Hodge theory of cubic fourfolds, their Fano varieties, and associated K3 categories

after Hassett, Addington–Thomas, Beauville, Donagi, Voisin, Galkin–Shinder, . . .

Hodge theory of Kuznetsov's category

Recall: $$(S,L) \sim X \;\Leftrightarrow\; \pi[(S,L)] = [X]$$

for $\pi\colon M_d \longrightarrow \mathcal{C}$.

Weaker: $$\Rightarrow\; H^2(S,\mathbb{Z})_{L\text{-pr}} \hookrightarrow H^4(X,\mathbb{Z})_{\rm pr}(-1).$$

Goal: Introduce $\widetilde{H}(\mathcal{A}_X,\mathbb{Z})$ to distinguish the two notions!

Need $\widetilde{H}(\mathcal{A}_X,\mathbb{Z})$ to capture more than $H^2(S,\mathbb{Z})_{L\text{-pr}}$ & $H^4(X,\mathbb{Z})_{\rm pr}$.

$$\begin{array}{lcccl} \mathrm{sign} = (2,21) & \widetilde{\Gamma} & \overset{??}{\longleftrightarrow} & \widetilde{\Lambda} \supset \Lambda,\ A_2 & \mathrm{sign} = (4,20) \\ & \cup & & \cup & \\ \mathrm{sign} = (2,20) & \Gamma & \simeq & A_2^{\perp} & \end{array}$$

$\widetilde{\Gamma}$ has wrong signature, odd lattice,...

Idea: Replace $\widetilde{\Gamma} = H^4(X,\mathbb{Z})(-1)$ by $H^*(X,\mathbb{Z})(-1)$.

Recall: $\Gamma \simeq E \oplus U_1 \oplus U_2 \oplus A_2(-1)$ and $\Gamma \oplus A_2 \subset \widetilde{\Lambda}$ given by

$$A_2(-1) \oplus A_2 \subset U_3 \oplus U_4,$$

where $U_4 \simeq (H^0 \oplus H^4)(S, \mathbb{Z})$, i.e.

$$\Gamma \oplus A_2 \hookrightarrow \widetilde{H}(S, \mathbb{Z}) \text{ extends to } \Gamma \oplus A_2 \subset \widetilde{\Lambda} \xrightarrow{\sim} \widetilde{H}(S, \mathbb{Z}).$$

Could try to use

$$A_2 \hookrightarrow U_3 \oplus U_4 \simeq H^{*\neq 4}(X, \mathbb{Z})$$

and extend

$$\Gamma \oplus A_2 \hookrightarrow H^4(X, \mathbb{Z})(-1) \oplus H^{*\neq 4}(X, \mathbb{Z}) \simeq H^*(X, \mathbb{Z})(-1)$$

to

$$\Gamma \oplus A_2 \subset \widetilde{\Lambda} \hookrightarrow H^*(X, \mathbb{Z})(-1).$$

But this time

$$A_2(-1) \oplus A_2 \hookrightarrow H^4(X, \mathbb{Z})(-1) \oplus U_3 \oplus U_4.$$

Need a different embedding of A_2!

Explicit construction $A_2 \hookrightarrow H^*(X, \mathbb{Q})(-1)$

$$\lambda_1 \longmapsto v(\lambda_1) := 3 + \frac{5}{4}H - \frac{7}{32}H^2 - \frac{77}{384}H^3 + \frac{41}{2048}H^4.$$

$$\lambda_2 \longmapsto v(\lambda_2) := -3 - \frac{1}{4}H + \frac{15}{32}H^2 + \frac{1}{384}H^3 - \frac{153}{2048}H^4.$$

Problems:

1. *Classes are not integral.*
2. *Intersection pairing on $H^*(X)$ not compatible with $(\ .\ )$ on A_2.*
3. *Why this choice?*

Mukai pairing on $H^*(S,\mathbb{Z}) = H^0 \oplus H^2 \oplus H^4 \rightsquigarrow \widetilde{H}(S,\mathbb{Z})$:

$$\begin{aligned}(\alpha_0+\alpha_2+\alpha_4)^2 &:= (\alpha_2)^2 - 2\,(\alpha_0.\alpha_4).\\ \text{or } (\alpha.\alpha') &:= -\textstyle\int \alpha^*\cdot\alpha'\end{aligned}$$

where $(\alpha_0+\alpha_2+\alpha_4)^* := \alpha_0 - \alpha_2 + \alpha_4$.

Mukai vector

$$v(E) := \mathrm{ch}(E)\sqrt{\mathrm{td}(S)}$$

for $E \in \mathrm{Coh}(S)$ or $E \in K_{\mathrm{top}}(S)$, where $\sqrt{\mathrm{td}(S)} = (1,0,1)$.

1. $v(E) \in \widetilde{H}(S,\mathbb{Z})$ *(integral!) for* $E \in \mathrm{Coh}(S)$ *and* $E \in K_{\mathrm{top}}(S)$.
2. $\chi(E,E') := \sum(-1)^i \mathrm{ext}^i(E,E') = -(v(E).v(E'))$.

Mukai pairing on $H^*(X,\mathbb{Q})$: $(\alpha.\alpha') := -\int e^{\frac{c_1(X)}{2}} \cdot \alpha^* \cdot \alpha'$, where

$$\begin{aligned}(\alpha_0+\alpha_2+\alpha_4+\alpha_6+\alpha_8)^* &:= \alpha_0-\alpha_2+\alpha_4-\alpha_6+\alpha_8\\ e^{\frac{c_1(X)}{2}} &= e^{\frac{3H}{2}}\end{aligned}$$

Neither symmetric nor integral on $H^*(X,\mathbb{Z})$!

Mukai vector

$$v(E) := \mathrm{ch}(E)\sqrt{\mathrm{td}(X)}$$

for $E \in \mathrm{Coh}(X)$ or $E \in K_{\mathrm{top}}(X)$, where

$$\sqrt{\mathrm{td}(X)} = 1 + \frac{3}{4}H + \frac{11}{32}H^2 + \frac{15}{128}H^3 + \frac{121}{6144}H^4.$$

Using $\sqrt{\mathrm{td}}^* = \sqrt{\mathrm{td}} \cdot e^{-\frac{c_1}{2}}$, one finds

$$\chi(E, E') = -(v(E).v(E')).$$

Examples:

$$\begin{aligned} w_0 := v(\mathcal{O}_X) &= 1 + \tfrac{3}{4}H + \tfrac{11}{32}H^2 + \tfrac{15}{128}H^3 + \tfrac{121}{6144}H^4. \\ w_1 := v(\mathcal{O}_X(1)) &= 1 + \tfrac{7}{4}H + \tfrac{51}{32}H^2 + \tfrac{385}{384}H^3 + \tfrac{2921}{6144}H^4. \\ w_2 := v(\mathcal{O}_X(2)) &= 1 + \tfrac{11}{4}H + \tfrac{132}{32}H^2 + \tfrac{1397}{384}H^3 + \tfrac{16025}{6144}H^4. \end{aligned}$$

Claim (K3 symmetry)

$(\ .\ )$ *is symmetric and of signature* $(4, 20)$ *on*

$$\{w_0, w_1, w_2\}^\perp \subset H^*(X, \mathbb{Q}).$$

Compute that $v(\lambda_1), v(\lambda_2) \in \{w_0, w_1, w_2\}^\perp \subset H^*(X, \mathbb{Q})$.

$\Rightarrow w_0, w_1, w_2, v(\lambda_1), v(\lambda_2) \in \mathbb{Q}[H] \subset H^*(X, \mathbb{Q})\ldots$
$\ldots$ and they are lin. independent!

$\Rightarrow$

$$\begin{aligned} H^4(X, \mathbb{Q})_{\mathrm{pr}} &= \{w_0, w_1, w_2, v(\lambda_1), v(\lambda_2)\}^\perp \\ &= {}^\perp\{w_0, w_1, w_2, v(\lambda_1), v(\lambda_2)\}. \end{aligned}$$

1. $(\ .\ )$ is symmetric on $H^4(X, \mathbb{Q})_{\mathrm{pr}}$ with signature $(2, 20)$.
2. $H^4(X, \mathbb{Q})_{\mathrm{pr}} \perp (\mathbb{Q}v(\lambda_1) \oplus \mathbb{Q}v(\lambda_2))$ are orthogonal.
3. $\mathbb{Z}v(\lambda_1) \oplus \mathbb{Z}v(\lambda_2) \simeq \mathbb{Z}\lambda_1 \oplus \mathbb{Z}\lambda_2 \simeq A_2$ by explicit computation (in particular symmetric).

Let $\mathbb{P}^1 \simeq \ell \subset X \subset \mathbb{P}^5$ be a line and define $v(\lambda_j)$ as the images of

$$u_j := v(\mathcal{O}_\ell(j)) = \begin{cases} \frac{1}{3}H^3 + \frac{5}{12}H^4 & j = 1 \\ \\ \frac{1}{3}H^3 + \frac{9}{12}H^4 & j = 2 \end{cases}$$

under the right orthogonal projection

$$p\colon H^*(X, \mathbb{Q}) \longrightarrow \{w_0, w_1, w_2\}^\perp, \; u_j \longmapsto v(\lambda_j).$$

Explicitly:

$v(\lambda_1) = u_1 - w_1 + 4w_0$ *and* $v(\lambda_2) = u_2 - w_2 + 4w_1 - 6w_0.$

Use:

$$\begin{aligned} (w_i.w_j) &= \chi(\mathcal{O}_X(i), \mathcal{O}_X(j)) &&= \chi(X, \mathcal{O}_X(j-i)), \\ (w_i.u_j) &= \chi(\mathcal{O}_X(i), \mathcal{O}_\ell(i)) &&= \chi(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(j-i)) \\ (u_i, u_j) &= 0 && \\ (u_i, w_j) &= \chi(\mathcal{O}_\ell(i), \mathcal{O}_X(j)) &&= \chi(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(i-j-3)) \end{aligned}$$

Can define (artificially) isometric embedding

$$A_2^{\perp} \oplus A_2 \hookrightarrow \widetilde{\Lambda} \hookrightarrow H^*(X, \mathbb{Q})(-1)$$

with $\widetilde{\Lambda}_{\mathbb{Q}}^{\perp} = \langle w_0, w_1, w_2 \rangle_{\mathbb{Q}}$.

By

$$v \colon A_2^{\perp} \simeq \Gamma \xrightarrow{\sim} H^4(X, \mathbb{Z})_{\mathrm{pr}}(-1)$$

$$v \colon A_2 \hookrightarrow \mathbb{Q}[H] \subset H^*(X, \mathbb{Q})(-1), \ \lambda_i \longmapsto v(\lambda_i)$$

and $\alpha := \frac{1}{3}(\mu_1 - \mu_2 - \lambda_1 + \lambda_2) \in \widetilde{\Lambda}$ is mapped to

$$\frac{1}{3}(v(\mu_1) - v(\mu_2) - v(\lambda_1) + v(\lambda_2)).$$

Better via topological K-theory [Add-Th]:

$$v := \mathrm{ch}(\ )\sqrt{\mathrm{td}(X)} \colon K_{\mathrm{top}}(X) \subset K_{\mathrm{top}}(X) \otimes \mathbb{Q} \xrightarrow{\sim} H^*(X, \mathbb{Q})(-1)$$

with induced $(\ .\ )$ which is only $\mathbb{Q}$-valued and not symmetric $\rightsquigarrow$

$$[\mathcal{O}_X], [\mathcal{O}_X(1)], [\mathcal{O}_X(2)], [\mathcal{O}_\ell(1)], [\mathcal{O}_\ell(2)] \in K_{\mathrm{top}}(X)$$

e.g. via $K(X) \longrightarrow K_{\mathrm{top}}(X)$. Then

$$K_{\mathrm{top}}(\mathcal{A}_X) := \{[\mathcal{O}_X], [\mathcal{O}_X(1)], [\mathcal{O}_X(2)]\}^\perp \subset K_{\mathrm{top}}(X)$$

with respect to $\chi(\ .\ ) := -(v(\ ).v(\ ))$. Clear:

$$v \colon K_{\mathrm{top}}(\mathcal{A}_X) \otimes \mathbb{Q} \xrightarrow{\sim} \langle w_0, w_1, w_2 \rangle_{\mathbb{Q}}^\perp \subset H^*(X, \mathbb{Q})(-1).$$

Theorem (Addington–Thomas)

$$(K_{\mathrm{top}}(\mathcal{A}_X), -\chi(\ .\ )) \simeq \widetilde{\Lambda}.$$

Proof. $\mathcal{A}_X \simeq \mathrm{D^b}(\mathrm{K3}) \rightsquigarrow K_{\mathrm{top}}(\mathcal{A}_X) \simeq K_{\mathrm{top}}(\mathrm{K3}) \simeq \widetilde{H}(\mathrm{K3}, \mathbb{Z})$.

Use (right) orthogonal projection $p\colon K_{\mathrm{top}}(X) \longrightarrow K_{\mathrm{top}}(\mathcal{A}_X)$.

$$\Rightarrow \ \mathbb{Z}\, p[\mathcal{O}_\ell(1)] \oplus \mathbb{Z}\, p[\mathcal{O}_\ell(2)] \simeq \mathbb{Z}\lambda_1 \oplus \mathbb{Z}\lambda_2 \simeq A_2.$$

Explicitly:

$$\begin{aligned} p[\mathcal{O}_\ell(1)] &= [\mathcal{O}_\ell(1)] - [\mathcal{O}_X(1)] + 4[\mathcal{O}_X] \\ p[\mathcal{O}_\ell(2)] &= [\mathcal{O}_\ell(2)] - [\mathcal{O}_X(2)] + 4[\mathcal{O}_X(1)] - 6[\mathcal{O}_X] \end{aligned}$$

1. $\rightsquigarrow \lambda_1, \lambda_2 \in K_{\mathrm{top}}(\mathcal{A}_X)$.
2. $\Gamma \simeq H^4(X,\mathbb{Z})_{\mathrm{pr}} \hookrightarrow K_{\mathrm{top}}(\mathcal{A}_X)$.
3. $\frac{v(\mu_1)-v(\mu_2)-v(\lambda_1)+v(\lambda_2)}{3}$ lifts to $\alpha \in K_{\mathrm{top}}(X) \subset K_{\mathrm{top}}(X) \otimes \mathbb{Q}$?
4. $(\ .\ )$ is integral on $\{\lambda_1,\lambda_2\}^\perp \subset K_{\mathrm{top}}(\mathcal{A}_X)$ and on α.

$\rightsquigarrow$

$$\widetilde{\Lambda} \hookrightarrow K_{\mathrm{top}}(\mathcal{A}_X)$$

is a finite index immersion of (integral!) lattices.

$\Rightarrow$ ($\widetilde{\Lambda}$ unimodular) $\qquad \widetilde{\Lambda} \xrightarrow{\sim} K_{\mathrm{top}}(\mathcal{A}_X)$.

Definition

For $X \subset \mathbb{P}^5$ smooth cubic, let $\widetilde{H}(\mathcal{A}_X, \mathbb{Z})$ be the weight two Hodge structure on the lattice

$$(K_{\mathrm{top}}(\mathcal{A}_X), -\chi(\ .\ )) \simeq \widetilde{\Lambda}$$

defined by

$$\widetilde{H}^{2,0}(\mathcal{A}_X) := v^{-1}(H^{3,1}(X)).$$

Then, there exists a natural isometric primitive embedding

$$A_2 \hookrightarrow \widetilde{H}^{1,1}(\mathcal{A}_X, \mathbb{Z})$$

with

$$v = \mathrm{c}_2 \colon A_2^{\perp} \xrightarrow{\sim} H^4(X, \mathbb{Z})_{\mathrm{pr}}(-1) \simeq \Gamma.$$

Hence

$$H^4(X, \mathbb{Z})_{\mathrm{pr}}(-1) \simeq A_2^{\perp} \subset \widetilde{H}(\mathcal{A}_X, \mathbb{Z})$$

is a natural sub-Hodge structure.

Recall: $(S, L) \in M_d \subset \mathcal{M}_d$ and $X \in M \cap \mathcal{C}_d \subset \mathcal{C}$ are associated if under $\pi\colon \mathcal{M}_d \longrightarrow \mathcal{C}$:

$$\pi[(S, L)] = [X]. \qquad (*)$$

Spelled out: Have fixed

$$\widetilde{\Lambda} \supset \Lambda \supset \Lambda_d = L_d^\perp \subset A_2^\perp = \Gamma \subset \widetilde{\Lambda}.$$

Then $(*)$ if and only if there exist markings

$$H^2(S, \mathbb{Z}) \simeq \Lambda \text{ and } H^4(X, \mathbb{Z})_{\mathrm{pr}}(-1) \simeq \Gamma = A_2^\perp$$

such that

$$\begin{array}{ccccccc} H^2(S,\mathbb{Z}) & \supset & H^2(S,\mathbb{Z})_{L\text{-pr}} & \simeq & K_d^\perp & \subset & H^4(X,\mathbb{Z})_{\mathrm{pr}}(-1) \\ \wr\| & & \wr\| & & \wr\| & & \wr\| \\ \Lambda & \supset & \Lambda_d & = & L_d^\perp & \subset & A_2^\perp \end{array}$$

Rephrased:

$$
\begin{array}{ccccccc}
\widetilde{H}(S,\mathbb{Z}) & \simeq & \widetilde{\Lambda} & = & \widetilde{\Lambda} & \simeq & \widetilde{H}(\mathcal{A}_X,\mathbb{Z}) \\
\cup & & \cup & & & & \cup \\
H^2(S,\mathbb{Z}) & \simeq & \Lambda & & \cup & & H^4(X,\mathbb{Z})_{\mathrm{pr}}(-1) \\
\cup & & \cup & & & & \wr| \\
H^2(S,\mathbb{Z})_{L\text{-pr}} & \simeq & \Lambda_d & = & L_d^{\perp} & \subset & A_2^{\perp}
\end{array}
$$

Corollary

(S,L) and X are associated if and only if there exists an Hodge isometry $\widetilde{H}(S,\mathbb{Z}) \simeq \widetilde{H}(\mathcal{A}_X,\mathbb{Z})$ such that

$$U_4 = (H^0 \oplus H^4)(S,\mathbb{Z}) \subset \widetilde{H}^{1,1}(S,\mathbb{Z}) \text{ and } A_2 \subset \widetilde{H}^{1,1}(\mathcal{A}_X,\mathbb{Z})$$

span a sublattice of rank three (finite index overlattice of $A_2 \oplus \mathbb{Z}v$) with $(\ )^{\perp} = H^2(S,\mathbb{Z})_{L\text{-}pr}$.

Alternatively:

Corollary

(S, L) and X are associated if and only if there exists an Hodge isometry

$$\begin{array}{ccc} \widetilde{H}(S,\mathbb{Z}) & \simeq & \widetilde{H}(\mathcal{A}_X,\mathbb{Z}) \\ \cup & & \cup \\ H^2(S,\mathbb{Z})_{L\text{-}pr} & \hookrightarrow & H^4(X,\mathbb{Z})_{\mathrm{pr}}(-1) \end{array}$$

Corollary

A smooth cubic fourfold X is associated with some polarized K3 surface (S, L) if and only if $\exists U \subset H^{1,1}(\mathcal{A}_X, \mathbb{Z})$

$$\text{(with } \mathrm{rk}(A_2 + U) = 3.)$$

Theorem (Addington)

There exists a natural isometry of Hodge structures

$$H^2(F(X), \mathbb{Z}) \simeq \lambda_1^{\perp} \subset \widetilde{H}(\mathcal{A}_X, \mathbb{Z}).$$