Hodge theory of cubic fourfolds, their Fano varieties, and associated K3 categories

after Hassett, Addington–Thomas, Beauville, Donagi, Voisin, Galkin–Shinder, ...

Hodge theory of the Fano variety

Let $X \subset \mathbb{P}^5$ smooth cubic and $F(X) =$ Fano variety of lines on X.
The universal family:

$$F(X) \xleftarrow{p} \mathbb{L} \xrightarrow{q} X.$$

$F(X) \subset \mathbb{G} = G(2,6)$ smooth, $\dim = 4$, $\omega_F \simeq \mathcal{O}_F$, [Al-Kl], [Ba-vV].

Theorem (Galkin–Shinder)

Let $X \subset \mathbb{P}^{n+1}$ be a smooth cubic hypersurface. Then in $K_0(\mathrm{Var})$:

$$[X^{[2]}] = [\mathbb{P}^n] \cdot [X] + \ell^2 \cdot [F(X)],$$

where $\ell = [\mathbb{A}^1]$.

Proof: $[X^{[2]}] = [\mathbb{P}^n] \cdot [X] + \ell^2 \cdot [F(X)]$

$$F(X) \xleftarrow{p} \mathbb{L} \xrightarrow{q} X \quad \text{and} \quad \mathbb{G} \xleftarrow{p} \mathbb{L}_{\mathbb{G}} \xrightarrow{q} \mathbb{P} = \mathbb{P}^5.$$

$$X^{[2]} \xrightarrow{\simeq} \mathbb{L}_{\mathbb{G}}|_X.$$

$$\{x, y\} \longmapsto (\overline{xy}, z = \text{res. pt}),$$

$$L \cap X \setminus z \longleftarrow (L, z).$$

Proof: $[X^{[2]}] = [\mathbb{P}^n] \cdot [X] + \ell^2 \cdot [F(X)]$

$$F(X) \overset{p}{\twoheadleftarrow} \mathbb{L} \overset{q}{\twoheadrightarrow} X \quad \text{and} \quad \mathbb{G} \overset{p}{\twoheadleftarrow} \mathbb{L}_{\mathbb{G}} \overset{q}{\twoheadrightarrow} \mathbb{P} = \mathbb{P}^5.$$

$$X^{[2]} \hookleftarrow X^{[2]} \setminus \mathbb{L}^{[2]} \xrightarrow{\simeq} \mathbb{L}_{\mathbb{G}}|_X \setminus \mathbb{L} \hookrightarrow \mathbb{L}_{\mathbb{G}}|_X.$$

$$\{x, y\} \longmapsto (\overline{xy}, z = \text{res. pt}),$$

$$L \cap X \setminus z \longleftarrow\!\shortmid (L, z).$$

1. $[\mathbb{L}] = [\mathbb{P}^1] \cdot [F(X)]$,
2. $[\mathbb{L}^{[2]}] = [\mathbb{P}^2] \cdot [F(X)]$,
3. $[\mathbb{L}_{\mathbb{G}}|_X] = [\mathbb{P}^n] \cdot [X]$ (use $\mathbb{L}_{\mathbb{G}}|_X \simeq \mathbb{P}(\mathcal{T}_{\mathbb{P}}|_X)$).

Corollary

Let $X \subset \mathbb{P}^{n+1}$ be a smooth cubic hypersurface. Then in $K_0(\mathrm{HS}_{\mathbb{Z}})$:

$$[H^*(X^{[2]}, \mathbb{Z})(2)] = [H^*(\mathbb{P}^n, \mathbb{Z})(2)] \cdot [H^*(X, \mathbb{Z})] + [H^*(F(X), \mathbb{Z})].$$

Corollary

Let $X \subset \mathbb{P}^5$ be a smooth cubic hypersurface. Then in $\mathrm{HS}_{\mathbb{Q}}$:

$$\begin{aligned} H^2(F(X), \mathbb{Q}) &\simeq H^4(X, \mathbb{Q})(1), \\ H^4(F(X), \mathbb{Q}) &\simeq S^2(H^4(X, \mathbb{Q})_{\mathrm{pr}})(2) \oplus H^4(X, \mathbb{Q})_{\mathrm{pr}} \oplus \mathbb{Q}(-2). \end{aligned}$$

Nothing $/\mathbb{Z}$ nor (.).

Let $h := H^2 \in H^4(X,\mathbb{Z})$ and $g = \mathrm{c}_1(\mathcal{O}_{\mathbb{G}}(1))$ (Plücker polarization).

$$\varphi := p_* \circ q^* \colon H^4(X,\mathbb{Z}) \longrightarrow H^2(F(X),\mathbb{Z})(-1)$$

1. *$\varphi(h) = g$,*
2. *$\varphi|_{H^4_{\mathrm{pr}}}$ is injective with*

$$(\alpha.\beta) = -\frac{1}{6}\int_{F(X)} \varphi(\alpha)\cdot\varphi(\beta)\cdot g^2,$$

3. [Beau–Don]*: $H^4(X,\mathbb{Z})_{\mathrm{pr}}(-1) \xrightarrow{\sim} H^2(F(X),\mathbb{Z})_{\mathrm{pr}}$ (with BB).*

Better:

$$\varphi(\ ) = p_* \circ (q^*(\ ).\mathrm{td}(p)),$$

but no difference on $H^4(X,\mathbb{Z})_{\mathrm{pr}}$.

Recall: $X \rightsquigarrow$ extension

$$\Gamma \simeq H^4(X, \mathbb{Z})_{\mathrm{pr}}(-1) \subset \widetilde{H}(\mathcal{A}_X, \mathbb{Z}) \simeq \widetilde{\Lambda}.$$

$\lambda_1 \in A_2 \subset \widetilde{H}^{1,1}(\mathcal{A}_X, \mathbb{Z}) \rightsquigarrow$ induced sub-Hodge structures:

$$A_2^{\perp} \subset \lambda_1^{\perp} \subset \widetilde{H}(\mathcal{A}_X, \mathbb{Z}).$$

Theorem (Addington)

There exists a natural isometry of Hodge structures

$$H^2(F(X), \mathbb{Z}) \simeq \lambda_1^{\perp} \subset \widetilde{H}(\mathcal{A}_X, \mathbb{Z}).$$

$$\begin{array}{ccc} K_{\mathrm{top}}(X) & \xrightarrow{p_* \circ q^*} & K_{\mathrm{top}}(F(X)) \\ \downarrow v & & \downarrow v \\ H^*(X, \mathbb{Q}) & \xrightarrow{\text{not graded!}} & H^*(F(X), \mathbb{Q}) \\ \alpha & \longmapsto & p_*(q^*\alpha \cdot \mathrm{td}(p)) \end{array}$$

Proposition (Addington)

$\lambda_1^\perp \subset K_{\mathrm{top}}(\mathcal{A}_X) \xrightarrow{p_* \circ q^*} K_{\mathrm{top}}(F(X)) \xrightarrow{c_1} H^2(F(X), \mathbb{Z})$ *is isometry.*

Hodge structure is no problem.

1st step:

$$
\begin{array}{ccccc}
A_2^{\perp} & \hookrightarrow & K_{\mathrm{top}}(X) & \xrightarrow{\ p_* \circ q^*\ } & K_{\mathrm{top}}(F(X)) \\
\downarrow \simeq & & \downarrow v & & \downarrow v \\
H^4(X,\mathbb{Z})_{\mathrm{pr}}(-1) & \hookrightarrow & H^*(X,\mathbb{Q}) & \xrightarrow{\ \text{not graded!}\ } & H^*(F(X),\mathbb{Q}) \\
& \searrow^{\varphi} & & \nearrow & \\
& & H^2(F(X),\mathbb{Z})_{\mathrm{pr}} & &
\end{array}
$$

2nd step: $\lambda_1 + 2\lambda_2 \longmapsto g$

Hence:

$$A_2^{\perp} \oplus \mathbb{Z}(\lambda_1 + 2\lambda_2) \subset \lambda_1^{\perp} \subset K_{\mathrm{top}}(\mathcal{A}_X) \longrightarrow H^2(F(X),\mathbb{Z})$$

and $\mathrm{disc}(\lambda_1^{\perp}) = 2 = \mathrm{disc}(H^2(F(X),\mathbb{Z}))$.

Theorem (Markman) *For any hyperkähler fourfold Y of $\mathrm{K3}^{[2]}$-type there exists a natural isometric primitive embedding*

$$\iota_Y\colon H^2(Y,\mathbb{Z}) \hookrightarrow \widetilde{\Lambda}$$

such that $Y \sim Y'$ if and only if there exists a Hodge isometry $H^2(Y,\mathbb{Z}) \simeq H^2(Y',\mathbb{Z})$ admitting an extension

$$\begin{array}{ccc} H^2(Y,\mathbb{Z}) & \xrightarrow{\simeq} & H^2(Y',\mathbb{Z}) \\ \iota_Y \downarrow & & \downarrow \iota_{Y'} \\ \widetilde{\Lambda} & \xrightarrow{\simeq} & \widetilde{\Lambda} \end{array}$$

Example 1: $Y = S^{[2]} \Rightarrow$

$$\iota_Y\colon H^2(S^{[2]},\mathbb{Z}) \hookrightarrow \widetilde{H}(S,\mathbb{Z}) \simeq \widetilde{\Lambda}$$

is $(1,0,-1)^{\perp}$.

Example 2: $X \subset \mathbb{P}^5 \rightsquigarrow Y = F(X)$ is a HK of $\mathrm{K3}^{[2]}$-type $\Rightarrow$

$$\iota_Y \colon H^2(F(X),\mathbb{Z}) \hookrightarrow \widetilde{H}(\mathcal{A}_X,\mathbb{Z}) \simeq \widetilde{\Lambda}.$$

is the Addington–Thomas extension.

Theorem (Addington)

The Fano variety $F(X)$ is birational to $S^{[2]}$ of some K3 surface S if and only if there exists $U \hookrightarrow H^{1,1}(\mathcal{A}_X,\mathbb{Z})$ such that $\lambda_1 \in U$.

Proof

$\Rightarrow$ Assume $F(X) \sim S^{[2]}$. Then

$$\begin{array}{ccc} \varphi\colon H^2(F(X),\mathbb{Z}) & \xrightarrow{\simeq} & H^2(S^{[2]},\mathbb{Z}) \\ \downarrow & & \downarrow \\ \tilde{\varphi}\colon \widetilde{\Lambda} & \xrightarrow{\simeq} & \widetilde{\Lambda}, \end{array} \qquad \lambda_1 \longmapsto \pm(1,0,-1).$$

$\rightsquigarrow$ $U := \tilde{\varphi}^{-1}(H^0 \oplus H^4)$ for which $\lambda_1 \in U$.

$\Leftarrow$ Assume $U \subset \widetilde{H}^{1,1}(\mathcal{A}_X,\mathbb{Z}) \Rightarrow \exists\, S$ with Hodge isometry

$$\tilde{\varphi} : \widetilde{H}(\mathcal{A}_X,\mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(S,\mathbb{Z})$$

such that $U = \varphi^{-1}(H^0 \oplus H^4)(S,\mathbb{Z})$. Then $\lambda_1 \in U$ implies $\tilde{\varphi}(\lambda_1) \in (H^0 \oplus H^4)(S,\mathbb{Z}) \Rightarrow \tilde{\varphi}(\lambda_1) = \pm(1,0,-1) \Rightarrow$

$$\begin{array}{ccc} \varphi\colon \lambda_1^{\perp} & \xrightarrow{\sim} & (1,0,-1)^{\perp} \\ \simeq H^2(F(X),\mathbb{Z}) & & \simeq H^2(S^{[2]},\mathbb{Z}) \end{array}$$

.

Corollary

Let $[X] \in M \subset \mathcal{C}$ be a smooth cubic fourfold. Then $X \in \mathcal{C}_d$ for some d satisfying $(\mathrm{K3}^{[2]}) \Leftrightarrow F(X) \sim S^{[2]}$ for some K3 surface S.

Question (Galkin–Schinder): *Is a smooth cubic fourfold $[X] \in M \subset \mathcal{C}$ rational $\Leftrightarrow [X] \in \mathcal{C}_d$ for some d satisfying $(\mathrm{K3}^{[2]})$*

$$\Leftrightarrow F(X) \sim S^{[2]}\,?$$

Compare to Hassett:
X rational $\Leftrightarrow X \in \mathcal{C}_d$ for some d satisfying (K3)?
First case with $(\mathrm{K3}) \not\Leftarrow (\mathrm{K3}^{[2]})$: $d = 74$.

Have seen:

$$\ell^2 \cdot [F] = [X^{[2]}] - [\mathbb{P}^4] \cdot [X] = S^2[X] - [\mathbb{P}^4] \cdot [X].$$

X rational $\Rightarrow [X] = [\mathbb{P}^4] + \ell \cdot \alpha$ with $\alpha = \sum a_i[T_i]$, T_i surfaces.
$\Rightarrow$

$$\begin{aligned} \ell^2 \cdot [F] &= S^2([\mathbb{P}^4] + \ell \cdot \alpha) - [\mathbb{P}^4] \cdot ([\mathbb{P}^4] + \ell \cdot \alpha) \\ &= \ell^2 \cdot S^2([\mathbb{P}^2] + \alpha) - \ell^4 \end{aligned}$$

(Use $S^2[\mathbb{P}^4] - (1 + \ell^4) \cdot [\mathbb{P}^4] = \ell^2 \cdot S^2[\mathbb{P}^2] - \ell^4$.)
Suppose ℓ is not a zero-divisor (for the classes involved).
$\Rightarrow$

$$[F] = S^2([\mathbb{P}^2] + \alpha) - \ell^2$$

and hence

$$F \sim T \times T' \text{ or } \sim T^{[2]}$$

with T, T' surfaces.

Use that $\sum h^{p,0}(F)t^p = 1 + t + t^2$ is irreducible in $\mathbb{Z}_{\geq 0}[t]$

$$\Rightarrow \quad F \sim T^{[2]} \quad \dots \quad \Rightarrow \quad F \sim S^{[2]}$$

with S a K3 surface.

Questions

1. *Any evidence for the converse?*
2. *Which one is it,* $(\mathrm{K3})$ *or* $(\mathrm{K3}^{[2]})$*?*
3. *...or something completely different?*