

FREE CURVES IN FANO HYPERSURFACES MUST HAVE HIGH DEGREE

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ABSTRACT. The purpose of this note is to show that the minimal e for which every smooth Fano hypersurface of dimension n contains a free rational curve of degree at most e cannot be bounded by a linear function in n when the base field has positive characteristic. This is done by providing a super-linear bound on the minimal possible degree of a free curve in certain Fermat hypersurfaces.

INTRODUCTION

The geometry of smooth projective Fano varieties is controlled by the rational curves they contain. Seminal work [KMM92, Cam92] of Kollár–Miyaoaka–Mori and Campana show that, over a field of characteristic 0, every smooth projective Fano variety X contains a rational curve $\varphi : \mathbf{P}^1 \rightarrow X$ that, informally speaking, can be deformed to pass through $r + 1$ general points of X for any chosen $r \geq 0$: in other words, X is *separably rationally connected*. The precise condition on the curve φ is that $H^1(\mathbf{P}^1, \varphi^* \mathcal{T}_X \otimes \mathcal{O}_{\mathbf{P}^1}(-r - 1)) = 0$; by way of terminology, φ is said to be *free* or *very free* when $r = 0$ or $r = 1$, respectively. See [Kol96, Deb01] for a presentation of this theory.

Whether smooth projective Fano varieties over a field of positive characteristic are separably rationally connected is a long-standing open question. Results are fragmentary even for smooth Fano hypersurfaces in projective space: The general Fano hypersurface is separably rationally connected by [Zhu11, CZ14, Tia15, CR19]; notably, the work of Tian reduces the problem of separable rational connectedness to *separable uniruledness*—that is, the existence of a free rational curve—a problem that often is simpler because free curves typically have significantly lower degree than very free curves. More recently, [STZ22, Theorems 3.10 and 3.24] shows that all smooth Fano hypersurfaces with degree less than the characteristic are separably rationally connected and even that, up to a minor condition, such hypersurfaces always contain either free lines or conics. See also [LP21, Theorem 34] and [BS23, Theorem 1.5] for related results.

The main result of this note is that, nevertheless and contrary to experience, the minimal e such that there exists a free rational curve of degree $\leq e$ on every smooth Fano hypersurface cannot be bounded by a linear function in the dimension (or degree); contrast this with the fact that every smooth Fano hypersurface in characteristic 0 contains either a free line or conic.

Theorem. — *For any algebraically closed field \mathbf{k} of characteristic $p > 0$,*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \inf \left\{ e \in \mathbf{Z} \left| \begin{array}{l} \text{for every smooth Fano hypersurface of dimension } n \text{ over } \mathbf{k} \\ \text{there exists a free rational curve of degree } \leq e \end{array} \right. \right\} = \infty.$$

It suffices to describe one sequence of increasingly high-dimensional hypersurfaces without low degree free curves, and this is given by the Fermat of degree $q + 1$ in \mathbf{P}^{q+1} with $q := p^\nu$ for $\nu \geq 1$. These hypersurfaces are exceptional and exemplify many positive characteristic phenomena: see [She12] for closely related work and [Che22, pp. 7–11] for a general survey regarding these hypersurfaces. What is fascinating is that these hypersurfaces contain many, many rational curves—they are unirational!—and so the challenge is to develop techniques to study their spaces of rational curves: see [Che23, Che24] for work in this direction. The method is to exploit a certain tension arising from the curious differential geometry of these hypersurfaces: the structure of the equation

implies that, on the one hand, free curves cannot lie in a hyperplane and, on the other hand, there are unexpected constraints on the coordinate functions of the curve.

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FREE CURVES IN THE FERMAT q -BIC

In what follows, let $X := V(T_0^{q+1} + \cdots + T_{q+1}^{q+1})$ be the Fermat hypersurface of degree $q+1$ in \mathbf{P}^{q+1} . The *embedded tangent bundle* \mathcal{E}_X is the vector bundle on X whose fibre at a point x is the linear space underlying the embedded tangent space of X at x ; it fits into a short exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{E}_X \rightarrow \mathcal{T}_X \rightarrow 0.$$

The extension class is the pullback via the tangent map $\mathcal{T}_X \rightarrow \mathcal{T}_{\mathbf{P}^{q+1}}|_X$ of the class of the Euler sequence, and so there is another short exact sequence

$$0 \rightarrow \mathcal{E}_X \rightarrow \mathcal{O}_X(1)^{\oplus q+2} \rightarrow \mathcal{N}_{X/\mathbf{P}^{q+1}} \rightarrow 0$$

where the second map is $(\psi_0, \dots, \psi_n) \mapsto \sum_{i=0}^{q+1} T_i^q \cdot \psi_i$. Remarkably, as already observed by [She12, Equation (2)], this sequence twisted down by $\mathcal{O}_X(-1)$ is isomorphic to the pullback of the dual Euler sequence by the q -power Frobenius morphism Fr . In particular, there is an isomorphism

$$\mathcal{E}_X(-1) \cong \text{Fr}^*(\Omega_{\mathbf{P}^{q+1}}^1(1)|_X).$$

Let $\varphi: \mathbf{P}^1 \rightarrow X$ be a nonconstant morphism of degree $e = mq + r$, where $m, r \in \mathbf{Z}$ and $0 \leq r \leq q-1$; to simplify notation, φ will sometimes be viewed as a morphism into \mathbf{P}^{q+1} . Viewing \mathcal{E}_X as an extension of \mathcal{T}_X , it follows that φ is free if and only if $H^1(\mathbf{P}^1, \varphi^* \mathcal{E}_X \otimes \mathcal{O}_{\mathbf{P}^1}(-1)) = 0$. Combined with the isomorphism above and the projection formula, this implies that, if φ is free,

$$\begin{aligned} 0 = H^1(\mathbf{P}^1, \varphi^* \mathcal{E}_X \otimes \mathcal{O}_{\mathbf{P}^1}(-1)) &= H^1(\mathbf{P}^1, \varphi^* \text{Fr}^*(\Omega_{\mathbf{P}^{q+1}}^1(1)|_X) \otimes \mathcal{O}_{\mathbf{P}^1}(e-1)) \\ &= H^1(\mathbf{P}^1, \varphi^*(\Omega_{\mathbf{P}^{q+1}}^1(1)) \otimes \text{Fr}_* \mathcal{O}_{\mathbf{P}^1}(e-1)). \end{aligned}$$

Since $\text{Fr}_* \mathcal{O}_{\mathbf{P}^1}(e-1) \cong \mathcal{O}_{\mathbf{P}^1}(m)^{\oplus r} \oplus \mathcal{O}_{\mathbf{P}^1}(m-1)^{\oplus q-r}$, this shows that:

1. Lemma. — *If $\varphi: \mathbf{P}^1 \rightarrow X$ is free, then $H^1(\mathbf{P}^1, \varphi^* \Omega_{\mathbf{P}^{q+1}}^1 \otimes \mathcal{O}_{\mathbf{P}^1}(e+m-1)) = 0$.* ■

The converse also holds. Since $\chi(\mathbf{P}^1, \varphi^* \Omega_{\mathbf{P}^{q+1}}^1 \otimes \mathcal{O}_{\mathbf{P}^1}(e+m-1)) = (q+2)m - e - m = m - r$, this gives a relation between m and r when φ is free, recovering [She12, Theorem 1.7]:

2. Lemma. — *If $\varphi: \mathbf{P}^1 \rightarrow X$ is free, then $r \leq m$.* ■

The following gives a geometric restriction on free curves in X , and stands in stark contrast to the fact, see [Kol96, V4.4], that a general smooth Fano hypersurface contains either a free line or conic:

3. Lemma. — *If $\varphi: \mathbf{P}^1 \rightarrow X$ is free, then $\varphi(\mathbf{P}^1)$ is not contained in any hyperplane.*

Proof. Identifying $\mathcal{E}_X = \ker(\mathcal{O}_X(1)^{\oplus q+2} \rightarrow \mathcal{O}_X(q+1))$ and using that φ is free shows that

$$\dim H^0(\mathbf{P}^1, \varphi^* \mathcal{E}_X \otimes \mathcal{O}_{\mathbf{P}^1}(-1)) = e.$$

If φ were contained in a hyperplane $\mathbf{P}^q \subset \mathbf{P}^{q+1}$, then $\varphi^*(\Omega_{\mathbf{P}^{q+1}}^1(1))$ would split as $\varphi^*(\Omega_{\mathbf{P}^q}^1(1)) \oplus \mathcal{O}_{\mathbf{P}^1}$. Combined with the fact that $\mathcal{E}_X(-1) \cong \text{Fr}^*(\Omega_{\mathbf{P}^{q+1}}^1(1))$, this would imply that

$$e = \dim H^0(\mathbf{P}^1, \varphi^* \mathcal{E}_X \otimes \mathcal{O}_{\mathbf{P}^1}(-1)) = \dim H^0(\mathbf{P}^1, \text{Fr}^* \varphi^*(\Omega_{\mathbf{P}^q}^1(1)) \otimes \mathcal{O}_{\mathbf{P}^1}(e-1)) + e.$$

Therefore $\text{Fr}^* \varphi^*(\Omega_{\mathbf{P}^q}^1(1)) \otimes \mathcal{O}_{\mathbf{P}^1}(e-1)$ could not have global sections; in fact, freeness of φ means that it can have no cohomology whatsoever, and so it must be isomorphic to $\mathcal{O}_{\mathbf{P}^1}(-1)^{\oplus q}$. However, the Euler sequence for $\varphi^*(\Omega_{\mathbf{P}^q}^1(1))$ implies that

$$\text{Fr}^* \varphi^*(\Omega_{\mathbf{P}^q}^1(1)) \otimes \mathcal{O}_{\mathbf{P}^1}(e-1) \cong \bigoplus_{i=0}^q \mathcal{O}_{\mathbf{P}^1}(e - qa_i - 1)$$

for some integers $a_i \geq 0$ such that $a_0 + \dots + a_q = e$. So if $e = qa_i$ for all $i = 0, \dots, q$, then $a_0 = \dots = a_q = e = 0$, meaning that φ would be constant: contradiction! \blacksquare

4. — Viewing φ as a morphism into \mathbf{P}^{q+1} and letting $(\varphi_0 : \dots : \varphi_{q+1})$ be its components, **3** means that the φ_i are linearly independent in $H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(e))$. Already, this implies $e \geq q+1$. Now, that φ factors through X means that $\sum_{i=0}^{q+1} \varphi_i^q \cdot \varphi_i = 0$. Upon choosing homogeneous coordinates $(S_0 : S_1)$ for \mathbf{P}^1 , there is a unique decomposition

$$\varphi_i = \sum_{j=0}^r \zeta_{ij}^q \cdot S_0^j S_1^{r-j} + \sum_{k=1}^{q-r-1} \eta_{ik}^q \cdot S_0^{r+k} S_1^{q-k}$$

where $\zeta_{ij} \in H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(m))$ and $\eta_{ik} \in H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(m-1))$. Substituting this into the equation of X and using the fact that q is a power of the ground field characteristic shows that

$$0 = \sum_{j=0}^r \left(\sum_{i=0}^{q+1} \varphi_i \zeta_{ij} \right)^q \cdot S_0^j S_1^{r-j} + \sum_{k=1}^{q-r-1} \left(\sum_{i=0}^{q+1} \varphi_i \eta_{ik} \right)^q \cdot S_0^{r+k} S_1^{q-k}$$

and so, upon looking at exponents modulo q , $\sum_{i=0}^{q+1} \varphi_i \zeta_{ij} = \sum_{i=0}^{q+1} \varphi_i \eta_{ik} = 0$ for all j and k . These relations are at odds with linear independence of the φ_i , and give strong restrictions on the degree e :

5. Theorem. — *If $\varphi : \mathbf{P}^1 \rightarrow X$ is free, then $q+1 \leq m^2 + m + r$.*

Proof. Consider the linear map $\Phi : H^0(\mathbf{P}^{q+1}, \mathcal{O}_{\mathbf{P}^{q+1}}(1)) \rightarrow H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(e))$ defining $\varphi : \mathbf{P}^1 \rightarrow \mathbf{P}^{q+1}$, so that the i -th coordinate T_i maps to φ_i . This has rank $q+2$ by **3**. Let $\text{Fr} : \mathbf{P}^1 \rightarrow \mathbf{P}^1$ be the q -power Frobenius morphism, and identify the target of Φ as

$$H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(e)) \cong H^0(\mathbf{P}^1, \text{Fr}_* \mathcal{O}_{\mathbf{P}^1}(e)) \cong H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(m))^{\oplus r+1} \oplus H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(m-1))^{\oplus q-r-1}.$$

Let Φ_1 and Φ_2 be the linear map obtained by post-composing Φ with projection to $H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(m))^{\oplus r+1}$ and $H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(m-1))^{\oplus q-r-1}$, respectively. Elementary linear algebra gives

$$q+2 = \text{rank } \Phi \leq \text{rank } \Phi_1 + \text{rank } \Phi_2 \leq (r+1)(m+1) + \text{rank } \Phi_2,$$

so it remains to bound the rank of Φ_2 .

Let $\Phi_{2,k} : H^0(\mathbf{P}^{q+1}, \mathcal{O}_{\mathbf{P}^{q+1}}(1)) \rightarrow H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(m-1))$ be the further projection of Φ_2 to the k -th component of its target, so that $\Phi_{2,k}(T_i) = \eta_{ik}$, notation as in **4**. The discussion of **4** means that the $\Phi_{2,k}$ lie in the kernel of the linear map

$$\begin{aligned} \text{Hom}(H^0(\mathbf{P}^{q+1}, \mathcal{O}_{\mathbf{P}^{q+1}}(1)), H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(m-1))) &\longrightarrow H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(e+m-1)) \\ \Psi &\longmapsto \sum_{i=0}^{q+1} \varphi_i \cdot \Psi(T_i). \end{aligned}$$

Comparing with the Euler sequence shows that the kernel is $H^0(\mathbf{P}^1, \varphi^* \Omega_{\mathbf{P}^{q+1}}^1 \otimes \mathcal{O}_{\mathbf{P}^1}(e+m-1))$, and this has dimension $m-r$ by **1**. Thus at most $m-r$ of the $q-r-1$ components $\Phi_{2,k}$ of Φ_2 are linearly independent. Since each $\Phi_{2,k}$ has rank at most m , this implies that $\text{rank } \Phi_2 \leq (m-r)m$. Combining with the inequality above gives

$$q+2 \leq (r+1)(m+1) + (m-r)m = m^2 + m + r + 1$$

which, upon subtracting 1 from either side, gives the result. \blacksquare

Combining **2** with **5** gives $\sqrt{q+2} - 1 \leq m+1$. Writing $e = mq + r$ and grossly underestimating yields a super-linear bound on the minimal degree of a free curve in X :

6. Theorem. — *If $\varphi : \mathbf{P}^1 \rightarrow X$ is free, then $e \geq q^{3/2} - q$.* ■

The strongest restrictions on possible degrees of free curves are provided by 2 and 5. They give the following for the first few prime powers:

q	2	3	4	5	7	8	9	11	13	16	17	19	23	25	27	29	31	32
e_{\min}	3	6	8	10	16	24	27	33	41	64	68	76	96	125	135	145	157	163

Free curves achieving these lower bounds are known to exist in the first few cases: see [Mad06, p.6], [Con06, p.69], and [BDE⁺13] for $(q, e) = (2, 3), (3, 6), (4, 8)$, respectively. As far as I know, no free curves are known to exist when $q \geq 5$. Finally, observe that the arguments of 1 further imply that if $\varphi : \mathbf{P}^1 \rightarrow X$ is free, then it is r -free, in the sense that $H^1(\mathbf{P}^1, \varphi^* \mathcal{T}_X \otimes \mathcal{O}_{\mathbf{P}^1}(-r-1)) = 0$, so that for many q above, a minimal possible free curve is automatically very free.

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