CALCULUS II ASSIGNMENT 5

DUE FEBRUARY 28, 2019

Okay, let's play with some sequences!

1. Explain in your own words what it means for a sequence to converge or diverge. Give two examples of sequences which converge, and two examples of sequences which converge. Try to carefully justify, using the definition from class, why your sequences do what they do.

2. Limits of sequences can be handled using things you know about limits of functions. For instance, given a function $f(x)$, we can define a sequence $\{a_n\}_{n=1}^{\infty}$ by sampling f at, say, the integers; that is, set

$$a_n := f(n).$$

Graphically, this means that the a_n are dots on the graph of f above the points $x = n$. Given this, one sees that

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x).$$

Try to use this to determine whether the following sequences are convergent, and if they are, what their limits are.

(i) $a_n = \dfrac{5n^3+2n^2+1}{n^3+3}$, (iii) $a_n = \left(1+\dfrac{1}{n}\right)^n$,

(ii) $a_n = \dfrac{n^3}{n+1}$, (iv) $a_n = \dfrac{\log(n)^2}{n}$.

3. Determine whether or not the following sequences converge or diverge. If they converge, determine their limit.

(i) $a_n = 1-(0.12)^n$, (iii) $a_n = (-1)^n$,

(ii) $a_n = \log(n+1) - \log(n)$, (iv) $\{0,1,0,0,1,0,0,0,1,\ldots\}$.

4. The Squeeze Theorem is incredibly helpful for determining the convergence of certain sequences. Let's try to work in an example that is not as obvious as the ones given in class.

(i) Use the Pythagorean Theorem $\sin(x)^2+\cos(x)^2 = 1$ together with the fact that $y^2 \geq 0$ for any real number y to justify the inequalities

$$-1 \leq \sin(x) \leq 1 \quad \text{and} \quad -1 \leq \cos(x) \leq 1.$$

(ii) Use (i) to show that for any numbers α and β, that

$$-\alpha-\beta \leq \alpha\sin(x)+\beta\cos(x) \leq \alpha+\beta$$

for any x.

(iii) Now use the Squeeze Theorem to show that the sequence

$$a_n = \frac{5\sin(n)+7\cos(n)}{n^{3/5}}$$

is convergent. Find its limit along the way.

5. Decide whether or not the sequence

$$a_n = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{n!}$$

is convergent. It may help to graph the sequence. Try to justify carefully, using the definition of limits given in class, your conclusion.

6. The Monotone Convergence Theorem is a powerful way to determine whether or not a sequence has a limit.

(i) Suppose you had a increasing sequence $\{a_n\}$ whose terms are numbers between 0 and 10. Carefully explain why this has a limit L.

(ii) What can you say about the limit L? That is, do you know some inequalities on L?

(iii) Let T_n be the average temperature of the universe at year n—whatever that means! Apparently, there is an absolute coldest temperature, and if we assume that the universe is finite, then there is also some absolute hottest temperature somewhere. Use this to conclude that as the average temperature of the universe must have a limit as time goes off to infinity.

Regarding (iii), you might enjoy this short story by Isaac Asimov.

7. Once you know that a sequence has a limit, you can use formal methods to find the actual limit. Here are some examples.

(i) Find the limit of the sequence

$$\left\{ \sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots \right\}$$

by writing $a_n = \sqrt{2a_{n-1}}$.

(ii) Find the limit of the sequence with $a_1 = 1$ and

$$a_n = 3 - \frac{1}{a_{n-1}} \quad \text{for } n > 1.$$

In fact, you can try to prove this one is convergent by trying to show that it is increasing and that $a_n < 3$ for all n.

(iii) Let $\{f_n\}_n$ be the Fibonacci sequence, so that $f_1 = f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n > 2$. Let $a_n = f_{n+1}/f_n$. Check that

$$a_n = 1 + \frac{1}{a_{n-1}} \quad \text{for } n \geq 1$$

and find its limit. This is the asymptotic growth rate of the Fibonacci sequence and is known as the golden ratio.