Hodge theory of cubic fourfolds, their Fano varieties, and associated K3 categories

after Hassett, Addington–Thomas, Beauville, Donagi, Voisin, Galkin–Shinder, . . .

1. $X \subset \mathbb{P}^5$ smooth cubic hypersurface,
2. $\mathcal{A}_X \subset \mathrm{D}^{\mathrm{b}}(X)$ Kuznetsov category,
3. $F(X)$ Fano variety of lines,
4. S K3 surface.

Goal: Study and relate $H^*(S)$, $\widetilde{H}(\mathcal{A}_X)$, $H^4(X)$, $H^2(F(X))$.

Various aspects: Lattices, $\mathbb{Q}$- and $\mathbb{Z}$-Hodge structures,...

Plan:

1. Lattice theory
2. Period domains and moduli spaces
3. Rational Hodge structures
4. Hodge theory for $\mathcal{A}_X$

Lattice theory

Lattices for cubics $X \subset \mathbb{P}^5$:

$$\widetilde{\Gamma} := H^4(X,\mathbb{Z})(-1);\ \Gamma := H^4(X,\mathbb{Z})_{\mathrm{pr}}(-1);\ K_d \subset \widetilde{\Gamma}.$$

Lattices for K3 surfaces S:

$$\widetilde{\Lambda} := H^*(S,\mathbb{Z});\ \Lambda := H^2(S,\mathbb{Z});\ L_d \subset \widetilde{\Lambda}.$$

General lattice theory:

$$(N \simeq \mathbb{Z}^{\oplus r}, (\ .\ ))$$

with $(\ .\ )$ symmetric, $\mathbb{Z}$-valued, usually non-degenerate, often even (i.e. $(x)^2 = (x.x) \equiv 0\,(2)$).

$$\rightsquigarrow\ \operatorname{disc}(N) := |\det(\ .\ )| \text{ and } (n_+, n_-) := \operatorname{sign}(\ .\ ).$$

Discriminant: $A_N := N^*/N$, finite group $|A_N| = \operatorname{disc}(N)$:

$$N \hookrightarrow N^*,\ x \longmapsto (x.\ ).$$

$$(\ )^2 \colon N \longrightarrow \mathbb{Z}\ \rightsquigarrow\ (\ )^2 \colon N^* \longrightarrow \mathbb{Q}\ \rightsquigarrow\ (\ )^2 \colon A_N \longrightarrow \mathbb{Q}/\mathbb{Z}$$

For even $(\ .\ )$ it lifts to

$$q = (\ )^2 := A_N \longrightarrow \mathbb{Q}/2\mathbb{Z}.$$

Standard examples:

1. $\mathrm{I}_{n_+,n_-} = \mathbb{Z}^{\oplus n_+ + n_-}$ & $\mathrm{diag}(+1,\dots,+1,-1,\dots,-1)$,
$\simeq \mathbb{Z}^{\oplus n_+} \oplus \mathbb{Z}(-1)^{\oplus n_-}$, $\mathrm{disc} = (-1)^{n_-}$, $\mathrm{sign} = (n_+, n_-)$.
2. $U = \mathbb{Z}^{\oplus 2} = \mathbb{Z}e \oplus \mathbb{Z}f$ & $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\mathrm{disc} = 1$, $\mathrm{sign} = (1,1)$.
3. $A_2 = \mathbb{Z}\lambda_1 \oplus \mathbb{Z}\lambda_2$ & $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, $\mathrm{disc} = 3$, $\mathrm{sign} = (2,0)$,
$q\colon A_{A_2} \simeq \mathbb{Z}/3\mathbb{Z}^* \longrightarrow \mathbb{Q}/2\mathbb{Z},\ \lambda_1^* = \bar{1}, \lambda_2^* = \bar{2} \longmapsto (2/3)$.
$$A_2 \simeq (1,1,1)^\perp \subset \mathrm{I}_{3,0}$$
with basis $\lambda_1 = (1,-1,0)$ and $\lambda_2 = (0,1,-1)$.
4. $E_8 =$unique unimodular, even lattice of signature $(8,0)$. Let
$$E := E_8(-1)^{\oplus 2} \text{ (or just forget).}$$

Standard results:

1. $N_1 \hookrightarrow N$ finite index: $\mathrm{disc}(N_1) = \mathrm{disc}(N) \cdot [N : N_1]^2$
2. $N_1 \hookrightarrow N$: $\mathrm{disc}(N_1) \cdot \mathrm{disc}(N_1^\perp) = \mathrm{disc}(N) \cdot [N : N_1 \oplus N_1^\perp]^2$. N unimodular $\Rightarrow A_{N_1} \simeq A_{N_1^\perp}$.
3. N even, unimodular, $n_\pm > 0$: $N \simeq E_8(\pm 1)^{\oplus a} \oplus U^{\oplus b}$.
4. N odd, unimodular, $n_\pm > 0$: $N \simeq \mathrm{I}_{n_+, n_-}$.
5. N even, unimodular, $1 < n_\pm$: Primitive $\ell \in N$ with given $(\ell)^2$ is unique up to $\mathrm{O}(N)$.
6. $N = N' \oplus U^{\oplus 2}$ even: Primitive $\ell \in N$ with given $(\ell)^2$ and $(1/n)\ell \in A_N$, $(\ell.N) = n\mathbb{Z}$, is unique up to $\mathrm{O}(N)$. (Eichler).

K3 lattice: $\Lambda := H^2(S, \mathbb{Z})$ is even, unimodular, $\mathrm{sign}(3, 19)$. Hence,

$$\rightsquigarrow \quad \Lambda \simeq E \oplus U^{\oplus 3} = E \oplus U_1 \oplus U_2 \oplus U_3.$$

Polarized K3 lattice: $\Lambda_d := (e_2 + (d/2)f_2)^\perp \subset \Lambda$ is even, $\mathrm{sign}(2, 19)$.:

$$\Lambda_d \simeq E \oplus U_1 \oplus U_3 \oplus \mathbb{Z}(-d).$$

Mukai lattice: $\widetilde{\Lambda} := \widetilde{H}(S, \mathbb{Z})$ is even, unimodular, $\mathrm{sign} = (4, 20)$. Hence,

$$\rightsquigarrow \quad \widetilde{\Lambda} \simeq E \oplus U^{\oplus 4} = E \oplus U_1 \oplus U_2 \oplus U_3 \oplus U_4.$$

Convention: $U_4 = \mathbb{Z}e_4 \oplus \mathbb{Z}f_4$ & $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. Then

$$U_4 \simeq (H^0 \oplus H^4)(S, \mathbb{Z})$$

with Mukai pairing and $e_4 \longmapsto [S] \in H^0$, $f_4 \longmapsto [\mathrm{pt}] \in H^4$.

Cubic lattice: $\widetilde{\Gamma} := H^4(X, \mathbb{Z})(-1)$ is odd, unimodular, $\mathrm{sign} = (2, 21)$,

$$\rightsquigarrow \ \widetilde{\Gamma} \simeq E \oplus U_1 \oplus U_2 \oplus \mathbb{Z}(-1)^{\oplus 3}.$$

Primitive cubic lattice: $\Gamma := H^4(X, \mathbb{Z})_{\mathrm{pr}}(-1)$ even, $\mathrm{sign} = (2, 20)$, $\mathrm{disc} = 3$, $h := [H^2] \in H^4(X, \mathbb{Z})(-1)$ is

$$h = (-1, -1, -1) \in \mathbb{Z}(-1)^{\oplus 3}.$$

$$\rightsquigarrow \ \Gamma \simeq E \oplus U_1 \oplus U_2 \oplus A_2(-1).$$

Hassett: Via $H^4(X, \mathbb{Z})_{\mathrm{pr}}(-1) \simeq H^2(F(X), \mathbb{Z})_{\mathrm{pr}}$ [Beau-Don].
Beauville: Characteristic vectors in unimodular lattices [Wall].

$$
\begin{array}{ccccc}
\text{sign} = (2,21) & \widetilde{\Gamma} & \overset{??}{\longleftrightarrow} & \widetilde{\Lambda} \supset \Lambda, \ A_2 & \text{sign} = (4,20) \\
 & \cup & & \cup & \\
\text{sign} = (2,20) & \Gamma & \simeq & A_2^{\perp} & \text{sign} = (2,20)
\end{array}
$$

Explicit embedding: $A_2 \hookrightarrow \widetilde{\Lambda}, \quad \lambda_1 \longmapsto e_4 - f_4 \quad (= v(\mathcal{I}_{x,y})),$
$\lambda_2 \longmapsto e_3 + f_3 + f_4$

Then

$$A_2^{\perp} = E \oplus U_1 \oplus U_2 \oplus A_2(-1),$$

with $A_2(-1) = \mathbb{Z}\mu_1 \oplus \mathbb{Z}\mu_2$ & $\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$ and

$$\mu_1 = e_3 - f_3, \ \mu_2 = -e_3 - e_4 - f_4.$$

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$$\mu_1 = e_3 - f_3,\ \mu_2 = -e_3 - e_4 - f_4.$$

Then $A_2 \oplus A_2^{\perp} \subset \widetilde{\Lambda}$ is of index 3

Corollary

1. $\widetilde{\Gamma} \supset \Gamma \simeq A_2^{\perp} \subset \widetilde{\Lambda}$,
2. $\lambda_1^{\perp} \simeq \Lambda \oplus \mathbb{Z}(e_4 + f_4) \subset \widetilde{\Lambda}$. $\quad (\simeq H^2(S^{[2]}, \mathbb{Z}))$

Consider $v \in \Gamma \simeq A_2^{\perp} \subset \widetilde{\Lambda}$ primitive with $(v)^2 < 0$. Saturations:

$$\begin{aligned} L_v &:= \overline{A_2 \oplus \mathbb{Z}v} \subset \widetilde{\Lambda} \quad \mathrm{sign} = (2,1) \\ K_v &:= \overline{\mathbb{Z}h \oplus \mathbb{Z}v} \subset \widetilde{\Gamma} \quad \mathrm{sign} = (0,2). \end{aligned}$$

$$\begin{array}{ccccc} L_v^{\perp} & \subset & A_2^{\perp} & \subset & \widetilde{\Lambda} \\ |\wr & & |\wr & & \\ K_v^{\perp} & \subset & \Gamma & \subset & \widetilde{\Gamma}. \end{array}$$

1. $K_v = \mathbb{Z}h \oplus \mathbb{Z}x$ & $\begin{pmatrix} -3 & a \\ a & b \end{pmatrix}$. Hence, $\mathrm{disc}(K_v) \equiv 0, 2, 3, 5\,(6)$.
2. $\mathbb{Z}h \oplus \mathbb{Z}v \subset K_v \subset (1/3)\mathbb{Z}h \oplus (1/3)\mathbb{Z}v$ (use $(h)^2 = -3$).
   Hence, $[K_v : \mathbb{Z}h \oplus \mathbb{Z}v] = 1$ or 3.
3. $\mathrm{disc}(K_v) \cdot [K_v : \mathbb{Z}h \oplus \mathbb{Z}v]^2 = \mathrm{disc}(\mathbb{Z}h \oplus \mathbb{Z}v) = 3(v)^2$

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If $[K_v : \mathbb{Z}h \oplus \mathbb{Z}v] = 1$, then $\mathrm{disc}(K_v) = -3(v)^2 \equiv 0\,(6)$.
If $[K_v : \mathbb{Z}h \oplus \mathbb{Z}v] = 3$, then $3\,\mathrm{disc}(K_v) = -(v)^2 \equiv 0, 2, 4\,(6)$.

$$\Rightarrow d := \mathrm{disc}(K_v) \equiv 0, 2\,(6). \qquad (\mathrm{H})$$

Note: $d \equiv 0\,(6) \Rightarrow [K_v : \mathbb{Z}h \oplus \mathbb{Z}v] = 1$.

- $(v := sh + tx.h) = 0 \Rightarrow v = \lambda(ah + 3x)$.
- v primitive $\Rightarrow \lambda = \pm 1, \pm(1/3)$.
- $d \equiv 0\,(6) \Rightarrow a \equiv 0\,(3) \Rightarrow \pm v = h + x$ and $3(v)^2 = -d$.

$$\begin{aligned} d \equiv 0\,(6) \quad &\Rightarrow \quad (v)^2 = -d/3 \equiv 0 \text{ or } 2\,(6) \\ d \equiv 2\,(6) \quad &\Rightarrow \quad (v)^2 = -3d \equiv 0\,(6). \end{aligned}$$

$$\begin{aligned} (v)^2 \equiv 2, 4\,(6) \quad &\Rightarrow \quad d \equiv 0\,(6) \\ (v)^2 \equiv 0\,(6) \quad &\Rightarrow \quad d \equiv 0 \text{ or } 2\,(6) \end{aligned}$$

$d \equiv 0\,(6)$. Let $v_d := e_1 - (d/6)f_1 \in U_1 \subset \Gamma \simeq A_2^{\perp}$. $\rightsquigarrow$

$$L_d := L_{v_d} \subset \widetilde{\Lambda} \text{ and } K_d := K_{v_d} \subset \widetilde{\Gamma}.$$

$$(v_d)^2 = -d/3.$$

– $A_2 \subset U_3 \oplus U_4$ and $v_d \in U_1 \Rightarrow A_2 \oplus \mathbb{Z}v_d \subset \widetilde{\Lambda}$ is saturated.

$$\Rightarrow \operatorname{disc}(L_d) = \operatorname{disc}(A_2 \oplus \mathbb{Z}v_d) = -3(v_d)^2 = d.$$

– $h \in \mathbb{Z}(-1)^{\oplus 3}$ and $v_d \in U_1 \Rightarrow K_d = \mathbb{Z}h \oplus \mathbb{Z}v_d$ (or directly $\operatorname{disc}(K_d) = \operatorname{disc}(L_d)$).

$$L_d^{\perp} \simeq K_d^{\perp} \simeq E \oplus U_2 \oplus A_2(-1) \oplus \mathbb{Z}(e_1 + (d/6)f_1).$$

$d \equiv 2\,(6)$. Let $v_d := 3(e_1 - \frac{d-2}{6} f_1) + \mu_1 - \mu_2 \in U_1 \oplus A_2(-1)$. $\rightsquigarrow$

$$L_d := L_{v_d} \subset \widetilde{\Lambda} \text{ and } K_d := K_{v_d} \subset \widetilde{\Gamma}.$$

$$(v_d)^2 = -3d.$$

Recall
$\lambda_1 = e_4 - f_4,\ \lambda_2 = e_3 + f_3 + f_4,\ \mu_1 = e_3 - f_3,\ \mu_2 = -e_3 - e_4 - f_4.$
Check $v_d - \lambda_1 + \lambda_2$ divisible by 3. $\Rightarrow [L_d : A_2 \oplus \mathbb{Z}v] = 3$

$$\Rightarrow \operatorname{disc}(L_d) = -d.$$

$$L_d^\perp \simeq K_d^\perp \simeq E \oplus U_2 \oplus B.$$

with $\operatorname{sign}(B) = (1, 2)$.

Proposition (Hassett–Nikulin)

All L_v, K_v are of the form L_d, K_d up to $\tilde{\mathrm{O}}(\Gamma)$-action.

Proof:

Apply Eichler criterion to $v \in \Gamma = E \oplus U_1 \oplus U_2 \oplus A_2(-1)$.

$(v)^2 \equiv 2, 4\,(6) \Rightarrow (v.\Gamma) = \mathbb{Z}$, i.e. $n = 1$, and $v = 0 \in A_\Gamma \simeq \mathbb{Z}/3\mathbb{Z}$

$\Rightarrow$ one choice for $\mathrm{O}(\Gamma)v$.

$(v)^2 \equiv 0\,(6) \Rightarrow$ 2 cases:

(i) $n = 1$ & $v = 0$ in $A_\Gamma \simeq \mathbb{Z}/3\mathbb{Z}$,

(ii) $n = 3$ & $(1/3)v = \pm 1$ in $A_\Gamma \simeq \mathbb{Z}/3\mathbb{Z}$.

Conditions on L_d, K_d, or d

$$
\begin{array}{lll}
(\mathrm{H}) & = (*) & A_2 \oplus \mathbb{Z}v_d \subset L_d \subset \widetilde{\Lambda} \text{ and } \mathbb{Z}h \oplus \mathbb{Z}v_d \subset K_d \subset \widetilde{\Gamma}, \\
(\mathrm{K3}') & = (**') & \exists \;\; U(n) \hookrightarrow L_d, \\
(\mathrm{K3}) & = (**) & \exists \;\; U \hookrightarrow L_d \;\Leftrightarrow\; K_d^{\perp} \simeq L_d^{\perp} \simeq \Lambda_d, \\
(\mathrm{K3}^{[2]}) & = (***) & \exists \;\; U \hookrightarrow L_d \text{ with } \lambda_1 \in U.
\end{array}
$$

$$
(\mathrm{K3}^{[2]}) \Rightarrow (\mathrm{K3}) \Rightarrow (\mathrm{K3}') \Rightarrow (\mathrm{H}).
$$

Write $(\mathrm{H})_0$ *and* $(\mathrm{H})_2$ *for* $d \equiv 0\,(6)$ *resp.* $d \equiv 2\,(6)$*. Similarly,* $(\mathrm{K3})_0$ *and* $(\mathrm{K3})_2$*, etc.*

$$
\begin{aligned}
&(\mathrm{H}) && = (*) && \Leftrightarrow\ d \equiv 0, 2\,(6),\\
&(\mathrm{K3}') && = (**') && \Leftrightarrow\ \exists\, w \in A_2 \colon (w)^2 = d,\\
& && && \Leftrightarrow \tfrac{d}{2} = \prod p^{n_p} \text{ with } n_p \equiv 0\,(2)\ \forall\, p \equiv 2\,(3),\\
&(\mathrm{K3}) && = (**) && \Leftrightarrow\ \exists \text{ primitive } w \in A_2 \colon (w)^2 = d,\\
& && && \Leftrightarrow \tfrac{d}{2} = \prod p^{n_p} \text{ with } n_p = 0\ \forall\, p \equiv 2\,(3) \text{ and } n_3 \leq 1,\\
& && && \Leftrightarrow d = \tfrac{2n^2+2n+2}{a},\ \ a, n \in \mathbb{Z},\\
&(\mathrm{K3}^{[2]}) && = (***) && \Leftrightarrow d = \tfrac{2n^2+2n+2}{a^2},\ \ a, n \in \mathbb{Z}.
\end{aligned}
$$

$$
(\mathrm{K3}^{[2]}) \Rightarrow (\mathrm{K3}) \Rightarrow (\mathrm{K3}') \Rightarrow (\mathrm{H}).
$$

$(\mathrm{K3}^{[2]})$			14				26				38	42
(K3)			14				26				38	42
(K3')	8		14	18		24	26		32		38	42
(H)	8	12	14	18	20	24	26	30	32	36	38	42

$(\mathrm{K3}^{[2]})$							62					
(K3)							62				74	78
(K3')			50				62		68		74	78
(H)	44	48	50	54	56	60	62	66	68	72	74	78