diagonal of $M_{1,\dots,k}$, so that the root coordinates are given by those above the anti-diagonal of $M_{1,\dots,k}$ and the variables in the central $k \times (n-2k)$ block of M. Globally, X is cut out of $\mathbf{Gr}(k,n)$ by the regular section induced by q of the second symmetric power $\mathrm{Sym}^2 \mathscr{S}^\vee$ of the tautological rank k subbundle, so that

$$\omega_X^{-1} \cong \omega_{\mathbf{Gr}(k,n)}^{-1} \otimes \det \mathrm{Sym}^2 \mathscr{S}|_X \cong \mathscr{L}^{\otimes n-k-1}$$

where $\mathscr{L}$ is the restriction of the Plücker line bundle onto X. If $k < m$, then $\mathscr{L} = \mathscr{O}_X(1)$ is the ample generator of $\mathrm{Pic}\,X$ and otherwise $\mathscr{L} = \mathscr{O}_X(2)$ in the case $k = m$: see 4.11 for details.

4.9. — Consider first the case where $1 \leq k < \lfloor \frac{n-1}{2} \rfloor$; in terms of Dynkin diagrams, this means that the parabolic $\mathbf{P}$ is not one of the spinor nodes on the right. In this case, $\mathscr{O}_X(1)$ is the Plücker line bundle and a Levi factor $\mathbf{L}$ is isomorphic to $\mathbf{GL}_k \times \mathbf{SO}_{n-2k}$. Computations analogous to the symplectic case shows that the level decomposition of the Lie algebra of $\mathbf{U}$ is

$$\mathfrak{u} \cong \mathrm{Std}_{\mathbf{GL}_k} \otimes \mathrm{Std}_{\mathbf{SO}_{n-2k}} \oplus \wedge^2 \mathrm{Std}_{\mathbf{GL}_k}$$

where level 1 is the tensor product of the standard representations arising as the action on the central $k \times (n-2k)$ block of M via row operations by $\mathbf{GL}_k$ and column operations by $\mathbf{SO}_{n-2k}$; and level 2 comes from the action of $\mathbf{GL}_k$ on the anti-diagonally anti-symmetric matrix $M_{1,\dots,k}$. The wedge square is always irreducible, whereas the tensor product of standard representations is irreducible except in the case the characteristic $p = 2$ and $n = 2m+1$ is odd; we treat this exceptional case in 4.10. In the remaining cases, it suffices to construct sections of $\mathscr{O}_X((p-1)(n-k-2)-1)$ realizing the elements $\partial_{1,k+2}\tau$ and $\partial_{1,1}\tau$. We proceed in three steps:

Step 1. To construct $\partial_{1,k+2}\tau$, consider the $k \times (n-3)$ matrix

$$B := \begin{pmatrix} 0 & \cdots & 0 & x_{1,k+2} & \cdots & x_{1,n-k-1} & x_{1,1} & \cdots & x_{1,k} \\ 1 & \cdots & 0 & x_{2,k+2} & \cdots & x_{2,n-k-1} & x_{2,1} & \cdots & x_{2,k} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & x_{k,k+2} & \cdots & x_{k,n-k-1} & x_{k,1} & \cdots & x_{k,k} \end{pmatrix}$$

where the pair of columns with the variables $x_{i,k+1}$ and $x_{i,n-k}$ do not appear; the essential property of the missing columns is that they are matched by the quadratic form q. Applying 4.4 gives

$$(\mathrm{roll}\det B)^{p-1} = \Big(\prod\nolimits_{j=k+2}^{n-k-1} \prod\nolimits_{i=1}^{k} x_{i,j}\Big)^{p-1} \Big(\prod\nolimits_{j=1}^{k} \prod\nolimits_{i=1}^{k+1-j} x_{i,j}\Big)^{p-1} \in \mathbf{k}[\mathbf{U}_1]$$

which is the $(p-1)$-power of the product of all the variables in M excluding the $x_{i,k+1}$, $x_{i,n-k}$, and those strictly below the anti-diagonal in $M_{1,\dots,k}$. Consider the product of the anti-diagonal matrix entries $x_{i,k+1-i}$ in the second product: the first set of equations of X in 4.8 reads

$$x_{i,k+1-i} = -x_{i,k+1}x_{i,n-k} - q_i' \text{ for each } i = 1, \dots, k$$

where q_i' is a quadratic form in the variables $x_{i,j}$ with $k+2 \leq j \leq n-k-1$. Since each variable in q_i' already appears in the rolling determinant with exponent $p-1$, each $x_{i,k+1-i}$ only contributes

$x_{i,k+1}x_{i,n-k}$ into the product. Therefore

$$\Big(\prod_{j=k+2}^{n-k-1}\prod_{i=1}^{k} x_{i,j}\Big)^{p-1}\Big(\prod_{j=1}^{k-1}\prod_{i=1}^{k-j} x_{i,j}\Big)^{p-1}\Big(\prod_{i=1}^{k} x_{i,k+1-i}\Big)^{p-1}$$
$$= \Big(\prod_{j=k+2}^{n-k-1}\prod_{i=1}^{k} x_{i,j}\Big)^{p-1}\Big(\prod_{j=1}^{k-1}\prod_{i=1}^{k-j} x_{i,j}\Big)^{p-1}\Big(\prod_{i=1}^{k} x_{i,k+1}x_{i,n-k}\Big)^{p-1}$$

and this is precisely the trace element τ in $\mathbf{k}[\mathbf{U}_1]$. Since the weight of $x_{1,k+2}$ is indecomposable in $\Phi^+_{\mathbf{P}}$, dividing this expression for τ by $\det B_{1,\dots,k} = \pm x_{1,k+2}$ provides a suitable expression for $\partial_{1,k+2}\tau$. Irreducibility of the $\mathbf{L}$-representation

$$\mathbf{k}[\mathbf{U}_1]_{N-1} \cong \mathfrak{u}_1\cdot\tau \cong \mathrm{Std}_{\mathbf{GL}_k}\otimes \mathrm{Std}_{\mathbf{SO}_{n-2k}}$$

now implies that each of the elements $\partial_{i,j}\tau$ for $1 \le i \le k$ and $k+2 \le j \le n-k-1$ may be realized via a section of $\omega_X^{\otimes 1-p}\otimes \mathcal{O}_X(-p)$.

Step 2. We now show that the elements $\partial_{1,j}\partial_{k,n+1-j}\tau$ with $k+1 \le j \le n-k$ may also be realized by sections of $\omega_X^{\otimes 1-p}\otimes\mathcal{O}_X(-p)$. As discussed in 3.6, the restriction map res' is equivariant for the action of the Lie algebra $\mathfrak{u}$, so it suffices to show that the $\partial_{1,j}\partial_{k,n+1-j}\tau$ may be obtained from the $\partial_{i,j}\tau$ constructed in step 1 via the action of $\mathfrak{u}$. For each $k+1\le j\le n-k$, consider the Lie algebra elements $X_{1,j}$ and $X_{k,n+1-j}$ dual to the root coordinates $x_{1,j}$ and $x_{k,n+1-j}$, respectively. Weight considerations show that they act on $\mathbf{k}[\mathbf{U}_1]$ as derivations of the form

$$X_{1,j} = \partial_{1,j} + c x_{k,n+1-j}\partial_{1,1} + D \;\text{ and }\; X_{k,n+1-j} = \partial_{k,n+1-j} + c' x_{1,j}\partial_{1,1} + D'$$

for scalars $c, c' \in \mathbf{k}$, and derivations D and D' which do not involve any of $\partial_{1,j}$, $\partial_{k,n+1-j}$, and $\partial_{1,1}$. Compute the commutator $[X_{1,j}, X_{k,n+1-j}]$ in two ways: On the one hand, a direct computation with the derivations gives the first equality in

$$(c'-c)X_{1,1} = [X_{1,j}, X_{k,n+1-j}] = c'' X_{1,1}$$

where $X_{1,1} = \partial_{1,1}$ is the derivation dual to the root subgroup with coordinate $x_{1,1}$. On the other hand, $\mathfrak{u}$ is a subalgebra of $\mathfrak{so}_n$, so the commutator must be of the form $c''X_{1,1}$ where the coefficient is a multiple of Chevalley's structure constant indexed by the weights of $x_{1,j}$ and $x_{k,n+1-j}$. Consulting [Bou05, Ch. VIII §2.4, Prop. 7], for example, shows that all structure constants are ± 1 except in the case $n = 2m+1$ and $j = n-m = m+1$, wherein the structure constants between $x_{1,m+1}$ and $x_{k,m+1}$ is ± 2. Since we treat the odd n case in characteristic $p = 2$ separately in 4.10, we have $c'' \neq 0$. A direct calculation now shows that

$$\partial_{1,j}\partial_{k,n+1-j}\tau = \begin{cases} X_{1,j}\cdot \partial_{k,n+1-j}\tau & \text{if } c = 0,\\ X_{k,n+1-j}\cdot\partial_{1,j}\tau & \text{if } c' = 0, \text{ and}\\ (c'X_{1,j}\cdot\partial_{k,n+1,j}\tau - cX_{k,n+1-j}\cdot\partial_{1,j}\tau)/c'' & \text{if } cc' \neq 0.\end{cases}$$

In all cases, this shows that $\partial_{i,j}\partial_{k,n+1-j}\tau$ is in the image of res'.

Step 3. Finally, to construct $\partial_{1,1}\tau$, let B' be the $k\times(n-3)$ matrix obtained from B by moving the $x_{i,1}$ column to the k-th position. Computing as in step 1 together with weight considerations give

$$\frac{(\mathrm{roll}\det B')^{p-1}}{\det B'_{1,\dots,k}} = \pm\partial_{1,1}\tau + \sum_{j=k+1}^{n-k} c_j\cdot\partial_{1,j}\partial_{k,n+1-j}\tau.$$

for some scalars $c_1,\dots,c_{n-k}\in\mathbf{k}$ which are typically not all zero. Step 2 shows that each term in the second summand already lies in the image of res', whence this implies $\partial_{1,1}\tau$ does too.

4.10. — To complete the proof of 4.1 for orthogonal Grassmannian other than the spinor varieties, it remains to consider the odd orthogonal Grassmannian $X = \mathbf{OGr}(k, 2m+1)$ with $1 \leq k \leq m-2$ in characteristic $p = 2$. Each of the steps in 4.9 requires a slight modification:

Step 1′. The standard representation $\mathrm{Std}_{\mathbf{SO}_{2m+1-2k}}$ of the odd orthogonal groups is not simple: $\mathbf{SO}_{2m+1-2k}$ preserves the 1-dimensional kernel of the polar form b_q associated with the quadratic form q; see 5.1 for details. Thus there is a short exact sequence

$$0 \to \mathrm{Std}_{\mathbf{GL}_k} \otimes \operatorname{rad} b_q \to \mathrm{Std}_{\mathbf{GL}_k} \otimes \mathrm{Std}_{\mathbf{SO}_{2m+1-2k}} \to \mathrm{Std}_{\mathbf{GL}_k} \otimes \mathrm{Std}_{\mathbf{SO}_{2m+1-2k}} / \operatorname{rad} b_q \to 0$$

where the bilinear radical on the left is the 1-dimensional trivial representation for $\mathbf{SO}_{2m+1-2k}$ spanned by the vector dual to the coordinate x_{m+1}. This implies that when $k \neq m-1$, the element $\partial_{1,k+2}\tau$ realized in step 1 of 4.9 lies in the quotient above; a section of $\omega_X^{\otimes 1-p} \otimes \mathscr{O}_X(-p)$ realizing $\partial_{1,m+1}\tau$ in the subrepresentation may then be constructed as usual from the $(p-1)$-power rolling determinant of the matrix obtained from B by moving the $x_{i,m+1}$ column to the k-th position. Since the sub- and quotient representations are individually simple for $\mathbf{L}$, this produces each of the elements $\partial_{i,j}\tau$ for $1 \leq i \leq k$ and $k+1 \leq j \leq n-k$.

Step 2′. The argument in step 2 of 4.9 still shows that the elements $\partial_{1,j}\partial_{k,n+1-j}\tau$ lie in the image of res' for $k+1 \leq j \leq n-k$, except maybe for $\partial_{1,m+1}\partial_{k,m+1}\tau$. In fact, a more careful calculation shows that, although $c'' = 0$ in this last case, $cc' = 0$ and so the argument goes through; rather than explaining this, we instead do a more careful calculation in the next step.

Step 3′. To obtain $\partial_{1,1}\tau$ in this case, the divided rolling determinant nonetheless gives

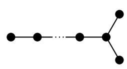
$$\beta' := \prod\nolimits_{\ell=2}^{2m-k-1} \det B'_{\ell,\dots,\ell+k-1} = \partial_{1,1}\tau + \sum\nolimits_{j=k+1}^{2m+1-k} c_j \cdot \partial_{1,j}\partial_{k,2m+2-j}\tau.$$

It suffices to show that $c_{m+1} = 0$, at which point the argument of step 3 from 4.9 yields the result. Applying 4.4 shows that β' has total degree

$$\deg \beta' = k(n-2k-2) + k(k+1)/2 - 1 = \dim X - k - 1$$

in the entries of B'. Since B' contains only $\dim X - 2k$ root coordinates of X, every term in β' contains at least $k-1$ of the $x_{j,k+1-i}$ with $1 \leq i \leq j \leq k$ on or below the anti-diagonal of $M_{1,\dots,k}$. The equations of 4.8 show that the $x_{j,k+1-i}$ are quadratic in the root coordinates, so degree considerations imply that each term of β' contains either k or $k-1$ contributions from these quadratic terms; moreover, in the former case, all nonzero terms are proportional to $\partial_{1,1}\tau$. In other words, each of degree $\dim X - 2$ terms in β' are obtained from τ by omitting a quadratic term appearing in the lower anti-triangular variables $x_{j,k+1-i}$. Examining the equations in 4.8 shows that the variables in the $m+1$ column either comes as a square $x_{i,m+1}^2$ or else with an even coefficient as $2x_{i,m+1}x_{j,m+1}$; in both cases, these vanish in $\mathbf{k}[\mathbf{U}_1]$. Thus the $x_{i,m+1}$ cannot be omitted in the rolling determinant above, so $c_{m+1} = 0$. This completes the proof of 4.1 for $X = \mathbf{OGr}(k,n)$ with $1 \leq k < \lfloor \frac{n}{2} \rfloor$.

4.11. Spinor variety. — Consider the orthogonal Grassmannian corresponding to a maximal parabolic indexed by one of the right-most vertices in the Dynkin diagrams

B_{m-1}: and D_m: 

these are the *spinor varieties*, all of which are isomorphic:

$$X := \mathbf{OGr}(m-1, 2m-1) = \mathbf{OGr}_+(m, 2m) = \mathbf{OGr}_-(m, 2m).$$

Geometrically, we view the spinor variety as parameterizing totally isotropic subspaces of dimension m for a quadratic form q of dimension $n = 2m$. The matrix M of local coordinates from 4.2 on the Schubert cell provided by $\mathbf{U}$ is therefore of size $m \times 2m$, and the equations in 4.8 simply say that the

$$\begin{pmatrix} 0 & y_{1,2} & y_{1,3} & y_{1,4} & y_{1,5} & y_{1,6} \\ & 0 & y_{2,3} & y_{2,4} & y_{2,5} & y_{2,6} \\ & & 0 & y_{3,4} & y_{3,5} & y_{3,6} \\ & & & 0 & y_{4,5} & y_{4,6} \\ & & & & 0 & y_{5,6} \\ & & & & & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & y_{1,2} & y_{1,3} & y_{1,4} & y_{1,5} & y_{1,6} & y_{1,7} \\ & 0 & y_{2,3} & y_{2,4} & y_{2,5} & y_{2,6} & y_{2,7} \\ & & 0 & y_{3,4} & y_{3,5} & y_{3,6} & y_{3,7} \\ & & & 0 & y_{4,5} & y_{4,6} & y_{4,7} \\ & & & & 0 & y_{5,6} & y_{5,7} \\ & & & & & 0 & y_{6,7} \\ & & & & & & 0 \end{pmatrix}$$

FIGURE 1. The matrices appearing in the rolling Pfaffian are the smallest anti-symmetric principal submatrices that contain the anti-diagonals starting from the top and right border of the matrix D, as is illustrated in the cases $m = 6$ and $m = 7$.

initial $M_{1,\dots,m}$ is anti-symmetric along its anti-diagonal; in what follows, it is slightly more convenient to reorder the coordinates and work with the anti-symmetric conjugate

$$D := \begin{pmatrix} 0 & x_{1,m-1} & \cdots & x_{1,2} & x_{1,1} \\ & 0 & \cdots & x_{2,2} & x_{2,1} \\ & & \ddots & \vdots & \vdots \\ & & & 0 & x_{m-1,1} \\ & & & & 0 \end{pmatrix} =: \begin{pmatrix} 0 & y_{1,2} & \cdots & y_{1,m-1} & y_{1,m} \\ & 0 & \cdots & y_{2,m-1} & y_{2,m} \\ & & \ddots & \vdots & \vdots \\ & & & 0 & y_{m-1,m} \\ & & & & 0 \end{pmatrix}$$

where we display only the upper triangular half; for the computation below, it will also be convenient to rename the variables $x_{i,m+1-j}$ to the $y_{i,j}$ which are indexed with respect to their usual position in D. A Levi factor is $\mathbf{L} = \mathbf{GL}_m$ and its adjoint representation on $\mathfrak{u} \cong \wedge^2 \mathrm{Std}_{\mathbf{GL}_m}$ is irreducible, being the alternating square of the standard representation arising from its action on alternating matrices.

Global sections of the ample generator $\mathcal{O}_X(1) \in \operatorname{Pic} X$ are now spanned by *Pfaffians* of principal anti-symmetric submatrices of D; see [FP98, Appendix D] for more on Pfaffians. That the restriction of the Plücker line bundle satisfies $\mathcal{L} \cong \mathcal{O}_X(2)$ reflects the identity $\det D = (\operatorname{pf} D)^2$. The canonical bundle computation of 4.8 means here that we are to construct the trace element τ as a polynomial of degree $(p-1)(2m-3)$ in the Pfaffians of anti-symmetric submatrices of D.

To describe the construction, use the following notation: For integers $1 \leq i \leq j \leq m$, write $[i,j]$ for the set of integers between i and j; for a subset $I \subseteq [1,m]$, let D_I denote the principal anti-symmetric submatrix of D consisting of those rows and columns indexed by I. Define the *rolling Pfaffian* of D depending on the parity of m. If $m = 2k$ is even, let

$$\operatorname{roll}\operatorname{pf} D := \Big(\prod\nolimits_{s=1}^{k-1} \operatorname{pf} D_{[1,2s]} \cdot \operatorname{pf} D_{[1,2s+1]\setminus(s+1)}\Big) \cdot \operatorname{pf} D \cdot \Big(\prod\nolimits_{s=1}^{k-1} \operatorname{pf} D_{[2s,2k]\setminus(k+s)} \cdot \operatorname{pf} D_{[2s+1,2k]}\Big).$$

In the case $m = 2k+1$ is odd, define

$$\begin{aligned} \operatorname{roll}\operatorname{pf} D := \Big(\prod\nolimits_{s=1}^{k-1} \operatorname{pf} D_{[1,2s]} \cdot \operatorname{pf} D_{[1,2s+1]\setminus(s+1)}\Big) \cdot \operatorname{pf} D_{[1,2k]} \\ \cdot \operatorname{pf} D_{[1,2k+1]\setminus(k+1)} \cdot \operatorname{pf} D_{[2,2k+1]} \cdot \Big(\prod\nolimits_{s=1}^{k-1} \operatorname{pf} D_{[2s,2k+1]} \cdot \operatorname{pf} D_{[2s+1,2k+1]\setminus(k+s+1)}\Big). \end{aligned}$$

The matrix appearing in the t-th term of the rolling Pfaffian is the smallest principal submatrix of D containing the anti-diagonal starting from $y_{1,t+1}$ for $1 \leq t \leq m-1$ and $y_{t+2-m,m}$ for $m \leq t \leq 2m-3$; see Figure 1. The $(p-1)$-power rolling Pfaffian of D realizes the trace element:

4.12. Lemma. — $(\operatorname{roll}\operatorname{pf} D)^{p-1} = \Big(\prod\nolimits_{i=1}^{m-1} \prod\nolimits_{j=1}^{m-i} x_{i,j}\Big)^{p-1} \in \mathbf{k}[\mathbf{U}_1].$

This may be established with an elementary inductive argument. As with 4.4, the main observation being that each successive term in the product is the Pfaffian of an anti-symmetric matrix which is triangular along its lower anti-diagonal. For example, if $m \geq 5$, the first four terms appearing in the $(p-1)$-power rolling Pfaffian give, in the ring $\mathbf{k}[y_{i,j} : 1 \leq i < j \leq m]/(y_{i,j}^p : 1 \leq i < j \leq m)$,

$$\begin{vmatrix} 0 & y_{1,2} \\ & 0 \end{vmatrix}^{p-1} \begin{vmatrix} 0 & y_{1,3} \\ & 0 \end{vmatrix}^{p-1} \begin{vmatrix} 0 & y_{1,2} & y_{1,3} & y_{1,4} \\ & 0 & y_{2,3} & y_{2,4} \\ & & 0 & y_{3,4} \\ & & & 0 \end{vmatrix}^{p-1} \begin{vmatrix} 0 & y_{1,2} & y_{1,4} & y_{1,5} \\ & 0 & y_{2,4} & y_{2,5} \\ & & 0 & y_{4,5} \\ & & & 0 \end{vmatrix}^{p-1}$$

$$= (y_{1,2}y_{1,3})^{p-1} \begin{vmatrix} 0 & 0 & 0 & y_{1,4} \\ & 0 & y_{2,3} & y_{2,4} \\ & & 0 & y_{3,4} \\ & & & 0 \end{vmatrix}^{p-1} \begin{vmatrix} 0 & 0 & y_{1,4} & y_{1,5} \\ & 0 & y_{2,4} & y_{2,5} \\ & & 0 & y_{4,5} \\ & & & 0 \end{vmatrix}^{p-1} = (y_{1,2}y_{1,3}y_{1,4}y_{2,3}y_{1,5}y_{2,4})^{p-1}$$

where the vertical bars here denote the Pfaffian of the corresponding anti-symmetric matrix. We leave the proof of 4.12 to the reader.

To conclude, observe that $(\operatorname{roll}\operatorname{pf} D)^{p-1}$ defines a section of $\mathcal{O}_X((p-1)(2m-3))$ and is divisible by the Pfaffian $x_{1,m-2}$ of the 2×2 anti-symmetric submatrix $D_{[1,2]}$. Since the weight of this variable is indecomposable in $\Phi_{\mathbf{P}}^+$, the expression $(\operatorname{roll}\operatorname{pf} D)^{p-1} / \operatorname{pf} D_{[1,2]}$ provides a section of $\omega_X^{\otimes 1-p} \otimes \mathcal{O}_X(-p) \cong \mathcal{O}_X((p-1)(2m-3)-1)$ realizing $\partial_{1,m-1}\tau$ in $\mathbf{k}[\mathbf{U}_1]$. Since $\mathfrak{u} \cong \wedge^2 \mathrm{Std}_{\mathbf{GL}_m}$ is an irreducible representation for $\mathbf{L} \cong \mathbf{GL}_m$, all the sections $\partial_{i,j}\tau$ are in the image of the restriction map, and so the conclusion follows from 3.7. This completes the proof of 4.1 in types B and D. ■

4.13. Example. — When $k = 1$, the orthogonal Grassmannian $X = \mathbf{OGr}(1,n)$ is but a smooth quadric hypersurface in $\mathbf{P}^{n-1}$. Thus 4.1 says that $\mathcal{E}_X(-1)$ is globally generated for smooth quadric hypersurfaces with $\dim X \geq 3$, except maybe in the case $\dim X = 3$ and $p = 2$. And indeed, $\mathcal{E}_X(-1)$ is not globally generated in the latter case: This follows, for instance, from the discussion of 3.4. More directly, observe that the restriction map of 3.5 in this case is a linear map of the form

$$\mathrm{res}' : \mathrm{H}^0(X, \mathcal{O}_X(1)) \to \mathbf{k}[\mathbf{U}_1]_{<\tau}.$$

This cannot be surjective since, the source has dimension 5 whereas the target has dimension 7.

4.14. Type G_2. — Let $\mathbf{G}$ be a semisimple algebraic group with Dynkin diagram of type G_2. Perhaps the most conceptual description of $\mathbf{G}$, due to Cartan in 1914, is as the automorphism group of an octonion algebra: see [SV00, Theorem 2.3.5] for a modern algebraic proof. A more suitable realization of $\mathbf{G}$ for our purposes, due to Engels in 1900, is as the stabilizer of a generic 3-form on a 7-dimensional $\mathbf{k}$-vector space V; a convenient choice of 3-form, found in [And11, Eq. (3.13)], is

$$\gamma := \mathrm{d}x_1 \wedge \mathrm{d}x_4 \wedge \mathrm{d}x_7 + \mathrm{d}x_2 \wedge \mathrm{d}x_4 \wedge \mathrm{d}x_6 + \mathrm{d}x_3 \wedge \mathrm{d}x_4 \wedge \mathrm{d}x_5 - \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_7 - \mathrm{d}x_1 \wedge \mathrm{d}x_5 \wedge \mathrm{d}x_6.$$

The simple roots α_1 and α_2 of $\mathbf{G}$ have different lengths. The homogeneous space for the maximal parabolic in $\mathbf{G}$ omitting the short root α_1 is isomorphic to a 5-dimensional quadric; omitting the long root α_2 instead gives the G_2-Grassmannian X, which may be described as the closed subvariety of $\mathbf{Gr}(2,V)$ parameterizing 2-dimensional spaces that are isotropic for γ:

$$X := \{[U] \in \mathbf{Gr}(2,V) : \gamma(u,v,-) = 0 \text{ where } U = \operatorname{span}\{u,v\}\}.$$

With the choice of γ, the Schubert cell provided by the opposite unipotent radical $\mathbf{U}$ may be described, as is done in [And09, Appendix D], in terms of the matrix

$$M := \begin{pmatrix} x_{\alpha_6} & -(x_{\alpha_3}x_{\alpha_5} + x_{\alpha_4}^2) & x_{\alpha_3} & x_{\alpha_4} & x_{\alpha_5} & 1 & 0 \\ x_{\alpha_2}x_{\alpha_4} - x_{\alpha_3}^2 & -(x_{\alpha_6} + x_{\alpha_3}x_{\alpha_4} + x_{\alpha_2}x_{\alpha_5}) & x_{\alpha_2} & x_{\alpha_3} & -x_{\alpha_4} & 0 & 1 \end{pmatrix}$$

where the coordinates are indexed by their weights α_2, $\alpha_3 := \alpha_1+\alpha_2$, $\alpha_4 := 2\alpha_1+\alpha_2$, $\alpha_5 := 3\alpha_1+\alpha_2$, and $\alpha_6 := 3\alpha_1+2\alpha_2$; the weights may be deduced by comparing the relations amongst the roots in type G_2 with the quadratic entries in M. A Levi factor is the group $\mathbf{L} \cong \mathbf{GL}_2$ corresponding to the root subsystem generated by α_1 and so, at least when $p \geq 5$, inspecting weights shows that the level decomposition of the adjoint representation of $\mathbf{L}$ on $\mathfrak{u}$ decomposes as

$$\mathfrak{u} \cong \mathrm{Sym}^3 \mathrm{Std}_{\mathbf{GL}_2} \otimes \mathbf{k}_{-\alpha_2} \oplus \mathbf{k}_{-\alpha_6}.$$

Globally, X may be realized as a set-theoretic linear section of the Grassmannian, given by the non-transverse intersection $\mathbf{Gr}(2,V) \cap \mathbf{P}\mathfrak{g}$, where the Lie algebra $\mathfrak{g}$ of $\mathbf{G}$ canonically embeds into the ambient Plücker space via the splitting $\wedge^2 V \cong V \oplus \mathfrak{g}$ as $\mathbf{G}$-representation: see [FH91, §22.3] for this latter fact. Better, the 3-form γ induces a regular morphism of vector bundles $\wedge^2\mathscr{S} \to \mathscr{Q}^\vee$ on $\mathbf{Gr}(2,V)$, where $\mathscr{S}$ and $\mathscr{Q}$ are the tautological sub- and quotient bundles of ranks 2 and 5, respectively, whose vanishing locus is precisely X. The canonical bundle of X is therefore

$$\omega_X \cong \omega_{\mathbf{Gr}(2,7)} \otimes \det \mathscr{Q}^\vee(1)|_X \cong \mathscr{O}_X(-3).$$

4.15. — To prove 4.1 for X, we must construct the elements $\partial_{\alpha_i}\tau$ as polynomials of degree $2p-3$ in the minors of M. We first construct the following element in the first irreducible summand of $\mathfrak{u}\cdot\tau$:

$$-\partial_{\alpha_3}\tau = x_{\alpha_3}^{p-2}(x_{\alpha_2}x_{\alpha_4}x_{\alpha_5}x_{\alpha_6})^{p-1} \in \mathbf{k}[\mathbf{U}_1] = \mathbf{k}[x_{\alpha_2},\ldots,x_{\alpha_6}]/(x_{\alpha_2}^p,\ldots,x_{\alpha_6}^p).$$

Since $\alpha_3 = \alpha_1+\alpha_2$ is indecomposable amongst the set of roots $\Phi^+_{\mathbf{P}} = \{\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6\}$, the displayed monomial spans the $\lambda-\alpha_3 = 3(p-1)(3\alpha_1+2\alpha_2)-(\alpha_1+\alpha_2)$ weight space in $\mathbf{k}[\mathbf{U}_1]$. This already implies that the following weight $\lambda-\alpha_3$ section of $\mathscr{O}_X(2p-3)$ is proportional to $\partial_{\alpha_3}\tau$:

$$(\det M_{1,7})^{p-3}(\det M_{1,5})(\det M_{1,2})^{p-1} = c x_{\alpha_3}^{p-2}(x_{\alpha_2}x_{\alpha_4}x_{\alpha_5}x_{\alpha_6})^{p-1} \in \mathbf{k}[\mathbf{U}_1].$$

Thus the task is to show that the scalar $c \in \mathbf{k}$ is nonzero. Write the first $p-2$ factors as

$$(\det M_{1,7})^{p-3}(\det M_{1,5}) = -x_{\alpha_4}x_{\alpha_6}^{p-2} - x_{\alpha_5}x_{\alpha_6}^{p-3}(x_{\alpha_2}x_{\alpha_4}-x_{\alpha_3}^2)$$

and separately compute the product of the two summands with

$$(\det M_{1,2})^{p-1} = \big(-x_{\alpha_6}(x_{\alpha_6}+x_{\alpha_3}x_{\alpha_4}+x_{\alpha_2}x_{\alpha_5}) + (x_{\alpha_3}x_{\alpha_5}+x_{\alpha_4}^2)(x_{\alpha_2}x_{\alpha_4}-x_{\alpha_3}^2)\big)^{p-1}.$$

In the product with $-x_{\alpha_4}x_{\alpha_6}^{p-2}$, observe that only terms in $(\det M_{1,2})^{p-1}$ which are linear in x_{α_6} may contribute, and so this gives the first line of

$$\begin{aligned}-x_{\alpha_4}x_{\alpha_6}^{p-2}(\det M_{1,2})^{p-1} &= -x_{\alpha_4}x_{\alpha_6}^{p-1}(x_{\alpha_3}x_{\alpha_4}+x_{\alpha_2}x_{\alpha_5})(x_{\alpha_3}x_{\alpha_5}+x_{\alpha_4}^2)^{p-2}(-x_{\alpha_3}^2+x_{\alpha_2}x_{\alpha_4})^{p-2} \\ &= -x_{\alpha_3}^{p-2}(x_{\alpha_2}x_{\alpha_4}x_{\alpha_5}x_{\alpha_6})^{p-1}.\end{aligned}$$

Examining powers of x_{α_2} shows that the nonzero contributions from the three binomials must involve the $x_{\alpha_2}x_{\alpha_5}$ in the first binomial and the $(p-2)$-nd power of $x_{\alpha_2}x_{\alpha_4}$ in the third binomial. With this, the second equality follows.

In the product with $-x_{\alpha_5}x_{\alpha_6}^{p-3}(x_{\alpha_2}x_{\alpha_4}-x_{\alpha_3}^2)$, only terms in $(\det M_{1,2})^{p-1}$ which are quadratic in x_{α_6} may contribute, and so this gives

$$\begin{aligned}-x_{\alpha_5}x_{\alpha_6}^{p-3}(x_{\alpha_2}x_{\alpha_4}-x_{\alpha_3}^2)\Big[&x_{\alpha_6}^2(x_{\alpha_3}x_{\alpha_5}+x_{\alpha_4}^2)^{p-2}(x_{\alpha_2}x_{\alpha_4}-x_{\alpha_3}^2)^{p-2} \\ &+ x_{\alpha_6}^2(x_{\alpha_3}x_{\alpha_4}+x_{\alpha_2}x_{\alpha_5})^2(x_{\alpha_3}x_{\alpha_5}+x_{\alpha_4}^2)^{p-3}(x_{\alpha_2}x_{\alpha_4}-x_{\alpha_3}^2)^{p-3}\Big].\end{aligned}$$

For the first summand in the brackets, examining powers of x_{α_2} shows that it contributes a coefficient of -1 to the result. For the second summand, observe that $(x_{\alpha_3}x_{\alpha_4}+x_{\alpha_2}x_{\alpha_5})^2$ must contribute either a linear or quadratic term in x_{α_5}, and so its product with the initial factor outside the brackets is

$$-x_{\alpha_5}x_{\alpha_6}^{p-1}(x_{\alpha_2}x_{\alpha_4}-x_{\alpha_3}^2)^{p-2}(2x_{\alpha_2}x_{\alpha_3}^{p-2}x_{\alpha_4}x_{\alpha_5}^{p-2} - 3x_{\alpha_2}^2x_{\alpha_3}^{p-4}x_{\alpha_4}^2x_{\alpha_5}^{p-2}) = 4x_{\alpha_3}^{p-2}(x_{\alpha_2}x_{\alpha_4}x_{\alpha_5}x_{\alpha_6})^{p-1}.$$

In sum, this gives $-x_{\alpha_5}x_{\alpha_6}^{p-3}(x_{\alpha_2}x_{\alpha_4}-x_{\alpha_3}^2)(\det M_{1,2})^{p-1} = 3x_{\alpha_3}^{p-2}(x_{\alpha_2}x_{\alpha_4}x_{\alpha_5}x_{\alpha_6})^{p-1}$ and so that $c=-1+3=2$, which is nonzero since $p\geq 5$.

It remains to construct the basis element $-\partial_{\alpha_6}\tau = (x_{\alpha_2}x_{\alpha_3}x_{\alpha_4}x_{\alpha_5})^{p-1}x_{\alpha_6}^{p-2}\in \mathbf{k}[\mathbf{U}_1]$. Since $\alpha_6 = \alpha_2+\alpha_5=\alpha_3+\alpha_4$, the $\lambda-\alpha_6$ weight space of $\mathbf{k}[\mathbf{U}_1]$ is 3-dimensional, and also contains the monomials $\partial_{\alpha_2}\partial_{\alpha_5}\tau$ and $\partial_{\alpha_3}\partial_{\alpha_4}\tau$. A straightforward computation shows that

$$\begin{aligned}(\det M_{1,7})^{p-2}(\det M_{1,2})^{p-1} &= (x_{\alpha_2}x_{\alpha_3}x_{\alpha_4}x_{\alpha_5})^{p-1}x_{\alpha_6}^{p-2} \\ &\quad + (x_{\alpha_2}x_{\alpha_5}x_{\alpha_6})^{p-1}(x_{\alpha_3}x_{\alpha_4})^{p-2} - 3(x_{\alpha_3}x_{\alpha_4}x_{\alpha_6})^{p-1}(x_{\alpha_2}x_{\alpha_5})^{p-2}.\end{aligned}$$

To conclude, we show that $\partial_{\alpha_2}\partial_{\alpha_5}\tau$ and $\partial_{\alpha_3}\partial_{\alpha_4}\tau$ also lie in the image of the restriction map res' using the action the Lie algebra $\mathfrak{u}$. Let $X_{\alpha_6}=\partial_{\alpha_6}$ be the Lie algebra element dual to x_{α_6}, and write

$$X_{\alpha_2} = \partial_{\alpha_2} + cx_{\alpha_5}\partial_{\alpha_6},\quad X_{\alpha_3} = \partial_{\alpha_3}+dx_{\alpha_4}\partial_{\alpha_6},\quad X_{\alpha_4} = \partial_{\alpha_4}+d'x_{\alpha_3}\partial_{\alpha_6},\quad X_{\alpha_5}=\partial_{\alpha_5}+c'x_{\alpha_2}\partial_{\alpha_6},$$

for those corresponding to the root coordinates x_{α_2}, x_{α_3}, x_{α_4}, and x_{α_5}, respectively. Directly computing commutators of the derivations shows that

$$[X_{\alpha_2},X_{\alpha_5}] = (c'-c)X_{\alpha_6} \text{ and } [X_{\alpha_3},X_{\alpha_4}] = (d'-d)X_{\alpha_6}.$$

Examining the structure constants in type G_2 shows that $c'-c$ is a nonzero multiple of ±1, whereas $d'-d$ is a nonzero multiple of ±3. Since the characteristic p is ≥ 5, both are nonzero. Arguing as in step 2 of 4.9 now shows that $\partial_{\alpha_2}\partial_{\alpha_5}\tau$ and $\partial_{\alpha_3}\partial_{\alpha_4}\tau$ lie in the image of the restriction map. Combined with the determinant computation above, this implies that $\partial_{\alpha_6}\tau$ also lies in the image of res'. This completes the proof of 4.1 in the case of the G_2-Grassmannian. ■

5. Exceptions

The exceptions in 4.1 are truly exceptions: namely, let X be either the symplectic Grassmannian $\mathbf{SGr}(k,2m)$ for $2\leq k\leq m$ with $p=2$; or the odd orthogonal Grassmannian $\mathbf{OGr}(m-1,2m+1)$ with $p=2$; or the G_2-Grassmannian with $p=2$ or $p=3$. In all of these cases, $\mathcal{E}_X$ is *not* ample: see 5.3, 5.6, and 5.9, respectively. In most cases, non-ampleness is explained by the existence of an exotic isogeny between the relevant algebraic groups: between $\mathbf{SO}_{2m+1}$ and $\mathbf{Sp}_{2m}$ in characteristic $p=2$ for the first two cases, and for a group of type G_2 with itself in characteristic $p=3$.

5.1. Orthogonal and symplectic Grassmannian. — Suppose $\mathbf{k}$ has characteristic $p=2$, let V be a $\mathbf{k}$-vector space of dimension $2m+1$, and let $q\in \mathrm{Sym}^2 V^\vee$ be a nonsingular quadratic form, by which we mean that the associated quadric hypersurface $Q:=\mathrm{V}(q)\subset \mathbf{P}V$ is smooth. Since $-1=1$ in $\mathbf{k}$, the polar form $b_q\colon V\times V\to\mathbf{k}$ associated with q, that is, the symmetric bilinear form defined by

$$b_q(u,v) := q(u+v)-q(u)-q(v) \text{ for } u,v\in V,$$

is also alternating, and hence has even rank; its *radical*

$$\mathrm{rad}\, b_q := \{v\in V : b_q(u,v)=0 \text{ for all } u\in V\}$$

is therefore nonzero and nonsingularity of q implies that it is 1-dimensional, and b_q induces a nondegenerate symplectic form ω on the $2m$-dimensional quotient $W:=V/\mathrm{rad}\, b_q$. The representation of $\mathbf{SO}(V,q)$ on W through the quotient map $\pi\colon V\to W$ preserves ω, yielding a purely inseparable isogeny $\mathbf{SO}(V,q)\to\mathbf{Sp}(W,\omega)$ of height 1 and degree 2^{2m}: see [Bor91, §23.5] for more details. This induces morphisms of the corresponding homogeneous spaces, the simplest being the projection

$$Q := \{(x_0:\cdots:x_{2m})\in\mathbf{P}^{2m} : x_0x_1+\cdots+x_{2m-2}x_{2m-1}+x_{2m}^2=0\}\to\mathbf{P}^{2m-1}$$

from the strange point $(0:\cdots:0:1)$ of a smooth quadric in $\mathbf{P}^{2m}$ to the $\mathbf{P}^{2m-1}$ of the first $2m$ coordinates. More generally and geometrically, we have:

5.2. Lemma. — *The image under $\pi\colon V \to W$ of a k-dimensional q-isotropic subspace $U \subset V$ is a k-dimensional ω-isotropic subspace $\pi(U) \subset W$. The assignment $U \mapsto \pi(U)$ defines a $\mathbf{k}$-morphism*

$$\varphi\colon \mathbf{OGr}(k,V,q) \to \mathbf{SGr}(k,W,\omega)$$

which is purely inseparable of height 1, finite of degree 2^k, and satisfies $\varphi^*\mathcal{O}_{\mathbf{SGr}(k,W,\omega)}(1) \cong \mathcal{O}_{\mathbf{OGr}(k,V,q)}(1)$.

Proof. Nonsingularity of q implies that q-isotropic subspaces $U \subset V$ do not contain $\operatorname{rad} b_q$. Combined with the fact that ω is induced by b_q, it follows that $\pi\colon V \to W$ sends q-isotropic subspaces to ω-isotropic ones of the same dimension. Applying π to the universal subbundle on $\mathbf{OGr}(k,V,q)$ induces the morphism $\varphi\colon \mathbf{OGr}(k,V,q) \to \mathbf{SGr}(k,W,\omega)$ of schemes over $\mathbf{k}$; moreover, since φ is induced by linear projection, it respects the Plücker line bundles.

It remains to show that φ is finite, purely inseparable of height 1, and of degree 2^k. Consider a $\mathbf{k}$-point of $\mathbf{SGr}(k,W,\omega)$ corresponding to a k-dimensional ω-isotropic $\bar{U} \subset W$, and consider its scheme-theoretic inverse image $\varphi^{-1}([\bar{U}])$. Its $\mathbf{k}$-points are given by q-isotropic subspaces in the $(k+1)$-dimensional space $\pi^{-1}(\bar{U}) \subset V$. Since this contains $\operatorname{rad} b_q$ and since $\bar{U}$ is isotropic for the symplectic form ω induced by b_q, the restriction of q thereon has rank 1, and so it contains a unique k-dimensional subspace U such that $\pi(U) = \bar{U}$. This already shows that φ is a bijection on $\mathbf{k}$-points, whence finite and purely inseparable: see [Stacks, 01S4].

To understand the schematic structure of $\varphi^{-1}([\bar{U}])$, it suffices to consider its points valued in Artinian local $\mathbf{k}$-algebras A: these are given by free rank k A-submodules of $\pi^{-1}(\bar{U}) \otimes_{\mathbf{k}} A$ which are q-isotropic and which restrict to the subspace U upon reduction modulo the maximal ideal. We may choose a basis $\pi^{-1}(\bar{U}) \otimes_{\mathbf{k}} A \cong Au_1 \oplus \cdots \oplus Au_k \oplus Av$ such that the restriction of q is the quadratic form y^2, the square of the coordinate dual to v. With this, it is straightforward to see that

$$\varphi^{-1}([\bar{U}])(A) \cong \{A(u_1 + a_1 v) \oplus \cdots \oplus A(u_k + a_k v) : a_1, \ldots, a_k \in A \text{ such that } a_i^2 = 0\}.$$

This implies that $\varphi^{-1}([U]) \cong \operatorname{Spec} \mathbf{k}[\alpha_1, \ldots, \alpha_k]/(\alpha_1^2, \ldots, \alpha_k^2)$, from which the result follows. ■

5.3. Proposition. — *If $p = 2$, then $\mathcal{E}_X$ is not ample for any $X := \mathbf{SGr}(k,2m)$ with $2 \leq k \leq m$.*

Proof. Set $Y := \mathbf{OGr}(k,2m+1)$ and consider the morphism $\varphi\colon Y \to X$ from 5.2. Its Tschirnhausen bundle $\mathcal{E}_\varphi$ is thus a vector bundle of rank $\deg\varphi - 1 = 2^k - 1$ and a Grothendieck–Riemann–Roch computation using the canonical bundle formulae found in 4.7 and 4.8 shows that

$$\begin{aligned} c_1(\mathcal{E}_\varphi) = -c_1(\varphi_*\mathcal{O}_Y) &= 2^{k-1}c_1(\mathcal{T}_X) - \frac{1}{2}\varphi_* c_1(\mathcal{T}_Y) \\ &= 2^{k-1}(2m+1-k)c_1(\mathcal{O}_X(1)) - \frac{1}{2}(2m-k)\varphi_* c_1(\mathcal{O}_Y(1)) \end{aligned}$$

where $\mathcal{O}_X(1)$ and $\mathcal{O}_Y(1)$ are the Plücker line bundles on the two isotropic Grassmannian. Since φ preserves Plücker line bundles, the projection formula gives

$$\varphi_* c_1(\mathcal{O}_Y(1)) = \varphi_* c_1(\varphi^*\mathcal{O}_X(1)) = \varphi_*\varphi^* c_1(\mathcal{O}_X(1)) = 2^k c_1(\mathcal{O}_X(1)).$$

Therefore $c_1(\mathcal{E}_\varphi) = 2^{k-1}c_1(\mathcal{O}_X(1))$. Now, X contains lines in its Plücker embedding, and the restriction of $\mathcal{E}_\varphi$ to any such is a vector bundle on $\mathbf{P}^1$ whose rank $2^k - 1$ is smaller than its degree 2^{k-1} whenever $k \geq 2$: this is not be ample, hence $\mathcal{E}_\varphi$ is not ample, and thus $\mathcal{E}_X$ is not ample by 1.2. ■

5.4. — Consider the morphism $\psi\colon \mathbf{SGr}(k,W,\omega) \to \mathbf{OGr}(k,V,q)$ dual to φ; that is, the unique finite purely inseparable morphism of height 1 such that $\psi \circ \varphi$ and $\varphi \circ \psi$ are the Frobenius morphisms of $\mathbf{OGr}(k,V,q)$ and $\mathbf{SGr}(k,W,\omega)$, respectively. In particular, combined with the properties of φ from

5.2, this implies that $\psi^*\mathscr{O}_{\mathbf{OGr}(k,V,q)}(1) \cong \mathscr{O}_{\mathbf{SGr}(k,W,\omega)}(2)$ and that $\deg\psi = 2^{N-k}$ where

$$N := \dim \mathbf{OGr}(k,V,q) = \dim \mathbf{SGr}(k,W,\omega) = \frac{1}{2}k(4m-3k+1).$$

A Grothendieck–Riemann–Roch computation as in the proof of 5.3 shows that

$$c_1(\mathscr{E}_\psi) = (2m-k-1)2^{N-k-2}c_1(\mathscr{O}_{\mathbf{OGr}(k,V,q)}(1))$$

In the case $k = m-1$, the restriction of $\mathscr{E}_\psi$ to a line has rank $2^{N-m+1}-1$ and degree $m2^{N-m-1}$, whereby the argument of 5.3 shows that the Frobenius-trace kernel of $\mathbf{OGr}(m-1,V,q) = \mathbf{OGr}(m-1,2m+1)$ is not ample when $m \leq 3$. In fact, 5.6 below shows that the Frobenius-trace kernel is not ample for these odd orthogonal Grassmannian for all $m \geq 2$. The proof relies on finer geometric properties of the morphism ψ which, in terms of foliations as in [Eke87], says that the leaves of the foliation associated with ψ are isomorphic to the even orthogonal Grassmannian $\mathbf{OGr}(k,2m)$:

5.5. Proposition. — *If $p = 2$, there exists, for any $1 \leq k \leq m-1$, a Cartesian commutative diagram*

$$\begin{tikzcd} \mathbf{OGr}(k,2m) \arrow[r] \arrow[d, "F"'] & \mathbf{SGr}(k,2m) \arrow[d, "\psi"] \\ \mathbf{OGr}(k,2m) \arrow[r] & \mathbf{OGr}(k,2m+1) \end{tikzcd}$$

where the horizontal arrows are closed immersions.

Proof. Consider first the case $k = 1$, so that $\mathbf{SGr}(1,2m) = \mathbf{P}^{2m-1}$ and $\mathbf{OGr}(1,2m+1) = Q$ is a smooth quadric in $\mathbf{P}^{2m}$. Choosing coordinates as in 5.1, the morphism $\varphi\colon Q \to \mathbf{P}^{2m-1}$ is linear projection onto the first $2m$ coordinates. The dual morphism $\psi\colon \mathbf{P}^{2m-1} \to Q$ is then given by

$$\psi(x_0 : \cdots : x_{2m-1}) = (x_0^2 : \cdots : x_{2m-1}^2 : x_0x_1 + \cdots + x_{2m-2}x_{2m-1}).$$

Let $H \subset \mathbf{P}^{2m}$ be the hyperplane defined by $x_{2m} = 0$ and consider the hyperplane section $Q' := Q \cap H$. Linear projection onto the first $2m$ coordinates provides an isomorphism $H \cong \mathbf{P}^{2m-1}$, whence a closed immersion $Q' \subset \mathbf{P}^{2m-1}$ identifying it as the quadric defined by $x_0x_1 + \cdots + x_{2m-2}x_{2m-1} = 0$. Thus the restriction of $\psi\colon \mathbf{P}^{2m-1} \to Q$ to $Q' \subset \mathbf{P}^{2m-1}$ factors through $H \cap Q = Q'$ and is simply the Frobenius morphism. The degree of both $F\colon Q' \to Q'$ and $\psi\colon \mathbf{P}^{2m-1} \to Q$ is 2^{2m-2}, so a standard argument shows that the commutative diagram

$$\begin{tikzcd} Q' \arrow[r] \arrow[d, "F"'] & \mathbf{P}^{2m-1} \arrow[d, "\psi"] \\ Q' \arrow[r] & Q \end{tikzcd}$$

is Cartesian. Recognizing $Q' = \mathbf{OGr}(1,2m)$ now completes the proof in the case $k = 1$.

For $k \geq 2$, we claim that the morphism dual to $\varphi\colon \mathbf{OGr}(k,2m+1) \to \mathbf{SGr}(k,2m)$ from 5.2 is, in fact, the morphism induced by $\psi\colon \mathbf{P}^{2m-1} \to Q$ given above. Observe first that ψ sends a line $\ell \subset \mathbf{P}^{2m-1}$ isotropic for the symplectic form $\omega = \mathrm{d}x_0 \wedge \mathrm{d}x_1 + \cdots + \mathrm{d}x_{2m-2} \wedge \mathrm{d}x_{2m-1}$ to a line in Q: Given distinct points $x = (x_0 : \cdots : x_{2m-1})$ and $y = (y_0 : \cdots : y_{2m-1})$ on ℓ and any $\lambda, \mu \in \mathbf{k}$, the polar identity applied to the quadratic form $q'(x) := x_0x_1 + \cdots + x_{2m-2}x_{2m-1}$ defining Q' in $\mathbf{P}^{2m-1}$ gives

$$q'(\lambda x + \mu y) = \lambda\mu b_{q'}(x,y) - \lambda^2 q'(x) - \mu^2 q'(y).$$

Since the characteristic of $\mathbf{k}$ is $p = 2$, the polar form $b_{q'}$ associated with q' is also symplectic, and is in fact none other than ω. Since ℓ is ω-isotropic, $b_{q'}(x,y) = \omega(x,y) = 0$ and so

$$\psi(\lambda x + \mu y) = (\lambda^2 x_0^2 + \mu^2 y_0^2 : \cdots : \lambda^2 x_{2m-1}^2 + y_{2m-1}^2 : q'(\lambda x + \mu y)) = \lambda^2\psi(x) + \mu^2\psi(y)$$

meaning that $\psi(\ell) \subset Q$ is a line. This implies that ψ sends ω-isotropic $(k-1)$-planes in $\mathbf{P}^{2m-1}$ to $(k-1)$-planes in Q, and so induces a morphism $\mathbf{SGr}(k,2m) \to \mathbf{OGr}(k,2m+1)$; abusing notation, this morphism will also be denoted by ψ. A direct calculation shows that $\psi \circ \varphi$ and $\varphi \circ \psi$ are the endomorphisms of $\mathbf{OGr}(k,2m+1)$ and $\mathbf{SGr}(k,2m)$ which take a $(k-1)$-plane to itself, albeit squaring all the coordinates in the process: these are the Frobenius maps, verifying that ψ is the dual of φ.

Finally, identify $\mathbf{OGr}(k,2m)$ as the Fano scheme of linear spaces in the quadric $(2m-2)$-fold Q'. Embed this into $\mathbf{OGr}(k,2m+1)$ and $\mathbf{SGr}(k,2m)$ by viewing Q' as, respectively, a hyperplane section of Q and as a quadric in $\mathbf{P}^{2m-1}$ whose polar form $b_{q'}$ is the symplectic form ω. The $k=1$ case shows that the restriction to Q' of $\psi\colon \mathbf{P}^{2m-1} \to Q$ is the Frobenius morphism of Q', so functoriality of Hilbert schemes implies that the restriction of $\psi\colon \mathbf{SGr}(k,2m) \to \mathbf{OGr}(k,2m+1)$ to the compatibly embedded pair of $\mathbf{OGr}(k,2m)$ is also the Frobenius, providing the commutative diagram in the statement. That the diagram is also Cartesian now follows from the fact that

$$\deg(F\colon \mathbf{OGr}(k,2m) \to \mathbf{OGr}(k,2m)) = 2^{\frac{1}{2}k(4m-3k-1)} = \deg(\psi\colon \mathbf{SGr}(k,2m) \to \mathbf{OGr}(k,2m+1)). \qquad \blacksquare$$

5.6. Proposition. — *If $p=2$, then $\mathscr{E}_Y$ is not ample for $Y := \mathbf{OGr}(m-1,2m+1)$.*

Proof. Let $X := \mathbf{SGr}(m-1,2m)$ and let $\psi\colon X \to Y$ be the morphism dual to that from 5.3. By 5.5, there exists embeddings $Z \subset X$ and $Z \subset Y$ of $Z := \mathbf{OGr}(m-1,2m)$ such that ψ restricts to the Frobenius morphism $F\colon Z \to Z$. Since ψ is finite locally free, this implies that

$$\psi_*\mathscr{O}_X|_Z \cong F_*\mathscr{O}_Z$$

yielding an isomorphism $\mathscr{E}_\psi|_Z \cong \mathscr{E}_Z$. However, Z is a homogeneous space of Picard rank 2: In a nonsingular quadratic space (V',q') of dimension $2m$, an isotropic subspace $U \subset V'$ of dimension $m-1$ uniquely determines two m-dimensional isotropic subspaces $U_\pm \subset U^\perp \subset V'$ with the property that $U = U_+ \cap U_-$. The maps $[U] \mapsto [U_\pm]$ then exhibits $Z = \mathbf{OGr}(m-1,2m)$ as a projective bundle over the two spinor varieties $\mathbf{OGr}_\pm(m,2m)$ from 4.11. This implies that $\mathscr{E}_Z$ cannot be ample, for instance by 2.4, whence $\mathscr{E}_\psi$ is not ample. Therefore $\mathscr{E}_Y$ is not ample by 1.2. $\blacksquare$

5.7. Type G_2. — Consider now the case X is the G_2-Grassmannian as in 4.14. Since $\omega_X^{-1} \cong \mathscr{O}_X(3)$, 1.4 implies that $\mathscr{E}_X$ is not ample when $p=2$ since X contains lines for $\mathscr{O}_X(1)$: see [Str02, Lemma 3], whose proof also works in characteristic 2, though see also [CC98] and [LM03, §4.2]. The remaining case is more interesting and pertains to the exotic isogeny $\phi\colon \mathbf{G} \to \mathbf{G}$ of algebraic groups of type G_2 over fields of characteristic $p=3$. Existence proofs using general methods from the theory of reductive groups may be found, for example, in [Ste16, Chapter 10] and [$\mathrm{SGA}_{\mathrm{III}}$, Exposé XXI]. Chevalley's original construction is more explicit and better suited for our purposes; we sketch it below and refer to [Che05, no.21][1] for the details:

Let $\mathbf{G}$ be a semisimple algebraic group of type G_2. The basic observation is that, when $\mathbf{k}$ has characteristic $p=3$, the adjoint representation $\mathfrak{g}$ fits into a nonsplit short exact sequence

$$0 \to V \to \mathfrak{g} \to V' \to 0$$

where V is the 7-dimensional representation of $\mathbf{G}$ whose highest weight is the first fundamental weight $\varpi_1 = \alpha_4 = 2\alpha_1 + \alpha_2$: this may be seen directly by inspecting the multiplication table of the Lie algebra $\mathfrak{g}$—see [FH91, §22] for instance—and observing that the root spaces of weights $\pm\alpha_1$, $\pm\alpha_3$, and $\pm\alpha_4$ generate a p-Lie subalgebra due to vanishing of certain structure constants. Chevalley shows this more conceptually by observing that the Killing form on $\mathfrak{g}$ degenerates when $p=3$ and identifies V as the radical of the form. A useful consequence of this is that the induced bilinear form on V' is nondegenerate and $\mathbf{G}$-equivariant, which may be used to show that V' is also a simple

[1] See the original notes from Chevalley's seminar at `https://www.numdam.org/item/SCC_1956-1958__2_/`.

module for $\mathbf{G}$. This implies that the morphism $\mathbf{G} \to \mathbf{GL}(V')$ is an isogeny onto its image $\mathbf{G}'$. General results on semisimple algebraic groups now imply that $\mathbf{G}'$ is isomorphic to $\mathbf{G}$, yielding the isogeny $\phi : \mathbf{G} \to \mathbf{G}' \cong \mathbf{G}$. Properties of the induced morphisms on associated homogeneous spaces may be obtained from a careful analysis of the isogeny ϕ; instead, we give a direct geometric argument:

5.8. Lemma. — *If $p = 3$, there exists a dual pair of finite purely inseparable morphisms of height 1*

$$\varphi : X \to Q \text{ and } \psi : Q \to X$$

with $\deg \varphi = 9$ and $\deg \psi = 27$, and such that $\varphi^* \mathcal{O}_Q(1) \cong \mathcal{O}_X(1)$ and $\psi^* \mathcal{O}_X(1) \cong \mathcal{O}_Q(3)$.

Proof. To construct the morphism $\varphi : X \to Q$, observe that X naturally embeds into $\mathbf{P}\mathfrak{g}$ and may be identified as the closed orbit of $\mathbf{G}$ acting on a vector of highest weight $\alpha_6 = 3\alpha_1 + 2\alpha_2$. That $\mathfrak{g}$ contains the subrepresentation V means that $\mathbf{P}\mathfrak{g}$ contains another $\mathbf{G}$-orbit whose closure is $\mathbf{P}V$. Since X is not linearly degenerate in $\mathbf{P}\mathfrak{g}$, it must be disjoint from the linear space $\mathbf{P}V$. Linear projection with centre $\mathbf{P}V$ then induces a finite morphism $X \to \mathbf{P}V'$. This map is $\mathbf{G}$-equivariant and so it factors through the closed $\mathbf{G}$-orbit $Q \subset \mathbf{P}V'$, providing the morphism $\varphi : X \to Q$. Since φ is obtained from linear projection, it satisfies $\varphi^* \mathcal{O}_Q(1) \cong \mathcal{O}_X(1)$.

For the remaining properties, explicitly compute φ in terms of the coordinates of the Schubert cell as in 4.14. For each pair $1 \leq i < j \leq 7$, let $\wp_{i,j} := \det M_{i,j}$ be the Plücker coordinate associated with the i-th and j-th columns of the coordinate matrix M. By construction, φ projects onto the Plücker coordinates of weights $\pm\alpha_2$, $\pm\alpha_5$, $\pm\alpha_6$, and one coordinate of weight 0; note that, as in 3.1, the weight of the Plücker coordinates are related to that of the local affine coordinates x_{α_i} by a shift of $-\alpha_6 = -3\alpha_1 - 2\alpha_2$. A computation shows that

$$\wp_{1,2}\wp_{6,7} + \wp_{1,3}\wp_{5,7} + \wp_{2,5}\wp_{3,6} + (\wp_{1,7} + \wp_{3,5})^2 = -3(x_{\alpha_2}x_{\alpha_5}x_{\alpha_6} + x_{\alpha_3}x_{\alpha_4}x_{\alpha_6}) = 0$$

and so the map $\varphi : X \to Q := \{(y_0 : \cdots : y_6) \in \mathbf{P}^6 : y_0y_6 + y_1y_5 + y_2y_4 + y_3^2 = 0\}$ is given by

$$\varphi(x) = (\wp_{1,2}(x) : \wp_{1,3}(x) : \wp_{2,5}(x) : \wp_{1,7}(x) + \wp_{3,5}(x) : \wp_{3,6}(x) : \wp_{5,7}(x) : \wp_{6,7}(x)).$$

In particular, φ restricts to a map of affine open subschemes $X \setminus \mathrm{V}(\wp_{6,7}) \to Q \setminus \mathrm{V}(y_6)$ corresponding to the $\mathbf{k}$-algebra map $\varphi^\# : \mathbf{k}[y_1, \ldots, y_5] \to \mathbf{k}[x_{\alpha_2}, \ldots, x_{\alpha_6}]$ given by $y_4 \mapsto -x_{\alpha_2}$, $y_5 \mapsto x_{\alpha_5}$,

$$y_1 \mapsto x_{\alpha_3}^3 - x_{\alpha_2}x_{\alpha_3}x_{\alpha_4} + x_{\alpha_2}x_{\alpha_6}, \qquad y_3 \mapsto x_{\alpha_6} - x_{\alpha_2}x_{\alpha_5} - x_{\alpha_3}x_{\alpha_4},$$
$$y_2 \mapsto x_{\alpha_4}^3 - x_{\alpha_2}x_{\alpha_3}x_{\alpha_4} + x_{\alpha_2}x_{\alpha_5}^2 + x_{\alpha_5}x_{\alpha_6}.$$

Since $y_1 + y_3y_4 - y_4^2y_5 \mapsto x_{\alpha_3}^3$ and $y_2 - y_3y_5 - y_4y_5^2 \mapsto x_{\alpha_4}^3$, it follows that φ is purely inseparable of height 1 and degree 9. The properties of ψ are then deduced from the relation $\psi \circ \varphi = F$. ■

5.9. Proposition. — *If $p = 2$ or $p = 3$, then $\mathcal{E}_X$ is not ample for X the G_2-Grassmannian.*

Proof. It remains to handle the characteristic 3 case. Let $\psi : Y \to X$ be the morphism from 5.8. Then $\mathcal{E}_\psi$ is a vector bundle of rank $\deg \psi - 1 = 26$ and first Chern class

$$2c_1(\mathcal{E}_\psi) = -2c_1(\psi_*\mathcal{O}_Y) = 27c_1(\mathcal{T}_X) - \psi_*c_1(\mathcal{T}_Y) = 81c_1(\mathcal{O}_X(1)) - 5\psi_*c_1(\mathcal{O}_Y(1)).$$

Since $\psi^*\mathcal{O}_Y(1) \cong \mathcal{O}_X(3)$, a projection formula computation shows that $\psi_*c_1(\mathcal{O}_Y(1)) = 9c_1(\mathcal{O}_X(1))$, from which it follows that $c_1(\mathcal{E}_\psi) = 18c_1(\mathcal{O}_X(1))$. As before, X contains lines in its Plücker embedding, the restriction of $\mathcal{E}_\psi$ onto any such line is a vector bundle of rank $r = 26$ and degree $d = 18$, and so cannot be ample. Therefore $\mathcal{E}_\psi$ is itself not ample, whence $\mathcal{E}_X$ is not ample by 1.2. ■

It would be interesting to describe $\varphi : X \to Q$ appearing in 5.8 in terms of the trilinear form γ used to define X in 4.14. Towards this, it may be useful to view $\mathbf{G}$ as the automorphism group of the split octonionic Cayley algebra $C = \mathbf{k} \oplus V$ over $\mathbf{k}$, where the 7-dimensional representation V

may be identified as the subspace of purely imaginary octonions. When $p = 3$, the commutator with respect to the octonionic product satisfies the Jacobi identity—see [BEMN02, p.402] and also [Man25, Remark 3.4.1]—making C into a Lie algebra. Identifying $\mathfrak{g}$ as the Lie algebra of derivations on C, the adjoint representation of C on itself then provides an embedding of Lie algebras $V \subset C \to \mathfrak{g}$.

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