Hodge theory of cubic fourfolds, their Fano varieties, and associated K3 categories

after Hassett, Addington–Thomas, Beauville, Donagi, Voisin, Galkin–Shinder, . . .

Period domains and moduli spaces

$$\Gamma = H^4(X,\mathbb{Z})_{\mathrm{pr}}(-1),\ \Lambda = H^2(S,\mathbb{Z}) = E \oplus U_1 \oplus U_2 \oplus U_3,$$
$$\widetilde{\Lambda} = \widetilde{H}(S,\mathbb{Z}) = \Lambda \oplus U_4,\ A_2 \subset U_3 \oplus U_4$$

$\mathrm{sign} = (2,20)$ $\qquad$ $\Gamma \simeq A_2^\perp \simeq E \oplus U_1 \oplus U_2 \oplus A_2(-1).$

$\mathrm{sign} = (2,19)$

$$\begin{array}{ccccccccc} K_d^\perp & \subset & \Gamma & \subset & \widetilde{\Gamma} & & & & \\ \wr\| & & \wr\| & & & & & & \\ L_d^\perp & \subset & A_2^\perp & \subset & \widetilde{\Lambda} & \supset & \Lambda & \supset & \Lambda_d \end{array}$$

$(\mathrm{H})_0$: $\qquad$ $K_d^\perp \simeq L_d^\perp \simeq E \oplus U_2 \oplus A_2(-1) \oplus \mathbb{Z}(d/3).$

$(\mathrm{H})_2$:

(K3): $\qquad$ $K_d^\perp \simeq L_d^\perp \simeq \Lambda_d.$

$V = \mathbb{R}$ vector space & symmetric $(\ .\ )$, $\mathrm{sign} = (n_+ \geq 2, n_-)$

$$\mathrm{Gr}^{\mathrm{po}}(2, V) \simeq \{x \mid (x)^2 = 0,\ (x.\bar{x}) > 0\} \subset \mathbb{P}(V_{\mathbb{C}})$$

$$\simeq \mathrm{O}(n_+, n_-)/\mathrm{SO}(2) \times \mathrm{O}(n_+ - 2, n_-)$$

$\mathbb{P}(K_{d\mathbb{C}}^{\perp})$	$\subset$	$\mathbb{P}(\Gamma_{\mathbb{C}})$	$\subset$	$\mathbb{P}(\widetilde{\Lambda}_{\mathbb{C}})$	$\supset$	$\mathbb{P}(\Lambda_{\mathbb{C}})$	$\supset$	$\mathbb{P}(\Lambda_{d\mathbb{C}})$
20		21		23		21		20
D_d	$\subset$	D	$\subset$	$\widetilde{Q}$	$\supset$	Q	$\supset$	Q_d
19		20		22		20		19

- D_d, D, Q_d *have 2 connected components, e.g.* $D = D^+ \sqcup D^-$,
- $\widetilde{Q}, Q$ *are connected.*

$$\tilde{\mathrm{O}}(\Gamma) := \{g \in \mathrm{O}(\widetilde{\Gamma}) \mid g(h) = h\} \subset \mathrm{O}(\Gamma = h^\perp) \text{ index two}$$

$$\tilde{\mathrm{O}}^+(\Gamma) := \{g \in \tilde{\mathrm{O}}(\Gamma) \mid g|_\Gamma \text{ preserves orientation of } (2,0) \subset (2,19)\}$$
$$\Leftrightarrow \ g(D^+) = D^+$$

Similar: $\tilde{\mathrm{O}}^+(\Lambda_d) \subset \tilde{\mathrm{O}}(\Lambda_d = \ell^\perp)$, $\ell \in \Lambda$ with $(\ell)^2 = d$, and

$$\begin{array}{ccl} \tilde{\mathrm{O}}(\Gamma, K_d) & := & \{g \in \tilde{\mathrm{O}}(\Gamma) \mid g(K_d) = K_d \Leftrightarrow g(v_d) = \pm v_d\} \\ \bigcup & & \\ \tilde{\mathrm{O}}(\Gamma, v_d) & := & \{g \in \tilde{\mathrm{O}}(\Gamma) \mid g(v_d)|_{K_d} = \mathrm{id} \Leftrightarrow g(v_d) = v_d\}. \end{array}$$

Lemma (Hassett)

1. $(\mathrm{H})_0$, *i.e.* $d \equiv 0\,(6)$: $\tilde{\mathrm{O}}(\Gamma, v_d) \subset \tilde{\mathrm{O}}(\Gamma, K_d)$ *index 2.*
2. $(\mathrm{H})_2$, *i.e.* $d \equiv 2\,(6)$: $\tilde{\mathrm{O}}(\Gamma, v_d) = \tilde{\mathrm{O}}(\Gamma, K_d)$.

Lemma (Hassett)

1. $(\mathrm{H})_0$, *i.e.* $d \equiv 0\,(6)$: $\tilde{\mathrm{O}}(\Gamma, v_d) \subset \tilde{\mathrm{O}}(\Gamma, K_d)$ *index 2.*
2. $(\mathrm{H})_2$, *i.e.* $d \equiv 2\,(6)$: $\tilde{\mathrm{O}}(\Gamma, v_d) = \tilde{\mathrm{O}}(\Gamma, K_d)$.

Recall $(\mathrm{H})_0$: $\mathbb{Z}h \oplus \mathbb{Z}v_d = K_d \subset \mathbb{Z}(-1)^{\oplus 3} \oplus U_1$
$\Rightarrow \exists\, g = \mathrm{id} \oplus -\mathrm{id} \in \tilde{\mathrm{O}}(\Gamma, K_d) \setminus \tilde{\mathrm{O}}(\Gamma, v_d)$:

$$\widetilde{\Gamma} = \mathbb{Z}(-1)^{\oplus 3} \oplus E \oplus U_2 \oplus U_1$$

$(\mathrm{H})_2$: $\mathbb{Z}h \oplus \mathbb{Z}v_d \subset K_d$ index two with

$$v_d = 3(e_1 - \frac{d-2}{6} f_1) + \mu_1 - \mu_2$$

Here, $\mu_1 = (1, -1, 0), \mu_2 = (0, 1, -1) \in A_2(-1) \subset \mathbb{Z}(-1)^{\oplus 3}$ and $h = (1, 1, 1)$.

$$\Rightarrow \; (1/3)(v_d - h) \in K_d \text{ but } (1/3)(-v_d - h) \notin K_d.$$

If $V = N \otimes_{\mathbb{Z}} \mathbb{R}$ with $\mathrm{sign}(N) = (2, n_-)$, then $\mathrm{O}(N)$ acts properly discontinuous on period domain.

Applies to

1. $D \subset \mathbb{P}(\Gamma_{\mathbb{C}})$ and $\tilde{\mathrm{O}}(\Gamma)$,
2. $D_d \subset \mathbb{P}(K_{d\mathbb{C}}^{\perp})$ and $\tilde{\mathrm{O}}(\Gamma, v_d) \subset \tilde{\mathrm{O}}(\Gamma, K_d)$,
3. $Q_d \subset \mathbb{P}(\Lambda_{d\mathbb{C}})$ and $\tilde{\mathrm{O}}(\Lambda_d)$,

... but not to $\widetilde{Q} \subset \mathbb{P}(\widetilde{\Lambda}_{\mathbb{C}})$ and $Q \subset \mathbb{P}(\Lambda_{\mathbb{C}})$.

$$\begin{aligned}
\mathcal{C} &:= \tilde{\mathrm{O}}(\Gamma)\backslash D && \simeq \mathrm{O}(\Gamma)\backslash D \simeq \tilde{\mathrm{O}}^+(\Gamma)\backslash D^+, && \dim = 20 \\
\tilde{\mathcal{C}}_d &:= \tilde{\mathrm{O}}(\Gamma, K_d)\backslash D_d, && && \dim = 19 \\
\tilde{\tilde{\mathcal{C}}}_d &:= \tilde{\mathrm{O}}(\Gamma, v_d)\backslash D_d. && && \dim = 19 \\
\mathcal{M}_d &:= \tilde{\mathrm{O}}(\Lambda_d)\backslash Q_d && && \dim = 19
\end{aligned}$$

All irreducible!

Theorem (Baily–Borel) *Assume* $\mathrm{sign}(N) = (2, n_-)$ *and* $G \subset \mathrm{O}(N)$ *of finite index and torsion free. Then*

$$G \backslash D$$

is a smooth, quasi-projective, complex variety.

$\mathrm{O}(N)$-action properly discontinuous $\Rightarrow$ stabilizers are finite and hence torsion $\Rightarrow$ G acts freely $\Rightarrow$ quotient is a complex manifold.

Lemma *For all finite index* $G \subset \mathrm{O}(N)$, *there exists a torsion free normal subgroup* $G_0 \lhd G$ *of finite index.* $\Rightarrow$ $G_0 \backslash D$ *is smooth and quasi-projective* $\Rightarrow$ *normal and quasi-projective:*

$$G \backslash D \simeq (G/G_0) \backslash (G_0 \backslash D).$$

Minkowski theorem: $\mathrm{Gl}(n, \mathbb{Z}) \longrightarrow \mathrm{Gl}(n, \mathbb{F}_p)$, $p > 2$, is injective on finite subgroups $\rightsquigarrow$ $G_0 := G \cap \mathrm{Gl}(n, \mathbb{Z})(p)$.

(H) $\tilde{\mathcal{C}}_d \neq \emptyset$

$(\mathrm{H})_0$: $\tilde{\tilde{\mathcal{C}}}_d \xrightarrow{2:1} \tilde{\mathcal{C}}_d \xrightarrow{1:1} \mathcal{C}_d \subset \mathcal{C}$ finite

$(\mathrm{H})_2$: $\tilde{\tilde{\mathcal{C}}}_d \xrightarrow{\simeq} \tilde{\mathcal{C}}_d \xrightarrow{1:1} \mathcal{C}_d \subset \mathcal{C}$ finite

Theorem (Borel)

All maps are algebraic. (Be aware of torsion!)

(K3) $\Leftrightarrow K_d^\perp \simeq L_d^\perp \simeq \Lambda_d \Rightarrow D_d \simeq Q_d$ & $\tilde{\mathrm{O}}(\Gamma, v_d) \simeq \tilde{\mathrm{O}}(\Lambda_d)$:

$$\begin{aligned} \tilde{\mathrm{O}}(\Gamma, v_d) &= \{g \in \mathrm{O}(K_d^\perp) \mid g = \mathrm{id} \text{ on } A_{K_d^\perp}\} \\ \tilde{\mathrm{O}}(\Lambda_d) &= \{g \in \mathrm{O}(\Lambda_d) \mid g = \mathrm{id} \text{ on } A_{\Lambda_d}\}. \end{aligned}$$

Theorem (Hassett) *For d satisfying (K3):*

$$\mathcal{M}_d \simeq \tilde{\tilde{\mathcal{C}}}_d$$

Corollary *For d satisfying (K3):*

$$(\mathrm{K3})_0: \quad \mathcal{M}_d \simeq \tilde{\tilde{\mathcal{C}}}_d \xrightarrow{2:1} \tilde{\mathcal{C}}_d \xrightarrow{1:1} \mathcal{C}_d \subset \mathcal{C} \quad \textit{finite,}$$

$$(\mathrm{K3})_2: \quad \mathcal{M}_d \simeq \tilde{\tilde{\mathcal{C}}}_d \simeq \tilde{\mathcal{C}}_d \xrightarrow{1:1} \mathcal{C}_d \subset \mathcal{C} \quad \textit{finite.}$$

Similarly for K3s

$$\mathcal{M}_d = \tilde{\mathrm{O}}(\Lambda_d)\backslash Q_d \longrightarrow \mathrm{O}(\Lambda_d)\backslash Q_d$$

is finite of degree > 1 for $d > 4$.

$M_d :=$ moduli space of polarized K3 surfaces (S, L), $(L)^2 = d$
$\rightsquigarrow$ period map:

$$(S, L) \longmapsto H^{2,0}(S) \subset H^2(S, \mathbb{Z})_{L\text{-pr}} \otimes \mathbb{C} \simeq \Lambda_d \otimes \mathbb{C}.$$

Theorem (Pjateckiĭ-Šapiro/Šafarevič, Friedman, Sha, ...)

$$M_d \hookrightarrow \mathcal{M}_d = \tilde{\mathrm{O}}(\Lambda_d) \backslash Q_d$$

is an open, algebraic embedding.

$$\mathrm{Aut}(S, L) \simeq \{g \colon H^2(S, \mathbb{Z}) \xrightarrow{\sim} H^2(S, \mathbb{Z}) \mid \text{Hodge isometry}, g(L) = L\}.$$

Ample cone:

$$M_d = \mathcal{M}_d \setminus \bigcup \delta^{\perp},$$

where $\delta \in \Lambda_d$ *with* $(\delta)^2 = -2$.

$M := |\mathcal{O}_{\mathbb{P}^5}(3)|_{\rm sm}/\mathrm{PGl}(6) \rightsquigarrow$ period map:

$$X \longmapsto H^{3,1}(X) \subset H^4(X,\mathbb{Z})_{\rm pr} \otimes \mathbb{C} \simeq \Gamma \otimes \mathbb{C}.$$

Theorem (Voisin,... Looijenga, Charles, Zheng,..., H.–Rennemo)

$$M \hookrightarrow \mathcal{C} = \tilde{\mathrm{O}}(\Gamma)\backslash D$$

is an open, algebraic embedding.

$$\mathrm{Aut}(X) \simeq \{g \colon H^4(X,\mathbb{Z}) \xrightarrow{\sim} H^4(X,\mathbb{Z}) \mid \text{Hodge isometry}, g(h) = h\}.$$

Theorem (Laza, Looijenga)

$$M = \mathcal{C} \setminus (\mathcal{C}_2 \cup \mathcal{C}_6).$$

(K3): Then

$$\begin{array}{ccccc} \pi\colon M_d \subset \mathcal{M}_d & \twoheadrightarrow & \mathcal{C}_d & \subset & \mathcal{C} \\ & & & & \cup \\ & & & & M = \mathcal{C} \setminus (\mathcal{C}_2 \cup \mathcal{C}_6) \end{array}$$

and $$M \cap \mathcal{C}_d \subset \pi(M_d).$$

A polarized K3 surface $(S, L) \in M_d$ and a smooth cubic fourfold $X \subset \mathbb{P}^5$ are associated *if $\pi[(S, L)] = [X] \in M$.*

Corollary *A cubic X is associated to some polarized K3 surface if and only if $X \in \mathcal{C}_d$ for some d satisfying (K3).*

For given $[X] \in M$ there may be more than one (S, L):

1. $\mathcal{M}_d \longrightarrow \tilde{\mathcal{C}}_d$ *is (2:1) for* $(\mathrm{K3})_0$,
2. $\tilde{\mathcal{C}}_d \twoheadrightarrow \mathcal{C}_d$ *is only generically injective,*
3. $[X] \in \pi(M_d) \cap \pi(M_{d'})$ *is possible.*

When (S, L) (or just S) and X could be called associated:

1. $\pi[(S, L)] = [X] \in M$.
2. $H^2(S, \mathbb{Z})_{L\text{-pr}} \simeq v^\perp \subset H^4(X, \mathbb{Z})_{\mathrm{pr}}(-1)$ (& Tate).
3. $T(S) \simeq T(X)$ $(:= H^{2,2}(X, \mathbb{Z})^\perp \subset H^4(X, \mathbb{Z})(-1)$ & Tate).
4. $\mathrm{D^b}(S) \simeq \mathcal{A}_X$.

Then

$$(1) \Longrightarrow (2) \Longrightarrow (3) \overset{\mathrm{AT\&Co}}{\Longleftrightarrow} (4).$$

– Note $(1) \Leftarrow (2)$ does not hold, not even when $\rho(S) = 1$.
– For $(2) \Leftarrow (3)$ one needs to find a line bundle on S?

With one X, there may be infinitely many associated (S, L) (of unbounded degree), but only finitely many S.

Alternatively:

$$\mathrm{D^b}(S) \simeq \mathcal{A}_X \quad \Leftrightarrow \quad \exists\, \widetilde{H}(S,\mathbb{Z}) \simeq \widetilde{H}(\mathcal{A}_X,\mathbb{Z})$$

and

$$(S,L) \sim X \quad \Leftrightarrow \quad \exists \quad \begin{array}{ccc} \widetilde{H}(S,\mathbb{Z}) & \simeq & \widetilde{H}(\mathcal{A}_X,\mathbb{Z}) \\ \cup & & \cup \\ H^2(S,\mathbb{Z})_{L\text{-pr}} & \simeq & H^4(X,\mathbb{Z})_{\mathrm{pr}} \end{array}$$

But what is $\widetilde{H}(\mathcal{A}_X,\mathbb{Z})$??

Further results:

1. $\bigcup \mathcal{C}_d \subset \mathcal{C}$ *is analytically dense*
   (with d satisfying $(\mathrm{K3})^{[2]}$ should be enough).
2. $[X] \in M \setminus \bigcup \mathcal{C}_d \Rightarrow H^{2,2}(X,\mathbb{Z})_{\mathrm{pr}} = 0$.
3. ...

Question (Hassett, Harris,...): *Is a smooth cubic fourfold $[X] \in M \subset \mathcal{C}$ rational if and only if $[X] \in \mathcal{C}_d$ for some d satisfying* (K3), *i.e. if X is associated to some (S, L)?*

So far: In $\mathrm{codim} = 1$ ok for $d = 14$ (Beauville–Donagi), $d = 26, 38$ (Russo–Staglianò).