CALCULUS II ASSIGNMENT 10

DUE APRIL 18, 2019

1. Find the general solution to the following differential equations:

(i) $y' = 3x^2y^2$,

(ii) $\dfrac{dy}{dx} - y = e^x$,

(iii) $\dfrac{dz}{dt} + e^{t+z} = 0$,

(iv) $xy' + y = \sqrt{x}$,

(v) $\dfrac{d\theta}{dt} = \dfrac{t\sec\theta}{\theta e^{t^2}}$,

(vi) $t^2\dfrac{dz}{dt} + 3tz = \sqrt{1+t^2}$.

2. Solve the following initial value problems:

(i) $\dfrac{dP}{dt} = \sqrt{Pt}$ with $P(1) = 2$,

(ii) $xy' + y = x\log(x)$ with $y(1) = 0$,

(iii) $\dfrac{dL}{dt} = kL^2\log(t)$ with $L(1) = -1$,

(iv) $xy' = y + x^2\sin(x)$ with $y(\pi) = 0$.

3. Sometimes, equations that are not obviously of a form that you know how to solve can be massaged into one by a clever change of variables. For instance, consider the equation

$$y' = x + y.$$

This is not separable, but it can be transformed into a separable equation using a change of variables, as follows:

(i) Consider the substitution $u = x + y$. Express dx in terms of du and dy.

(ii) Make the substitution $u = x + y$ in $y' = x + y$ and rearrange so that the equation becomes separable. Don't forget to change the differential in $y' = dy/dx$!

(iii) Solve the differential equation and express the final solution in terms of x.

(iv) Double check that your proposed solution satisfies the original differential equation.

4. In a similar vein to **3.**, consider the equation

$$\frac{dx}{dt} = \frac{t}{2}x + \frac{1}{2x}e^{t^2/2}.$$

This is not a linear equation, a fact that you should explain. However, consider the substitution $y(t) := x(t)^2$. Show that y satisfies the linear differential equation

$$\frac{dy}{dt} = ty + e^{t^2/2}$$

and use this to solve the original equation.

5. A pretty neat equation I just learned about is the Gompertz function. This type of equation is good for modelling things that start off and end off slow, but may change rapidly in the interim. For instance, one application is apparently in modelling the growth of a tumor.

In any case, consider the equation

$$\frac{dV}{dt} = a(\log(b) - \log(V))V$$

where a and b are positive constants.

(i) Find the general solution to this equation.

(ii) Try to come up with a situation—one which is not covered in Wikipedia's list!—that this equation could serve as a useful model.

6. In class, we discussed a very simple model for population growth of a system. Of course, it was too simple and unrealistic: there are physical constraints to the population of a system. A slightly better model is given by the logistic differential equation:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

where k and M are positive constants. In modelling populations, k is a growth rate of the system, and M is the *carrying capacity* of the system: the maximum number of individuals a system can support with its resources.

(i) Solve the logistic equation by recognizing it as a separable equation.

(ii) Solve the logistic equation, again, by making the substitution $z = 1/P$ to obtain the linear differential equation

$$z' + kz = \frac{k}{M}.$$

(iii) Revisit the model that we discussed in class with the logistic model: let

$$P(t) := \text{population of the Earth in year } 2019 + t \text{ in billions}$$

so that the current population is $P(0) = 7.7$. Assume that the annual rate of population growth remains stead at 1.07%, or $k = 0.0107$. One estimate for the carrying capacity of the Earth is $M = 10$. Using these parameters, use your answers above to write down a function that models the population of the Earth for the next century. Sketch a graph representing this.