CALCULUS II ASSIGNMENT 9

DUE APRIL 11, 2019

1. Suppose that f is a function such that $f^{(n)}(0) = (n+1)!)$ for all nonnegative integers n. Find the Taylor series of f centred at 0 and find its radius of convergence.

2. Find the Taylor series of $f(x) := 1/x$ centred at $a = -3$. What is the radius of convergence of this representation?

3. Let $f(x) := (1+x)^{\alpha}$, where α is any fixed real number—in particular, it does not have to be an integer!

(i) Compute $f^{(n)}(x)$ and $f^{(n)}(0)$ for $n = 0, 1, 2, 3, 4$.
(ii) Guess a pattern for $f^{(n)}(0)$ and try to justify it.
(iii) Use your guess in (ii) to write down a Taylor series for $f(x)$ centred at 0.
(iv) Find the radius of convergence of your power series expansion.
(v) Use your power series to find a power series expansion of $f(x) := (1-x)^{3/4}$.

It might be helpful to use the following notation: for any α and any positive integer n, set

$$\binom{\alpha}{n} := \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}.$$

4. Let f_n denote the n^{th} Fibonacci number. Recall that f_n is defined recursively by setting $f_0 = f_1 = 1$ and for $n \geq 2$, $f_n = f_{n-1} + f_{n-2}$. Let

$$F(x) := \sum_{n=0}^{\infty} f_n x^n = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + f_4 x^4 + \cdots = 1 + x + 2x^2 + 3x^3 + 5x^4 + \cdots.$$

(i) Find the radius of convergence of $F(x)$. It might be helpful to look at **7**.(iii), HW5.
(ii) Use the recurrence relation for the Fibonacci numbers to show that

$$F(x) = 1 + xF(x) + x^2 F(x).$$

(iii) Rearrange the relation in (ii) to show that

$$F(x) = \frac{1}{1-x-x^2},$$

at least within the interval of convergence of $F(x)$.
(iv) Let

$$1 - x - x^2 = (\phi_+ - x)(\phi_- - x) \quad \text{where } \phi_\pm := \frac{1 \pm \sqrt{5}}{2}.$$

Find the partial fraction expansion of $\frac{1}{1-x-x^2}$ in terms of $\frac{1}{\phi_+ - x}$ and $\frac{1}{\phi_- - x}$
(v) Use the geometric series formula to find a power series expansion of $\frac{1}{\phi_\pm - x}$.
(vi) Put together (iii)–(v) to find an explicit formula for the n^{th} Fibonacci number f_n.