CALCULUS II MIDTERM II PRACTICE

This is a sample of what the midterm will look like. Try to work out these problems on your own before consulting the solutions I have provided. Note that the problems here are more challenging than those that will appear on the midterm. For extra practice, consider working through the Review Section at the end of Chapter 11.

1. Consider the sequence $\{a_n\}_n$ defined recursively by $a = 1$ and for $n \geq 2$,

$$a_n = \frac{1}{3}(a_{n-1} + 4).$$

(i) Assuming that the sequence $\{a_n\}_n$ is convergent, compute its limit.
(ii) State the Monotone Convergence Theorem.
(iii) Show that $\{a_n\}_n$ is increasing and that $a_n < 2$ for all n by considering the function $f(x) := \frac{1}{3}(x+4)$ and its derivative.[1]
(iv) Deduce that the sequence $\{a_n\}_n$ is convergent.

2. Determine whether the series

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\log(n)}}$$

is convergent or divergent. Explain.

3. Is the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n}}{n+1}$$

convergent or divergent? Justify your answer.

4. Does the series

$$\sum_{n=1}^{\infty} \log\left(\frac{n}{3n+1}\right)$$

converge or diverge? Why?

5. Determine the radius and interval of convergence of the power series

$$\sum_{n=1}^{\infty} n^3 x^n.$$

Bonus: Find its sum.

6. Let p be a number. Consider the series

[1]This might not be so easy! Try to consider the graph of the function $f(x)$ and compare with the graph of $y = x$.

(i) Find the values of p for which the series

$$S_p := \sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots.$$

is convergent.

(ii) Consider the power series

$$\mathrm{Li}_p(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^p}.$$

What is the radius of convergence of this power series?

(iii) What is the interval of convergence of the power series in (ii)? Your answer will depend on p.

(iv) Using the power series for $\frac{1}{1-x}$, find a Taylor series for $-\log(1-x)$ centred at 0.

(v) Show that

$$\mathrm{Li}_2(x) = \int \frac{-\log(1-x)}{x}\, dx$$

perhaps up to a constant of integration.

7. Suppose that you have a function $f(x)$ such that $f(0) = \pi$ and

$$\frac{d}{dx} f(x) = \log(1+x).$$

Use this relationship to derive a Taylor series centred at 0 for $f(x)$. Find the radius and interval of convergence for the resulting power series.