CALCULUS II FINAL PRACTICE

This is a sample of what the final would look like. Try to work out these problems on your own before consulting the solutions I have provided. Note that the problems here are more challenging than those that will appear on the final.

1. Evaluate the following integrals:

$$\text{(i)} \int \frac{2}{x(x+2)}\,dx, \qquad \text{(ii)} \int \sqrt{4-x^2}\,dx, \qquad \text{(iii)} \int \frac{dx}{(1+x^2)^{5/2}}.$$

2. Find the surface area of the surface obtained by revolving $y = x^2/2$ between $0 \le x \le 4$ around the x-axis.

3. Let's talk about series.

(i) What does it precisely mean for a series $\sum_{n=1}^{\infty} a_n$ to converge?
(ii) State the Ratio Test.
(iii) Determine whether

$$\sum_{n=1}^{\infty} (-1)^n \frac{n!}{2^n}$$

is absolutely convergent, conditionally convergent, or divergent.
(iv) Determine whether

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{2^{n!}}$$

is absolutely convergent, conditionally convergent, or divergent.

4. Let $\log(x)$ be the natural logarithm function.

(i) Compute $\int \log(x)^n\,dx$ for $n = 1,2,3,4$.
(ii) Compute $\int_0^1 \log(x)^n\,dx$ for integers $n \ge 1$.

5. Let's work with some Taylor series.

(i) Let $f(x)$ be a function which is infinitely differentiable at a point a. What is the Taylor series of f centred at a?
(ii) Compute, using your answer to (i), the Taylor series of $\sin(x)$ centred at 0.
(iii) Find the Taylor series centred at 0 for $\cos(x)$.
(iv) Compute the limit

$$\lim_{x\to 0} \frac{\sin(x)}{x}$$

in two ways: first using your Taylor series for $\sin(x)$ from (ii), and then with l'Hôpital's Rule.

6. Let $r > 0$ be positive real numbers. Consider the curve defined by

$$x^{2/3} + y^{2/3} = r^2.$$

(i) Sketch the curve defined by this equation.
(ii) Find the area bound by this curve.
(iii) Find the arc length of this curve.

7. Consider the differential equation

$$y' = -y.$$

(i) Find the general solution to the above equation in any way you like.

Now forget your method in (i) and assume that the differential equation admits a solution $f(x)$ with a power series expansion $f(x) = \sum_{n=0}^{\infty} c_n x^n$, that is,

$$f'(x) = -f(x).$$

(ii) Substitute the power series expression for $f(x)$ in the above equation and compare coefficients of x^{n-1} to obtain a relationship between c_n and c_{n-1}.
(iii) Iterate the relationship you found in (ii) to express c_n in terms of c_0.
(iv) Use the relationship from (iii) to simplify the power series of f.
(v) What is the radius of convergence for the power series you found?

8. Consider the region R bound by the y-axis and the curve $y = e^x$ between $x = 0$ and $x = \log(2)$. Let S be the solid of revolution obtained by rotating R about the y-axis.

(i) Sketch the region R and the solid S.
(ii) Compute the area of R.
(iii) Compute the volume of S using the cross-sectional method.
(iv) Compute the volume of S using the cylindrical shell method.

9. Let's check some things for convergence.

(i) Is the improper integral

$$\int_1^{\infty} \frac{\sin(x)}{x^3}\, dx$$

convergent or divergent? If it is convergent, evaluate it.

(ii) Does the series

$$\sum_{n=1}^{\infty} \frac{\sin(n)}{n^3}$$

converge or diverge? Justify your response.

(iii) Does the series

$$\sum_{n=1}^{\infty} \frac{1}{n^3 + n + 1}$$

converge or diverge? Explain.